Chapter 22

Parametric Equations

Imagine a car is traveling along the highway and you look down at the situation from high above:

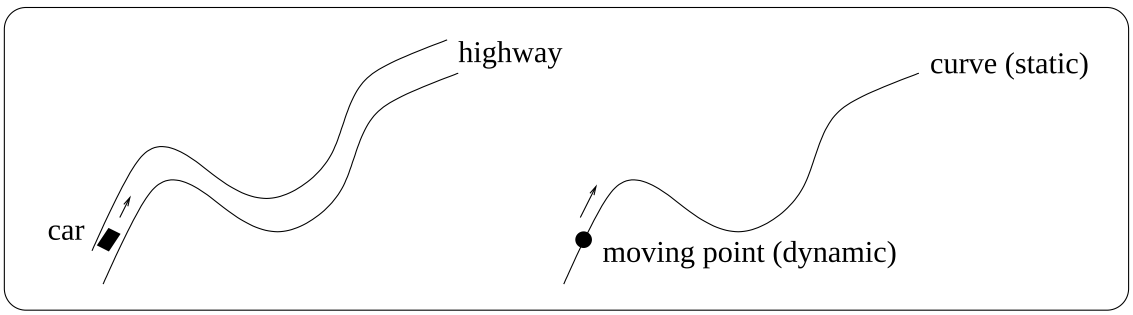
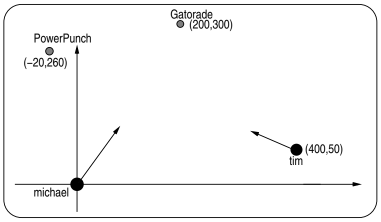


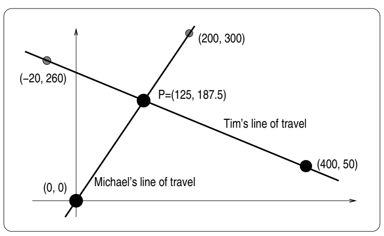
Figure 22.1: The dynamic motion of a car on a static highway.

We can adopt at least two different viewpoints: We can focus on the entire highway all at once, which is modeled by a curve in the plane; this is a "static viewpoint". We could study the movement of the car along the highway, which is modeled by a point moving along the curve; this is a "dynamic viewpoint". The ideas in this chapter are "dynamic", involving motion along a curve in the plane; in contrast, our previous work has tended to involve the "static" study of a curve in the plane. We will combine our understanding of linear functions, quadratic functions and circular functions to explore a variety of dynamic problems.

22.1 Parametric Equations



(a) Tim and Michael running toward refreshments.



(b) Modeling Tim and Michael as points moving on a path.

Figure 22.2: Visualizing moving points.

Example 22.1.1. *After a vigorous soccer match, Tim and Michael decide to have a glass of their favorite refreshment. They each run in a straight line along the indicated paths at a speed of 10 ft/sec. Will Tim and Michael collide?*

Solution. As a first step, we can model the lines along which both Tim and Michael will travel:

$$\text{Michael's line of travel: } f(x) = \frac{3}{2}x, \text{ and}$$
$$\text{Tim's line of travel: } g(x) = -\frac{1}{2}(x-400)+50.$$

It is an easy matter to determine where these two lines cross: Set $f(x) = g(x)$ and solve for x, getting $x = 125$, so the lines intersect at $P = (125, 187.5)$.

Unfortunately, we have NOT yet determined if the runners collide. The difficulty is that we have found where the two lines of travel cross, but we have not worried about the individual locations of Michael and Tim along the lines of travel. In fact, if we compute the distance from the starting point of each person to P, we find:

$$\begin{aligned} \text{dist(Mike, P)} &= \sqrt{(0-125)^2+(0-187.5)^2} \\ &= 225.35 \text{ feet} \\ \text{dist(Tim, P)} &= \sqrt{(400-125)^2+(50-187.5)^2} \\ &= 307.46 \text{ feet} \end{aligned}$$

Since these distances are different and both runners have the same speed, Tim and Michael do not collide! □

22.2 Motivation: Keeping track of a bug

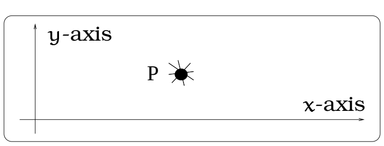


Figure 22.3: A bug on your desktop.

Imagine a bug is located on your desktop. How can you best study its motion as time passes?

Let's denote the location of the bug when you first observed it by P. If we let t represent time elapsed since first spotting the bug (say in units of seconds), then we can let $P(t)$ be the *new* location of the bug at time t. When $t = 0$, which is the instant you first spot the bug, the location $P(0) = P$ is the initial location. For example, the path followed by the bug might look something like the dashed path in the next Figure; we have indicated the bug's explicit position at four future times: $t_1 < t_2 < t_3 < t_4$.

How can we describe the curve in Figure 22.4? To start, lets define a couple of *new* functions. Given a time t, we have the point $P(t)$ in the plane, so we can define:

$$x(t) = x\text{-coordinate of } P(t) \text{ at time } t,$$
$$y(t) = y\text{-coordinate of } P(t) \text{ at time } t.$$

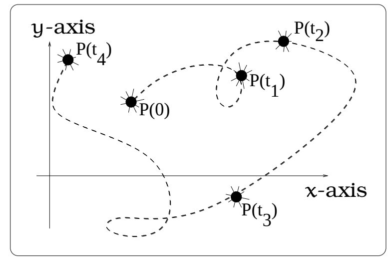


Figure 22.4: A bug's path.

In other words, the point $P(t)$ is described as

$$P(t) = (x(t), y(t)).$$

We usually call $x = x(t)$ and $y = y(t)$ the *coordinate functions* of $P(t)$. Also, it is common to call the pair of functions $x = x(t)$ and $y = y(t)$ the *parametric equations* for the curve. Anytime we describe a curve using parametric equations, we usually call it a *parametrized curve.*

Given parametric equations $x = x(t)$ and $y = y(t)$, the *domain* will be the set of t values we are allowed to plug in. Notice, we are using the same set of t-values to plug into both of the equations. Describing the curve in Figure 22.4 amounts to finding the parametric equations $x(t)$ and $y(t)$. In other words, we typically want to come up with "formulas" for the functions $x(t)$ and $y(t)$. Depending on the situation, this can be easy or very hard.

22.3 Examples of Parametrized Curves

We have already worked with some interesting examples of parametric equations.

Example 22.3.1. *A bug begins at the location (1,0) on the unit circle and moves counterclockwise with an angular speed of $\omega = 2$ rad/sec. What are the parametric equations for the motion of the bug during the first 5 seconds? Indicate, via "snapshots", the location of the bug at 1 second time intervals.*

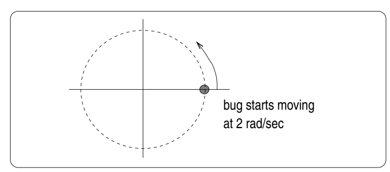


Figure 22.5: A circular path.

Solution. We can use Fact 14.2.2 to find the angle swept out after t seconds: $\theta = \omega t = 2t$ radians. The parametric equations are now easy to describe:

$$x = x(t) = \cos(2t)$$
$$y = y(t) = \sin(2t).$$

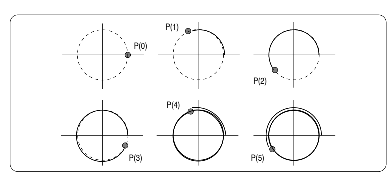


Figure 22.6: Six snapshots.

If we restrict t to the domain $[0, 5]$, then the location of the bug at time t is given by $P(t) = (\cos(2t), \sin(2t))$. We locate the bug via six one-second snapshots: □

When modeling motion along a curve in the plane, we would typically be given the curve and try to find the parametric equations. We can turn this around: Given a pair of functions $x = x(t)$ and $y = y(t)$, let

$$P(t) = (x(t), y(t)), \tag{22.1}$$

which assigns to each input t a point in the xy-plane. As t ranges over a given domain of allowed t values, we will obtain a collection of points in the plane. We refer to this as the *graph* of the parametric equations $\{x(t), y(t)\}$. Thus, we have now described a process which allows us to obtain a picture in the plane given a pair of equations in a common single variable t. Again, we call curves that arise in this way *parametrized curves*. The terminology comes from the fact we are describing the curve using an auxiliary variable t, which is called the describing "parameter". In applications, t often represents time.

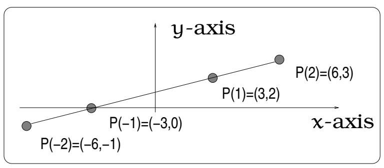


Figure 22.7: Observing the motion of P.

Example 22.3.2. *The graph of the parametric equations* $x(t) = 3t$ *and* $y(t) = t + 1$ *on the domain* $-2 \leq t \leq 2$ *is pictured; it is a line segment. As we let* t *increase from* -2 *to* 2*, we can observe the motion of the corresponding points on the curve.*

22.4 Function graphs

It is important to realize that the graph of every function can be thought of as a parametrized curve. Here is the reason why: Given a function $y = f(x)$, recall the graph consists of points $(x, f(x))$, where x runs over the allowed domain values. If we define

$$x = x(t) = t$$
$$y = y(t) = f(t),$$

then plotting the points $P(t) = (x(t), y(t)) = (t, f(t))$ gives us the graph of f. We gain one important thing with this new viewpoint: Letting $x = t$ increase in the domain, we now have the ability to dynamically view a point $P(t)$ moving along the function graph. See how this works in Example 22.4.1.

Example 22.4.1. *Consider the function* $y = x^2$ *on the domain* $-2 \leq x \leq 2$*. As a parametrized curve, we would view the graph of* $y = x^2$ *as all points of the form* $P(t) = (t, t^2)$*, where* $-2 \leq t \leq 2$*. If* t *increases from* -2 *to* 2*, the corresponding points* $P(t)$ *move along the curve as pictured:*

Solution. For example, $P(-2) = (-2, (-2)^2) = (-2, 4)$, $P(1) = (1, 1^2) = (1, 1)$, etc. ∎

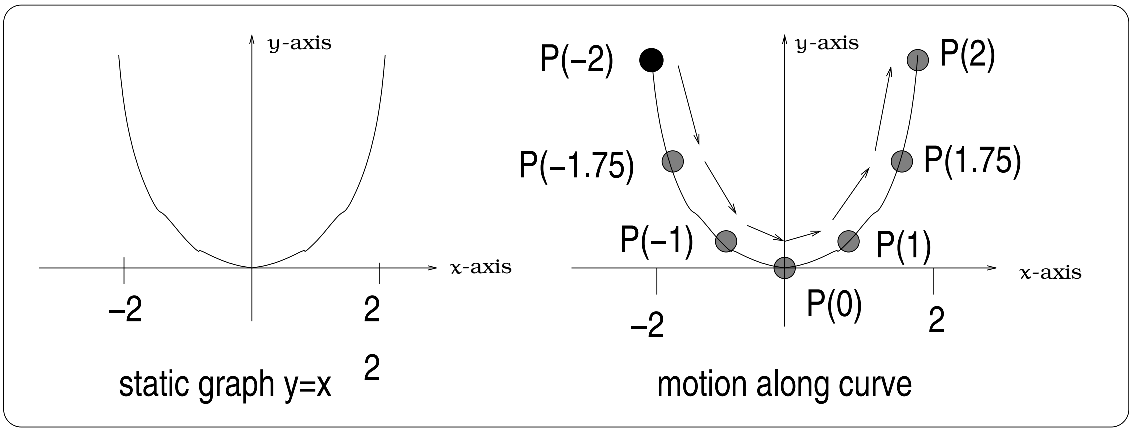


Figure 22.8: Visualizing dynamic motion along a static curve.

Not every parametrized curve is the graph of a function. For example, consider these possible curves in the plane: The second curve from the left is the graph of a function; the other curves violate the vertical line test.

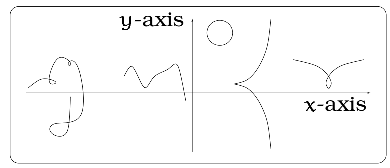


Figure 22.9: Some curves that are not functions.

22.4.1 A useful trick

There is an approach to understanding a parametrized curve which is sometimes useful: Begin with the equation $x = x(t)$. Solve the equation $x = x(t)$ for t in terms of the single variable x; i.e., obtain $t = g(x)$. Then substitute $t = g(x)$ into the other equation $y = y(t)$, leading to an equation involving only the variables x and y. If we were given the allowed t values, we can use the equation $x = x(t)$ to determine the allowed x values, which will be the domain of x values for the function $y = y(g(x))$. This may be a function with which we are familiar or can plot using available software.

Example 22.4.2. *Start with the parametrized curve given by the equations* $x = x(t) = 2t + 5$ *and* $y = y(t) = t^2$, *when* $0 \leq t \leq 10$. *Find a function* $y = f(x)$ *whose graph gives this parametrized curve.*

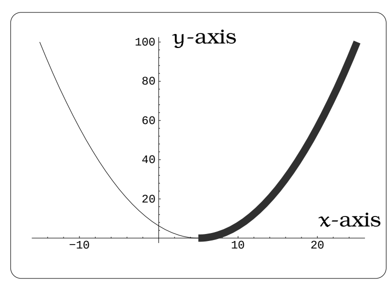


Figure 22.10: Finding the path equation.

Solution. Following the suggestion, we begin by solving $x = 2t + 5$ for t, giving $t = \frac{1}{2}(x - 5)$. Plugging this into the second equation gives $y = \left(\frac{1}{2}(x-5)\right)^2 = \left(\frac{1}{4}\right)(x-5)^2$. Conclude that (x, y) is on the parametrized curve if and only if the equation $y = f(x) = \left(\frac{1}{2}(x-5)\right)^2 = \left(\frac{1}{4}\right)(x-5)^2$ is satisfied. This is a quadratic function, so the graph will be an upward opening parabola with vertex (5,0).

Since the t domain is $0 \leq t \leq 10$, we get a new inequality for the x domain: $0 \leq \left(\frac{1}{2}\right)(x - 5) \leq 10$. Solving this, we get $0 \leq x - 5 \leq 2(10)$, so

$5 \le x \le 25$. This means the graph of the parametrized curve is the graph of the function $y = \left(\frac{1}{4}\right)(x-5)^2$, with the domain of x values $[5, 25]$. Here is a plot of the graph of $y = f(x)$; the thick portion is the parametrized curve we are studying. □

22.5 Circular motion

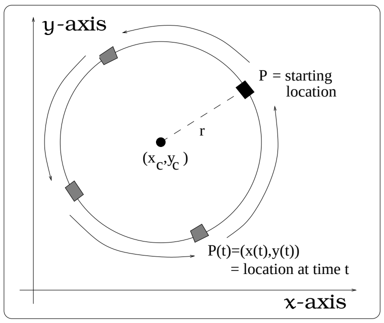


Figure 22.11: Circular motion.

We can describe the motion of an object around a circle using parametric equations. This will involve the trigonometric functions. The general setup to imagine is pictured: An object moving around a circle of radius r centered at a point (x_c, y_c) in the xy-plane. The path traced out is the circle. However, the location of the object at time t will depend on a number of things:

- The starting location P of the object;
- The angular speed ω of the object;
- The radius r and the center (x_c, y_c).

We will build up to the general solution by considering two cases, the first being a special case of the second.

22.5.1 Standard circular motion

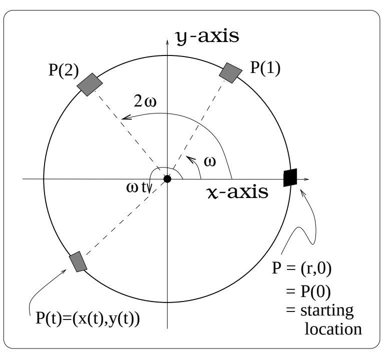


Figure 22.12: Standard circular motion.

As a first case to consider, assume that the center of the circle is $(0, 0)$ and the starting location $P = (r, 0)$, as pictured below. If the angular speed is ω, then the angle θ swept out in time t will be $\theta = \omega t$; this requires that the time units in ω agree with the time units of t! We denote by $P(t) = (x(t), y(t))$ the xy-coordinates of the object at time t. At time t, we can compute the coordinates of $P(t) = (x(t), y(t))$ using the circular functions:

$$x = x(t) = r\cos(\omega t)$$
$$y = y(t) = r\sin(\omega t).$$

This parametrizes motion starting at $P = (r, 0)$. Using the shifting technology of Chapter 9, we are led to a general description of this type of circular motion, which involves a circle of radius r centered at a point (x_c, y_c); we refer to this situation as *standard circular motion*.

Important Fact 22.5.1 (Standard circular motion). *Assume an object is moving around a circle of radius r centered at (x_c, y_c) with a constant angular speed of ω. Assume the object begins at $P = (x_c + r, y_c)$. Then the location of the object at time t is given by the parametric equations:* $x = x(t) = x_c + r\cos(\omega t)$ *and* $y = y(t) = y_c + r\sin(\omega t)$.

Example 22.5.2. *Cosmo the dog is tied to a 20 foot long tether, as in Figure 14.1. Assume Cosmo starts at the location "R" in the Figure and maintains a tight tether, moving around the circle at a constant angular speed $\omega = \frac{\pi}{5}$ radians/second. Parametrize Cosmos motion and determine where the dog is located after 3 seconds and after 3 minutes.*

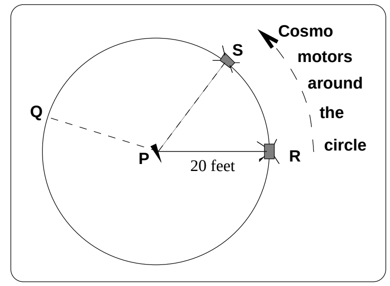


Figure 22.13: Cosmo on a running on a circular path.

Solution. Impose a coordinate system so that the pivot point of the tether is $(x_0, y_0) = (0, 0)$. Since $\omega > 0$, Cosmo is walking counterclockwise around the circle. By (4.1.6), the location of Cosmo after t seconds is $P(t) = (x(t), y(t)) = \left(20\cos\left(\frac{\pi}{5}t\right), 20\sin\left(\frac{\pi}{5}t\right)\right)$. After 3 seconds, Cosmo is located at $P(3) = \left(20\cos\left(\frac{3\pi}{5}\right), 20\sin\left(\frac{3\pi}{5}\right)\right) = (-6.18, 19.02)$. After 3 minutes = 180 seconds, the location of the dog will be $P(180) = \left(20\cos\left(\frac{180\pi}{5}\right), 20\sin\left(\frac{180\pi}{5}\right)\right) = (20{,}0)$, which is the original starting point. □

22.5.2 General circular motion

The circular motion of an object can begin at any location P on the circle. To handle the general case, we follow an earlier idea and introduce an auxiliary*relative coordinate system*: The $x_{rel}y_{rel}$-coordinates are obtained by drawing lines parallel to the xy-axis and passing through (x_c, y_c). We are using the subscript "rel" to stand for "relative". This new relative coordinate system has origin (x_c, y_c) and allows us to define the *initial angle* θ_0, which indexes the starting location P, as pictured below:

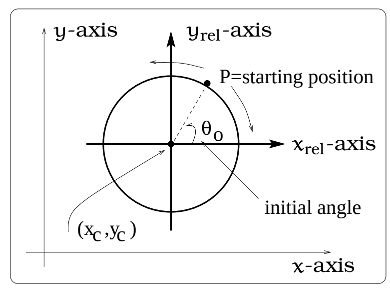


Figure 22.14: Initial angle and auxiliary axis.

Assume the object starts at P and is moving at a constant angular speed ω around the pictured circle of radius r. Then after time t has elapsed, the location of the object is indexed by sweeping out an angle ωt, **starting from** θ_0. In other words, the location after t units of time is going to be determined by the central standard angle $\theta_0 + \omega t$ with initial side the positive x_{rel}-axis. This means that if $P(t) = (x(t), y(t))$ is the location of the object at time t,

$$\begin{aligned} x &= x(t) = x_c + r\cos(\theta_0 + \omega t) \\ y &= y(t) = y_c + r\sin(\theta_0 + \omega t). \end{aligned}$$

Notice, the case of standard circular motion is just the scenario when $\theta_0 = 0$ and these parametric equations collapse to those of Fact 22.5.1.

Important Fact 22.5.3 (General circular motion). *Assume an object is moving around a circle of radius r centered at (x_c, y_c) with a constant angular speed of ω. Assume the object begins at the location P with initial* *angle θ_0, as in Figure 22.14. The object location at time t is given by: $x = x(t) = x_c + r\cos(\theta_0 + \omega t)$ and $y = y(t) = y_c + r\sin(\theta_0 + \omega t)$.*

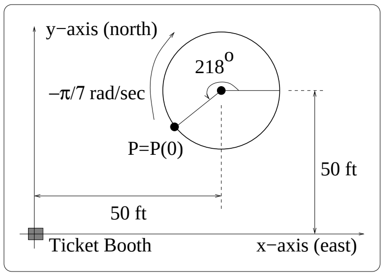


Figure 22.15: A rider jumps on a merry-go-round.

Example 22.5.4. *A rider jumps on a merry-go-round of radius 20 feet at the pictured location. The ride rotates at the constant angular speed of $\omega = -\frac{\pi}{7}$ radians/second. The center of the platform is located 50 feet East and 50 feet North of the ticket booth for the ride. What are the parametric equations describing the location of the rider? Where is the rider after 18 seconds have elapsed? How far from the ticket booth is the rider after 18 seconds have elapsed?*

Solution. In this example, since $\omega < 0$, the rotation is clockwise. Since the angular speed is given in radians, we need to convert the initial angle to radians as well: $218° = 3.805$ radians. Impose a coordinate system so that the center of the ride is $(50, 50)$ and its radius is 20 feet. By Fact 22.5.3, the parametric equations for the rider are given by $x = x(t) = 50 + 20\cos\left(3.805 - \frac{\pi}{7}t\right)$ and $y = y(t) = 50 + 20\sin\left(3.805 - \frac{\pi}{7}t\right)$. The location after 18 seconds will be

$$\begin{aligned} P(18) &= (x(18), y(18)) \\ &= (50 + 20\cos(-4.273), 50 + 20\sin(-4.273)) \\ &= (41.49, 68.10). \end{aligned}$$

The distance from $P(18)$ to the origin is

$$\begin{aligned} d &= \sqrt{(41.49)^2 + (68.10)^2} \\ &= 79.74 \text{ feet.} \end{aligned}$$

□