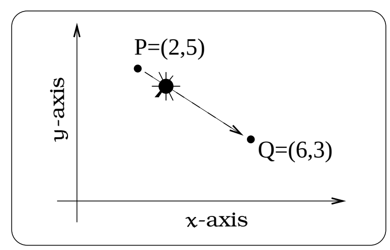
Chapter 23

Linear Motion

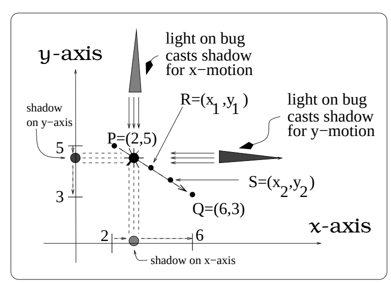
The simplest example of a parametrized curve arises when studying the motion of an object along a straight line in the plane. We will start by studying this kind of motion when the starting and ending locations are known.

23.1 Motion of a Bug

Example 23.1.1. *A bug is spotted at* $P = (2, 5)$ *in the* xy*-plane. The bug walks in a straight line from* P *to* $Q = (6, 3)$ *at a constant speed* s*. It takes the bug 5 seconds to reach* Q*. Assume the units of our coordinate system are feet. What is the speed* s *of the bug along the line connecting* P *and* Q*? Compute the horizontal and vertical speeds of the bug and show they are both constant.*



(a) A bug walking from P to Q.



(b) How to view motion in the coordinates.

Figure 23.1: Visualizing the model for the bug problem.

Solution. A standard technique in motion problems is to analyze the x and y-motion separately. This means we look at the projection of the bug location onto the x and y-axis separately, studying how each projection moves. We can think of these projections as "shadows" cast by a flashlight onto the two axes:

For the x-motion, we study the "shadow" on the x-axis which starts at "2" and moves toward "6" on the x-axis. For the y-motion, we study the "shadow" on the y-axis which starts at "5" and moves toward "3" along the y-axis.

In general, speed is computed by dividing distance by time elapsed, so

$$\begin{aligned} s &= \frac{\text{dist}(P, Q)}{5}\frac{\text{ft}}{\text{sec}} = \frac{\sqrt{(5-3)^2 + (2-6)^2}}{5}\frac{\text{ft}}{\text{sec}} \\ &= \frac{2\sqrt{5}}{5}\frac{\text{feet}}{\text{sec}}. \end{aligned} \tag{23.1}$$

This is the speed of the bug along the line connecting P and Q.

The hard part of this problem is to show that the speed of the horizontal and vertical shadows are also constant. This might seem obvious when you first think about it. In order to actually show it, let's take two intermediary positions $R = (x_1, y_1)$ and $S = (x_2, y_2)$ along the bugs path. We are going to relate the *horizontal speed* v_x of the bug between x_1 and x_2, the *vertical speed* v_y of the bug between y_1 and y_2 and the speed s of the bug from R to S. Actually, because there are positive or negative directions for the x and y axes, we will allow horizontal and vertical "speed" to be a $\pm$ quantity, with the obvious meaning. If it takes T seconds for the bug to travel from R to S, then T is the elapsed time for the horizontal motion from x_1 to x_2 and also the elapsed time for the vertical motion from y_1 to y_2. The horizontal speed v_x is the *directed distance* $\Delta x = (x_2 - x_1)$ divided by the time elapsed T, whereas the vertical speed v_y is the *directed distance* $\Delta y = (y_2 - y_1)$ divided by the time elapsed T.

We want to show that v_x and v_y are both constants! To do this, we have these three equations:

$$\begin{aligned}
v_x &= \frac{\Delta x}{T} = \frac{x_2 - x_1}{T} \\
v_y &= \frac{\Delta y}{T} = \frac{y_2 - y_1}{T} \\
s &= \frac{\text{distance}(R,S)}{T} = \frac{\sqrt{\Delta x^2 + \Delta y^2}}{T} = \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}{T}.
\end{aligned}$$

Now, square each side of the three equations and combine them to conclude: $s^2T^2 = v_x^2T^2 + v_y^2T^2$. We can multiply through by T^2 and that gives us the key equation:

$$s^2 = v_x^2 + v_y^2. \tag{23.2}$$

On the other hand, the ratio of the vertical and horizontal speed gives

$$\begin{aligned}
\frac{v_y}{v_x} &= \frac{\left(\frac{\Delta y}{T}\right)}{\left(\frac{\Delta x}{T}\right)} = \frac{\Delta y}{\Delta x} \\
&= \frac{y_2 - y_1}{x_2 - x_1} = \text{"slope of line connecting P and Q"} \\
&= -\frac{1}{2} \\
\therefore \frac{v_y}{v_x} &= -\frac{1}{2}.
\end{aligned} \tag{23.3}$$

Solving for v_y in terms of v_x we can write

$$v_y = -\frac{1}{2}v_x. \tag{23.4}$$

Since v_x is positive,

$$s = \sqrt{v_x^2 + v_y^2} = \sqrt{v_x^2 + \left(-\frac{1}{2}v_x\right)^2} = \frac{\sqrt{5}}{2}|v_x| = \frac{\sqrt{5}}{2}v_x;$$

that is,

$$v_x = \frac{2}{\sqrt{5}} s = \frac{2}{\sqrt{5}} \left(\frac{2\sqrt{5}}{5} \right) = \frac{4}{5} = 0.8\,\frac{\text{feet}}{\text{sec}}.$$

By (23.4), $v_y = \left(-\frac{1}{2}\right)\frac{4}{5} = -0.4\frac{\text{ft}}{\text{sec}}$. This shows the speed in both the horizontal and vertical directions is constant as the bug moves from R to S. Since R and S were *any* two intermediary points between P and Q, the horizontal and vertical bug speeds are constant. □

Example 23.1.2. *Parametrize the motion in the previous problem.*

Solution. We have shown $v_x = 0.8$ feet/sec and $v_y = -0.4$ feet/sec, so

$$\begin{aligned} x(t) &= (x\text{-coordinate of the bug at time } t) \\ &= (\text{beginning } x\text{-coordinate}) + \begin{pmatrix}\text{distance traveled in} \\ \text{the } x\text{-direction in } t \\ \text{seconds}\end{pmatrix} \\ &= 2 + 0.8t. \end{aligned}$$

$$\begin{aligned} y(t) &= (y\text{-coordinate of the bug at time } t) \\ &= (\text{beginning } y\text{-coordinate}) + \begin{pmatrix}\text{distance traveled in} \\ \text{the } y\text{-direction in } t \\ \text{seconds}\end{pmatrix} \\ &= 5 - 0.4t. \end{aligned}$$

As a check, notice that $P(0) = (2 + (0.8)0, 5 - (0.4)0) = (2, 5)$ is the starting location and $P(5) = (2+5(0.8), 5-5(0.4)) = (6, 3)$ is the ending location. □

23.2 General Setup

Given two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the plane, we can study motion of an object along the line connecting P and Q. In so doing, you need to first specify the starting location and the ending location of the object; lets say we start at P and proceed to Q. Fix the distance units used in the coordinate system (feet, inches, miles, meters, etc.) and the time units used (seconds, hours, years, etc.). As highlighted in the solution of Example 23.1.2, the key is to analyze the x-motion and y-motion separately.

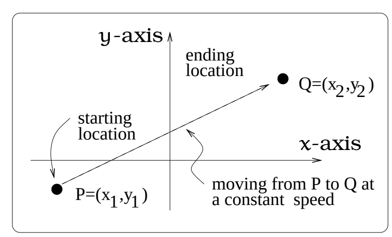


Figure 23.2: Typical scenario.

We will be imposing an assumption that the speed along the line connecting P and Q in Figure 23.1(b) is a constant. There is a crucial observation we need to make, a special case of which was the content of Example 23.1.1.

Important Facts 23.2.1 (Linear motion). *Assume that an object moves from P to Q along a straight line at a constant speed s, as in Figure 23.2.*

Then the speed v_x in the x-direction and the speed v_y in the y-direction are both constant. We also have two useful formulas:

$$s^2 = v_x^2 + v_y^2,$$
$$\frac{v_y}{v_x} = \begin{cases} \textit{slope of line of travel,} \\ \textit{when the line is non-} \\ \textit{vertical.} \end{cases}$$

This fact is established using the same reasoning as in Example 23.1.1. Let's make a few comments. To begin with, if the line of travel is either vertical or horizontal, then either $v_x = 0$ or $v_y = 0$ and Fact 23.2.1 isn't really saying anything of interest.

¡¡¡ CAUTION !!!

The formulas in Fact 23.2.1 only work for linear motion.

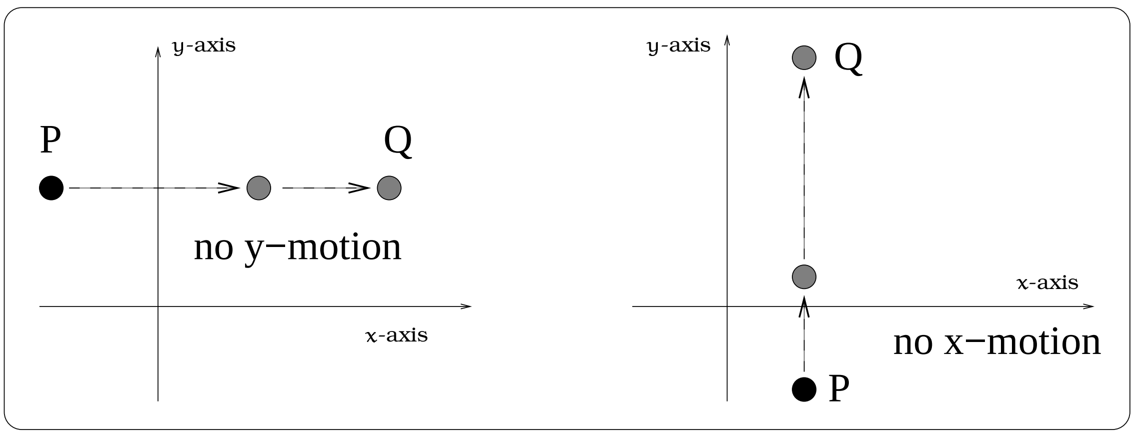


Figure 23.3: Horizontal or vertical motion.

For any other line of travel, we can use the reasoning used in Example 23.1.1. Pay attention that the *horizontal speed* v_x and the *vertical speed* v_y are both *directed quantities*; i.e. these can be positive or negative. The sign of v_x will indicate the direction of motion: If v_x is positive, then the horizontal motion is to the right and if v_x is negative, then the horizontal motion is to the left. Similarly, the sign of v_y tells us if the vertical motion is upward or downward.

Returning to Figure 23.2, to describe the x-motion, two pieces of information are needed: the starting location (in the x-direction) and the constant speed v_x in the x-direction. So,

$$\begin{aligned} x(t) &= (x\text{-coordinate of the object at time } t) \\ &= (\text{beginning } x\text{-coordinate}) + \begin{pmatrix} \text{distance traveled in} \\ \text{the } x\text{-direction in } t \\ \text{time units} \end{pmatrix} \\ &= x_1 + v_x \cdot t. \end{aligned}$$

If we are not given the horizontal velocity directly, rather the time T required to travel from P to Q, then we could compute v_x using the fact that the object starts at $x = x_1$ and travels to $x = x_2$:

$$\begin{aligned}
v_x &= \frac{\text{(directed horizontal distance traveled)}}{\text{(time required to travel this distance)}} \\
&= \frac{(\text{ending } x\text{-coordinate}) - (\text{starting } x\text{-coordinate})}{\text{(time required to travel this distance)}} \\
\therefore v_x &= \frac{\Delta x}{T} = \frac{x_2 - x_1}{T}.
\end{aligned}$$

To describe the y-motion in Figure 23.2, we proceed similarly. We will denote by v_y the constant vertical speed of the object, then after t time units the object has traveled $v_y \cdot t$ units. So,

$$\begin{aligned}
y(t) &= (y\text{-coordinate of object at time } t) \\
&= (\text{beginning } y\text{-coordinate}) + \begin{pmatrix}\text{distance traveled in} \\ \text{the } y\text{-direction in } t \\ \text{time units}\end{pmatrix} \\
&= y_1 + v_y \cdot t.
\end{aligned}$$

In summary,

Important Fact 23.2.2 (Linear motion). *Suppose an object begins at a point $P = (x_1, y_1)$ and moves at a constant speed s along a line connecting P to another point $Q = (x_2, y_2)$. Then the motion of this object will trace out a line segment which is parametrized by the equations:*

$$\begin{aligned}
x &= x(t) = x_1 + v_x \cdot t, \\
y &= y(t) = y_1 + v_y \cdot t.
\end{aligned}$$

Example 23.2.3. *Return to the linear motion problem studied in Example 23.1.1 and 23.1.2. However, now assume that the point $Q = (6, 3)$ is located at the center of a circular region of radius 1 ft. When and where does the bug enter this circular region?*

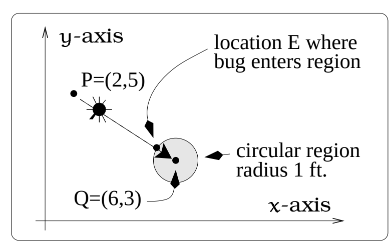


Figure 23.4: A bug crosses a circular boundary.

Solution. The parametric equations for the linear motion of the bug are given by:

$$\begin{aligned}
x = x(t) = 2 + 0.8t, \\
y = y(t) = 5 - 0.4t.
\end{aligned}$$

The equation of the boundary of the circular region centered at Q is given by

$$(x - 6)^2 + (y - 3)^2 = 1.$$

To find where and when the bug crosses into the circular region, we determine where and when the linear path and the circle equation have a simultaneous solution. To find such a location, we simply plug $x = x(t)$ and $y = y(t)$ into the circle equation:

$$\begin{aligned}
(x(t) - 6)^2 + (y(t) - 3)^2 &= 1 \\
(2 + 0.8t - 6)^2 + (5 - 0.4t - 3)^2 &= 1 \\
16 - 6.4t + 0.64t^2 + 4 - 1.6t + .16t^2 &= 1 \\
0.8t^2 - 8t + 19 &= 0.
\end{aligned}$$

Notice, this is an equation in the single variable t. Finding the solution of this equation will tell us *when* the bug crosses into the region. Once we know *when* the bug crosses into the region, we can determine the location by plugging this time value into our parametric equations. By the quadratic formula, we find the solutions are

$$t = \frac{8 \pm \sqrt{64 - 4(19)(0.8)}}{1.6} = 6.12 \text{ or } 3.88.$$

We know that the bug reaches the point Q in 5 seconds, so the second solution $t = 3.88$ seconds is the time when the bug crosses into the circular region. (If the bug had continued walking in a straight line directly through Q, then the time when the bug leaves the circular region would correspond to the other solution $t = 6.12$ seconds.) Finally, the location E of the bug when it crosses into the region is $E = (x(3.88), y(3.88)) = (5.10, 3.45)$. □