Brownian Motion with Darning

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Preface

This is an expanded version of a preliminary set of notes for a series of lectures that I give at Kyoto University from January to March, 2012. It contains more material than I have covered in these lectures.

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Chapter 1

Brownian Motion with Darning

1.1 What is Brownian motion with darning?

Let X be Brownian motion on $\mathbb{R}^d$. For a nearly Borel set $A \subset \mathbb{R}^d$, a point x is said to be regular for A if $\mathbb{P}_x(\sigma_A = 0) = 1$. Here $\sigma_A := \inf\{t > 0 : X_t \in A\}$ is the first hitting time of A by Brownian motion X. We use A^r to denote all the regular points of A. A nearly Borel set A in $\mathbb{R}^d$ is said to be polar if $\mathbb{P}_x(\sigma_A < \infty) = 0$ for all $x \in \mathbb{R}^d$. It is well-known that for every nearly Borel set A, $A \setminus A^r$ is polar (see [16, Proposition 6.3 on p.44]). It is also known that when $d = 1$, $A^r = \overline{A}$ (see [16, Proposition 3.2 on p.30]). Lebesgue showed that when $d = 2$, any connected subset B of $\mathbb{R}^2$ that contains at least two points is non-polar and $B \subset B^r$ (see [16, Proposition 7.2 on p.47]).

Suppose that E a domain (open connected subset) of $\mathbb{R}^d$, and $K_1, \dots, K_N$ are quasi-separated non-polar finely closed relatively compact subsets of E. Let $D = E \setminus \cup_{j=1}^N K_j$. Intuitively speaking, Brownian motion with darning on $D^* := D \cup \{a_1^*, \dots, a_N^*\}$ is a Brownian motion in E by "shorting" each K_j into a single point a_j^*. Sometimes we also use K_j^* to denote the point a_j^*. For such a purpose, we may assume without loss of generality[1] that $K_j \subset K_j^r$. But for the convenience of describing the topology on D^*, in this notes we assume that each K_j is compact but put no assumptions on the regular points of K_j, that is, we *do not* assume $K_j \subset K_j^r$.

Formally, by identifying each K_j with a single point a_j^*, we can get an induced topological space $D^* := D \cup \{a_1^*, \dots, a_N^*\}$ from E, with a neighborhood of each a_j^* defined as $(U \cap D) \cup \{a_j^*\}$ for some neighborhood U of K_j in E. Let m be the Lebesgue measure on D, extended to D^* by setting $m(K^*) = 0$, where $K^* := \{a_1^*, \dots, a_N^*\}$.

Definition 1.1.1 *Brownian motion with darning* (BMD in abbreviation) X^* is an m-symmetric diffusion on D^* such that

[1] In general, note that (cf. [2, Lemma A.2.18(i)]) $K_j \setminus K_j^r$ is semipolar and hence polar. Thus for every $x \in K_j \cap K_j^r$, since $\sigma_{K_j} = \sigma_{K_j \cap K_j^r} \wedge \sigma_{K_j \setminus K_j^r}$, $\mathbb{P}_x(\sigma_{K_j \cap K_j^r} = 0) = \mathbb{P}_x(\sigma_{K_j} = 0) = 1$; that is, every point of $K_j \cap K_j^r$ is regular for $K_j \cap K_j^r$. So we can take $K_j \cap K_j^r$ as new K_j, which is non-polar and *finely closed* (rather than closed) since $E \setminus K_j$ is finely open.

(i) its part process in D has the same law as Brownian motion in D;

(ii) it admits no killings on K^*.

Observe that it follows from the m-symmetry of X^* and the fact that $m(K^*) = 0$ that BMD X^* spends zero Lebesgue amount of time (i.e. zero sojourn time) at K^*. We point out that D can be disconnected.

Example 1.1.2 (One dimensional examples) Let $E = \mathbb{R}$.

(i) $N = 1$ and $K = [0, 1]$. In this case, $D^* \cong \mathbb{R}$ and BMD X^* on D^* is just the standard BM on $\mathbb{R}$ (see Figure 1.1).

Figure 1.1: Example 1.1.2(i)

(ii) $N = 1$ and $K = [0, 1/3] \cup [2/3, 1]$. D^* is homeomorphic to a knotted curve (see Figure 1.2) and X^* is BM on this graph.

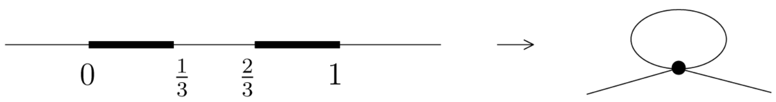

Figure 1.2: Example 1.1.2(ii)

(iii) $N = 1$ and $K = \{-1, 0, 1, 2\}$. The graph D^* has three knots hanging at the same point (see Figure 1.3). BMD X^* is BM on this graph.

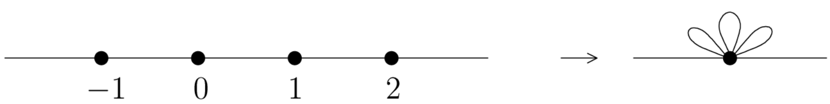

Figure 1.3: Example 1.1.2(iii)

(iv) $N = 2$, $K_1 = \{-1, 1\}$ and $K_2 = \{0, 2\}$. D^* is a graph consisting a circle and a line passing the center of the circle (see Figure 1.4). BMD X^* is BM on this graph.

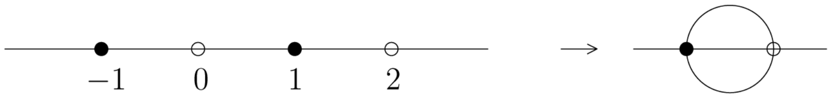

Figure 1.4: Example 1.1.2(iv)

(v) $N = 1$ and K is the Cantor subset of the unit interval $[0, 1]$. D^* is a graph with infinite degree (see Figure 1.5). BMD X^* is BM on this graph.

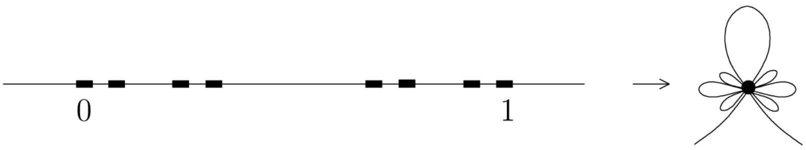

Figure 1.5: Example 1.1.2(v)

Example 1.1.3 (Multidimensional examples) Let $E = \mathbb{R}^d$ with $d \geq 2$.

(i) $N = 1$ and K is a non-polar connected compact subset of $\mathbb{R}^d$. See Figure 1.6.

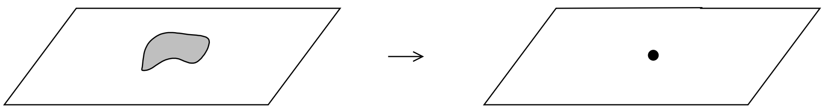

Figure 1.6: Example 1.1.3(i)

(ii) $N = 1$ and $K = \partial B(0, 1)$. D^* is homeomorphic to he plane with a sphere sitting on top of it. See Figure 1.7.

(iii) $N = 2$, $K_1 = B(0, 1)$ and $K_2 = B(x_0, 1)$ for some $x_0 \in \mathbb{R}^d$ with $|x_0| \geq 2$. See Figure 1.8.

(iv) $N = 2$, $K_1 = \partial B(0, 1)$ and $K_2 = \partial B(x_0, 2)$ for some $x_0 \in \mathbb{R}^d$ with $|x_0| \geq 4$. D^* is homeomorphic to the plane with a sphere sitting on top of it. See Figure 1.9.

(v) $N = 1$, $K = B(0, 1) \cup B(x_0, 1)$ for some $x_0 \in \mathbb{R}^d$ with $|x_0| \geq 2$.

(vi) $N = 1$ and $K = \partial B(0, 1) \cup \partial B(x_0, 2)$ for some $x_0 \in \mathbb{R}^d$ with $|x_0| \geq 4$. D^* is homeomorphic to the D^* in (iv) but with two points where the spheres touch the plane identified into one point.

(vii) $d = 2$, $N = 1$ and K is the Siepinski gasket or Siepinkski carpet in $\mathbb{R}^2$.

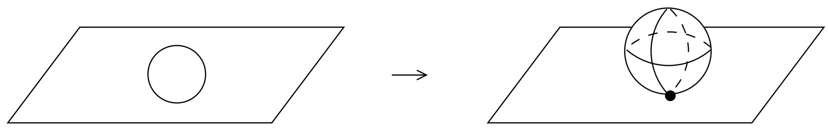

Figure 1.7: Example 1.1.3(ii)

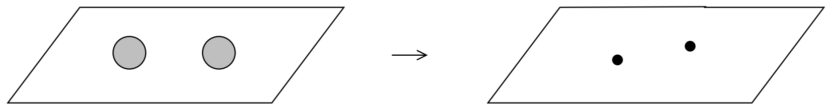

Figure 1.8: Example 1.1.3(iii)

Remark 1.1.4 (i) BMD with darning on $(E \setminus K)^*$ when K is a fractal-like set might be an interesting subject to study from the fractal geometry point of view.

(ii) One can also do darning (or shorting) for symmetric diffusions on $\mathbb{R}^d$ as well as on general state spaces. In fact, one can do darning for a large family of non-symmetric (possibly discontinuous) Markov processes. See [2, 4, 5, 6, 9]. The results developed in this lecture notes (except the conformal invariance property for planar BMD) can be easily adapted to be applicable to symmetric diffusions on general state spaces. □

1.2 Existence and Uniqueness

In this section, we show that BMD always exists and is unique in law.

As mentioned earlier, BMD on D^* can be intuitively thought of as obtained from Brownian motion on E by "shorting" each K_j. The Dirichlet form for the part process X^E of Brownian motion X killed upon leaving domain E is $(\mathbf{D}, W_0^{1,2}(E))$, where $\mathbf{D}(u,v) = \frac{1}{2}\int_E \nabla u(x) \cdot \nabla v(x) dx$ and $W_0^{1,2}(E)$ is the $\sqrt{\mathbf{D}_1}$-completion of $C_c^\infty(E)$. Here for $\alpha > 0$, $\mathbf{D}_\alpha(u,u) := \mathbf{D}(u,u) + \alpha \int_E u(x)^2 dx$. The quadratic form $(\mathbf{D}, W_0^{1,2}(E))$ is a regular Dirichlet form in $L^2(E; dx)$. For $u \in W_0^{1,2}(E)$, its energy measure

$$\mu_{\langle u \rangle}(dx) = |\nabla u(x)|^2 dx,$$

which is the same as its strongly local part $\mu^c_{\langle u \rangle}(dx)$ as the Dirichlet form $(\mathbf{D}, W_0^{1,2}(E))$ is strongly local.

Think $\mathbf{D}(u,u)$ as the energy for the potential (or voltage) u on E. "Shorting" on K_j means u is constant $\mathbf{D}$-q.e. on K_j. Denote by $(\mathcal{E}^*, \mathcal{F}^*)$ the Dirichlet form for BMD X^* on D^*. Then intuitively,

$$\mathcal{F}^* = \{u \in W_0^{1,2}(E) : u \text{ is constant } \mathbf{D}\text{-q.e. on each } K_j\}$$

and $\mathcal{E}^*(u,v) = \mathbf{D}(u,v)$ for $u, v \in \mathcal{F}^*$. Denote $K = \cup_{j=1}^N K_j$ and $\sigma_K := \inf\{t > 0 : X_t^E \in K\}$. It is well known that for every $u \in W_0^{1,2}(E)$ and $\alpha > 0$, $\mathbf{H}_K^\alpha u(x) := \mathbb{E}_x \left[e^{-\alpha \sigma_K} u(X^E_{\sigma_K}) \right]$ is in

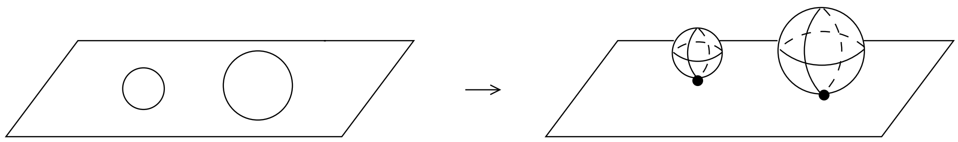

Figure 1.9: Example 1.1.3(iv)

$W_0^{1,2}(E)$ and $u - \mathbf{H}_K^\alpha u \in W_0^{1,2}(D)$. Moreover, $u - \mathbf{H}_K^\alpha u$ is the $\mathbf{D}_\alpha$-orthogonal projection of u into the closed subspace $W_0^{1,2}(D)$ of $W_0^{1,2}(E)$. Define for each j,

$$u_j(x) := \mathbb{E}_x \left[e^{-\sigma_K};\, X^E_{\sigma_K} \in K_j \right].$$

Since K_j is compact, $u_j = \mathbf{H}_K^1 f$ for any $f \in C_c^\infty(E)$ with $f = 1$ on K_j and $f = 0$ on other K_i's, so it is an element in $W_0^{1,2}(E)$ that is $\mathbf{D}_1$-orthogonal to $W_0^{1,2}(D)$. For $u \in \mathcal{F}^* \subset W_0^{1,2}(E)$, since u takes constant value, denoted as $u(K_j)$, $\mathbf{D}$-q.e. on each K_j, we have

$$\mathbf{H}_K^1 u(x) = \sum_{j=1}^N \mathbb{E}_x \left[e^{-\sigma_K} u(X^E_{\sigma_K});\, X^E_{\sigma_K} \in K_j \right] = \sum_{j=1}^N u(K_j) u_j(x).$$

As each K_j is non-polar, one has

$$\mathcal{F}^* = \text{linear span of } W_0^{1,2}(D) \text{ and } \{u_j, j = 1, \ldots, N\}$$

and for $u, v \in \mathcal{F}^*$,

$$\mathcal{E}^*(u,v) = \mathbf{D}(u,v) = \frac{1}{2} \int_D \nabla u(x) \cdot \nabla v(x) dx.$$

In the last equality, we used the fact that

$$\mu^c_{\langle u \rangle}(\cup_{j=1}^N K_j) = \int_{\cup_{j=1}^N K_j} |\nabla u(x)|^2 dx = 0 \quad \text{for any } u \in \mathcal{F}^*,$$

due to the following result that is valid for any quasi-regular Dirichlet form.

Theorem 1.2.1 *Let $(\mathcal{E}, \mathcal{F})$ be a generic quasi-regular Dirichlet form on $L^2(E; m)$, where E is a Lusin space. Suppose that $u \in b\mathcal{F}$. Then the push forward measure ν of $\mu^c_{\langle u \rangle}$ under map u defined by*

$$\nu(A) := \mu^c_{\langle u \rangle}(u^{-1}(A)), \quad A \in \mathcal{B}(\mathbb{R}),$$

is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. This in particular implies that $\mu^c_{\langle u \rangle}$ does not charge on level sets of u. Here $\mu^c_{\langle u \rangle}$ is the Revuz measure for $\langle M^{u,c} \rangle$, the predictable quadratic variation of the continuous part $M^{u,c}$ of the square-integrable martingale M^u appeared in Fukushima's decomposition of $u(X_t) - u(X_0)$.

Proof. It suffices to show that for any compact set $K \subset \mathbb{R}$ having zero Lebesgue measure, $\nu(K) = 0$. Let K be a compact set having zero Lebesgue measure. There exists a sequence $\{\phi_k, k \geq 1\}$ of continuous functions having compact support in $\mathbb{R}$ such that $|\phi_k| \leq 1$, $\lim_{k\to\infty} \phi_k(r) = 1_K(r)$ on $\mathbb{R}$, and

$$\int_0^\infty \phi_k(r)dr = \int_{-\infty}^0 \phi_k(r)dr = 0 \quad \text{for } k \geq 1.$$

The last display implies that each $\Phi_k(x) := \int_0^x \phi_k(r)dr$ is a C^1 function with compact support, $\Phi_k(0) = 0$ and $|\Phi_k'(x)| \leq 1$. Hence $\Phi_k(u)$ is a normal contraction of u and so $\Phi_k(u) \in \mathcal{F}$ with $\mathcal{E}(\Phi_k(u), \Phi_k(u)) \leq \mathcal{E}(u, u)$. Since $\lim_{k\to\infty} \Phi_k(r) = 0$ on $\mathbb{R}$, by dominated convergence theorem, $\Phi_k(u) \to 0$ in $L^2(E; m)$. Thus by Banach-Saks Theorem (see, e.g., [2, Theorem A.4.1]), taking the Cesàro mean sequence of a suitable subsequence of $\{\phi_k, k \geq 1\}$, and then redefining them as $\{\phi_k, k \geq 1\}$ if necessary, we may and do assume that $\Phi_k(u)$ is $\mathcal{E}_1$-convergent to $0 \in \mathcal{F}$. Now by Fatou's lemma and [2, Theorems 4.3.3(iii) and 4.3.7], we have

$$\begin{aligned} \nu(K) &\leq \lim_{k\to\infty} \int_{\mathbb{R}} \phi_k(r)^2 \nu(dr) = \lim_{k\to\infty} \int_E \phi_k(u(x))^2 \mu^c_{\langle u \rangle}(dx) \\ &= \lim_{k\to\infty} 2\mathcal{E}^c(\Phi_k(u), \Phi_k(u)) \leq 2 \lim_{k\to\infty} \mathcal{E}(\Phi_k(u), \Phi_k(u)) = 0. \end{aligned}$$

This completes the proof. □

Now we define

$$\mathcal{F}^* = \text{linear span of } W_0^{1,2}(D) \text{ and } \{u_j|_D, j = 1, \dots, N\} \tag{1.2.1}$$

and for $u, v \in \mathcal{F}^*$,

$$\mathcal{E}^*(u, v) = \frac{1}{2} \int_D \nabla u(x) \cdot \nabla v(x) dx. \tag{1.2.2}$$

Observe that

$$\mathcal{F}^* = \left\{ u|_D : \ u \in W_0^{1,2}(E), \ u \text{ is constant } \mathbf{D}\text{-q.e. on each } K_j \right\} \tag{1.2.3}$$

and

$$W_0^{1,2}(D) \subset \mathcal{F}^* \subset W^{1,2}(D) := \left\{ f \in L^2(D; dx) : \nabla f \in L^2(D; dx) \right\}.$$

Clearly, $(\mathcal{E}^*, \mathcal{F}^*)$ is a Dirichlet form on $L^2(D; dx) = L^2(D^*; m)$.

Theorem 1.2.2 *The quadratic form $(\mathcal{E}^*, \mathcal{F}^*)$ defined by (1.2.1)-(1.2.2) is a regular Dirichlet form on $L^2(D^*; m)$. It is strongly local and each a_j^* has positive capacity. Consequently, there is an m-symmetric diffusion X^* on D^* that starts from every point in D^* and admits no killings on D^*. The diffusion X^* is BMD on D^* and every a_j^* is regular for itself.*

Proof. Let $\mathcal{C} = \{u \in C_c^\infty(E) : u \text{ is constant on each } K_j\}$. By defining $u(a_j^*)$ to be the value of u on K_j, we can view $\mathcal{C}$ as a subspace of $C_c(D^*) \cap \mathcal{F}^*$. Since $\mathcal{C}$ is an algebra that separates points in D^*, by Stone-Weierstrass theorem, $\mathcal{C}$ is uniformly dense in $C_\infty(D^*)$. Next we show $\mathcal{C}$ is $\mathcal{E}_1^*$-dense in $\mathcal{F}^*$. For this, it suffices to establish that each u_j can be $\mathcal{E}_1$-approximated by elements in $\mathcal{C}$. Let $f_j \in C_c^\infty(E)$ so that $f_j = 1$ on K_j and $f_j = 0$ on K_i for $i \neq j$. Note that $u_j = \mathbf{H}_K^1 f_j = f_j - (f_j - \mathbf{H}_K^1 f_j)$ is a $\mathbf{D}_1$-orthogonal decomposition with $f_j - \mathbf{H}_K^1 f_j \in W_0^{1,2}(D)$. Since $(\mathbf{D}, W_0^{1,2}(D))$ is a regular Dirichlet form on $L^2(D; dx)$, there is a sequence $\{g_k, k \geq 1\} \subset C_c^\infty(D)$ that is $\mathbf{D}_1$-convergent to $f_j - \mathbf{H}_K^1 f_j$. Let $v_k := f_j - g_k$, which is in $\mathcal{C}$ and $\mathcal{E}_1^*$-convergent to u_j. Thus we have established that $(\mathcal{E}^*, \mathcal{F}^*)$ is a regular Dirichlet form on $L^2(D^*; m)$. Clearly it is strongly local and its part Dirichlet form on D is $(\mathbf{D}, W_0^{1,2}(D))$. So there is an m-symmetric diffusion X^* on D^* associated with $(\mathcal{E}^*, \mathcal{F}*)$, whose part process in D is the killed Brownian motion in D. The diffusion X^* is a BMD on D^*. Since Brownian motion X^E in E starting from $x \in D$ visits each K_j with positive probability, X^* starting from $x \in D$ visits each a_j^* with positive probability. This implies that each a_j^* has positive capacity. Consequently, X^* can be refined to start from every point in D^*. That each a_j^* is regular for itself follows from the general fact that for any nearly Borel measurable set A, $A \setminus A^r$ is semipolar and hence m-polar. $\square$

We point out that in the above theorem, we do not assume that every point of K_j is a regular point for K_j. If $K_j \subset K_j^r$ for every $j = 1, \ldots, N$, then each u_j is a continuous functions in $C_\infty(E)$ that takes constant value 1 on K_j and zero on other K_i. From it, one concludes immediately that $\mathcal{C}_1 := \{u \in W_0^{1,2}(E) \cap C_\infty(E) : u \text{ is constant on each } K_j\}$, after defining $u(a_j^*)$ to be the value of u on K_j for each $u \in \mathcal{C}_1$, is a core of $(\mathcal{E}^*, \mathcal{F}^*)$ and so $(\mathcal{E}^*, \mathcal{F}^*)$ is a regular Dirichlet form.

Every function in a regular Dirichlet form is known to admit a quasi-continuous version (see, e.g., [2]). We assume throughout this notes that every function u in the domain of a regular Dirichlet form is always represented by its quasi-continuous version.

Theorem 1.2.3 *BMD on D^* is unique in law.*

Proof. It suffices to show that if X^* is a BMD on D^*, its associated quasi-regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(D^*; m)$ has to be $(\mathcal{E}^*, \mathcal{F}^*)$. First note that according to the definition of BMD, each a_j^* is non-polar for X^* and that the part Dirichlet form $(\mathcal{E}, \mathcal{F}_D)$ of $(\mathcal{E}, \mathcal{F})$ in D is $(\mathbf{D}, W_0^{1,2}(D))$ (see [2, Theorem 3.3.8]). By the $\mathcal{E}_1$-orthogonal projection (see [2, Theorem 3.2.2]), for every $u \in \mathcal{F}$, $\mathbf{H}_{K^*}^1 u(x) := \mathbb{E}_x \left[e^{-\sigma^*} u(X_{\sigma^*}^*) \right] \in \mathcal{F}$ and $u - \mathbf{H}_{K^*}^1 u \in W_0^{1,2}(D)$. Here $K^* := \{a_1^*, \ldots, a_N^*\}$ and $\sigma^* := \inf\{t > 0 : X_t^* \in K^*\}$. Now

$$\mathbf{H}_{K^*}^1 u(x) = \sum_{j=1}^N u(a_j^*) \, \mathbb{E}_x \left[e^{-\sigma^*}; X_{\sigma^*}^* = a_j^* \right] \quad \text{for } x \in D.$$

By the continuity of X^*, the definition of a_j^* and the fact that $X^{*,D}$ has the same distribution as the subprocess of X^E killed upon leaving D, we see that

$$\mathbb{E}_x \left[e^{-\sigma^*}; X_{\sigma^*}^* = a_j^* \right] = \mathbb{E}_x \left[e^{-\sigma_E}; X_{\sigma_K}^E \in K_j \right] = u_j(x) \quad \text{for } x \in D.$$

It follows then $\mathbf{H}^1_{K^*}u = \sum_{j=1}^N u(a_j^*)u_j(x)$. As each a_j^* is non-polar,

$$\{(u(a_1^*), \dots, u(a_N^*)); u \in \mathcal{F}\} = \mathbb{R}^N$$

and so $\mathcal{F} = \mathcal{F}^*$. Note that $(\mathcal{E}, \mathcal{F})$ is strongly local so for every bounded $u \in \mathcal{F} = \mathcal{F}^*$,

$$\mathcal{E}(u,u) = \frac{1}{2}\mu^c_{\langle u \rangle}(D^*) = \frac{1}{2}\mu^c_{\langle u \rangle}(D) + \sum_{j=1}^N \mu^c_{\langle u \rangle}(a_j^*) = \frac{1}{2}\mu^c_{\langle u \rangle}(D),$$

where in the last equality, we used Theorem 1.2.1 with $A = \{u(a_j^*); j = 1, \dots, N\}$. For every relatively compact open subset U of D, there is a $\psi \in C_c^\infty(D)$ so that $\psi = 1$ on $\overline{U}$. Note that $u\psi \in \mathcal{F}_D = W_0^{1,2}(D)$ and $u\psi = u$ on U. As $(\mathcal{E}, \mathcal{F}_D) = (\mathbf{D}, W_0^{1,2}(D))$, by the strong local property of the energy measure $\mu^c_{\langle u \rangle}$ (see [2, Proposition 4.3.1]), we have

$$\mu^c_{\langle u \rangle}(dx) = \mu^c_{\langle u\psi \rangle}(dx) = |\nabla(u\psi)(x)|^2 dx = |\nabla u(x)|^2 dx \quad \text{on } U.$$

Consequently, we have $\mu^c_{\langle u \rangle}(dx) = |\nabla u(x)|^2 dx$ on D. So $\mathcal{E}(u,u) = \frac{1}{2}\int_D |\nabla u(x)|^2 dx$ for every bounded $u \in \mathcal{F}$ and hence for every $u \in \mathcal{F}$. This completes the proof that $(\mathcal{E}, \mathcal{F}) = (\mathcal{E}^*, \mathcal{F}^*)$. □

Remark 1.2.4 (i) The above procedure of constructing BMD works almost word for word for darning holes for symmetric diffusions on general state spaces. We will use this extension without further mention in Theorems 1.3.2 and 1.3.3.

(ii) Let D be a Euclidean domain in $\mathbb{R}^d$. In [11], Fukushima considered via Dirichlet form technique a process that amounts to darning reflected Brownian motion on $\overline{D}$ by "shorting" ∂D. □

Theorem 1.2.5 *Let*

$$\varphi_j(x) := \mathbb{P}_x(X^E_{\sigma_K} \in K_j), \quad j = 1, \dots, N,$$

and $(\mathcal{E}^*, \mathcal{F}_e^*)$ *the extended Dirichlet form of* $(\mathcal{E}^*, \mathcal{F}^*)$. *Then*

$$\begin{aligned} \mathcal{F}_e^* &= \text{linear span of } W_{0,e}^{1,2}(D) \text{ and } \{\varphi_j|_D, j = 1, \dots, N\}, \\ \mathcal{E}^*(u,v) &= \frac{1}{2}\int_D \nabla u(x) \cdot \nabla v(x) dx \qquad \text{for } u, v \in \mathcal{F}^*. \end{aligned}$$

Here $W_{0,e}^{1,2}(D)$ *denotes the extended Dirichlet space of* $(\mathbf{D}, W_0^{1,2}(D))$.

Proof. Clearly, $W_{0,e}^{1,2}(D) \subset \mathcal{F}_e^*$. Let $f_j \in C_c^\infty(E)$ so that $f_j = 1$ on K_j and $\text{supp}[f_j] \cap K_i = \emptyset$ for any $i \neq j$. Then $\phi_j(x) = \mathbf{H}_K f_j(x) := \mathbb{E}_x\left[f_j(X^E_{\sigma_K})\right]$. Since for $\alpha \in (0,1)$, $\mathbf{H}_k^\alpha f_j \in \mathcal{F}^*$ with

$$\mathcal{E}^*_\alpha(\mathbf{H}^\alpha_K f_j, \mathbf{H}^\alpha_K f_j) \leq \mathcal{E}^*_\alpha(f_j, f_j) \leq \mathbf{D}_\alpha(f_j, f_j)$$

and that $\lim_{\alpha\to 0} \mathbf{H}_K^\alpha f_j = \mathbf{H}_K f_j = \varphi_j$ on D, we conclude that $\varphi_j \in \mathcal{F}_e^*$. Hence we have shown

$$\mathcal{F}_e^* \supset \text{linear span of } W_{0,e}^{1,2}(D) \text{ and } \{\varphi_j|_D, j = 1, \dots, N\},$$

Now suppose that $u \in \mathcal{F}_e^*$. Then there is an $\mathcal{E}^*$-Cauchy sequence $\{w_k, k \geq 1\}$ in $\mathcal{F}^*$ that converges to u m-a.e. on D^*. For each $k \geq 1$, there is $f_k \in W_0^{1,2}(D)$ so that

$$w_k(x) = f_k(x) + \sum_{j=1}^N w_k(a_j^*) u_j(x) = f_k(x) + \sum_{j=1}^N w_k(a_j^*)(u_j(x) - \varphi_j(x)) + \sum_{j=1}^N w_k(a_j^*)\varphi_j(x).$$

Note that $h_k := \sum_{j=1}^N w_k(a_j^*)\varphi_j \in \mathcal{F}_e$ which is $\mathcal{E}^*$-orthogonal (or equivalently, $\mathbf{D}$-orthogonal) to $W_{0,e}^{1,2}(D)$, while $g_k := f_k + \sum_{j=1}^N w_k(a_j^*)(u_j - \varphi_j) \in W_{0,e}^{1,2}(D)$ due to the fact that $u_j - \varphi_j = \lim_{\alpha\to 0}(\mathbf{H}_K^1 f_j - \mathbf{H}_K^\alpha f_j)$ and $\mathbf{H}_K^1 f_j - \mathbf{H}_K^\alpha f_j \in W_0^{1,2}(D)$. Thus $\{g_k, k \geq 1\}$ is a $\mathbf{D}$-Cauchy sequence in the transient Dirichlet form $(\mathbf{D}, W_0^{1,2}(D))$ in $L^2(D; dx)$ and so $g_k \to g$ in the Hilbert space $(W_{0,e}^{1,2}(D), \mathbf{D})$ and a.e. on D for some $g \in W_{0,e}^{1,2}(D)$. Consequently, $h_k \to h := u - g$ m-a.e. on D^* as $k \to \infty$. It follows then $w_k(a_j^*)$ converges to some constant c_j as $k \to \infty$ because $\{\varphi_j(x), j = 1, \dots, N\}$ are linearly independent functions on D. We thus conclude that $h = \sum_{j=1}^N c_j \varphi_j$. As $u = g + h$, this completes proof of the theorem. □

Remark 1.2.6 Let $W_{0,e}^{1,2}(E)$ be the extended Dirichlet space of $(\mathbf{D}, W_0^{1,2}(E))$. Then we conclude by the same argument as those in the second paragraph of this section that

$$\mathcal{F}_e^* = \left\{ u|_D : u \in W_{0,e}^{1,2}(E),\ u \text{ is constant } \mathbf{D}\text{-q.e. on each } K_j \right\}.$$

1.3 Localization Properties

Suppose that E is a domain in $\mathbb{R}^d$ and $K_1, \dots, K_N$ are disjoint non-polar compact subsets of E. Suppose also that E_1 is a subdomain of E that contains $K_1, \dots, K_l$ for some $l \leq N$ and that $\overline{E}_1 \cap K_j = \emptyset$ for $j > l$. Let $D = E \setminus \cup_{j=1}^N K_j$ and $D_1 = E_1 \setminus \cup_{j=1}^l K_j$. Set $D^* := D \cup \{a_1^*, \dots, a_N^*\}$ and $D_1^* = D_1 \cup \{a_1^*, \dots, a_l^*\}$, and let X^* be BMD on D^*.

Theorem 1.3.1 *The part process X^{*,D_1^*} of X^* killed upon leaving D_1^* is the BMD on D_1^*.*

Proof. We will present two proofs for this theorem.

(i) Using Theorem 1.2.3 and by checking the definition of BMD in D_1^*, we see immediately that X^{*,D_1^*} is the BMD on D_1^*.

(ii) We now present a second proof by using Dirichlet form characterization of BMD in D_1^*. Let $(\mathcal{E}^*, \mathcal{F}^*)$ and $(\mathcal{E}, \mathcal{F})$ be the Dirichlet forms of BMD in D^* and D_1^*, respectively. Recall from (1.2.3) that

$$\mathcal{F}^* = \left\{ u|_D :\ u \in W_0^{1,2}(E),\ u \text{ is constant } \mathbf{D}\text{-q.e. on each } K_j \right\}.$$

It is known that X^{*,D_1^*} has Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*_{D^*})$ on $L^2(D_1^*; m)$, where

$$\mathcal{F}^*_{D_1^*} := \{u \in \mathcal{F}^* : u = 0 \ \mathcal{E}^*\text{-q.e. on } D^* \setminus D_1^*\}.$$

Since each a_j^* has positive capacity and $(\mathcal{E}^*, \mathcal{F}^*_D) = (\mathbf{D}, W_0^{1,2}(D))$, we conclude that

$$\begin{aligned} \mathcal{F}^*_{D_1^*} &= \Big\{u|_D : u \in W_0^{1,2}(E),\ u \text{ is constant } \mathbf{D}\text{-q.e. on } K_j \text{ for } j = 1, \dots, l \\ &\qquad \text{and } u = 0 \ \mathbf{D}\text{-q.e. on } E \setminus E_1\Big\} = \mathcal{F}. \end{aligned}$$

So $(\mathcal{E}^*, \mathcal{F}^*_{D_1^*}) = (\mathcal{E}, \mathcal{F})$, which establishes that X^{*,D_1^*} is the BMD on D_1^*. □

The next theorem says one can darn (or short) holes one by one.

Theorem 1.3.2 *Let Y be BMD on $O^* := (E \setminus \cup_{j=1}^{N-1} K_j) \cup \{a_1^*, \dots, a_{N-1}^*\}$ by darning (or shorting) the first $N-1$ holes. Let Z be the diffusion with darning on D^* obtained from Y by shoring K_N to a single point a_N^*. Then Z is BMD on D^*.*

Proof. Let $D_1 = E \setminus \cup_{j=1}^{N-1} K_j$ and denote by $(\mathcal{E}, \mathcal{F})$ the Dirichlet form of Y on $L^2(D_1^*; m)$. In view of (1.2.3) and Theorem 1.2.1,

$$\mathcal{F} = \Big\{u|_{D_1} :\ u \in W_0^{1,2}(E),\ u \text{ is constant } \mathbf{D}\text{-q.e. on } K_j \text{ for } j = 1, \dots, N-1\Big\}$$

and

$$\mathcal{E}(u, v) = \frac{1}{2} \int_{D_1} \nabla u(x) \cdot \nabla v(x) dx.$$

The Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $L^2(D^*; m)$ for Z is

$$\begin{aligned} \widetilde{\mathcal{F}} &= \{u|_D :\ u \in \mathcal{F},\ u \text{ is constant } \mathcal{E}\text{-q.e. on } K_N\} \\ &= \Big\{u|_D :\ u \in W_0^{1,2}(E),\ u \text{ is constant } \mathbf{D}\text{-q.e. on } K_j \text{ for } j = 1, \dots, N\Big\} \\ &= \mathcal{F}^* \end{aligned}$$

and, in view of Theorem 1.2.1, for $u, v \in \widetilde{\mathcal{F}}$,

$$\widetilde{\mathcal{E}}(u, v) = \mathcal{E}(u, v) = \frac{1}{2} \int_D \nabla u(x) \cdot \nabla v(x) dx.$$

This shows that $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}) = (\mathcal{E}^*, \mathcal{F}^*)$, which completes the proof of the theorem. □

Theorem 1.3.3 *Let $K = A \cup B$ be the union of two disjoint non-polar compact subsets of E. Let Y be BMD on $(E \setminus A)^*$ by darning A, and Z the diffusion with darning on $(E \setminus K)^*$ obtained from Y by darning (or shoring) $A^* \cup B$. Then Z is BMD on $(E \setminus K)^*$ by darning K into one single point.*

Proof. Let $(\mathcal{E},\mathcal{F})$ and $(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}})$ be the Dirichlet forms for the processes Y and Z on $L^2((E\setminus A)^*;m)$ and $L^2((E\setminus K)^*;m)$, respectively. Note that

$$\begin{aligned}\mathcal{F} &= \left\{u|_{E\setminus A}:\ u\in W_0^{1,2}(E),\ u \text{ is constant } \mathbf{D}\text{-q.e. on } A\right\},\\ \mathcal{E}(u,v) &= \frac{1}{2}\int_{E\setminus A}\nabla u(x)\cdot\nabla v(x)dx \qquad \text{for } u,v\in\mathcal{F},\end{aligned}$$

while

$$\begin{aligned}\widetilde{\mathcal{F}} &= \left\{u|_{E\setminus K}:\ u\in \mathcal{F},\ u \text{ is constant } \mathcal{E}\text{-q.e. on } A^*\cup B\right\}\\ &= \left\{u|_{E\setminus K}:\ u\in W_0^{1,2}(E),\ u \text{ is constant } \mathbf{D}\text{-q.e. on } K=A\cup B\right\}=\mathcal{F}^*,\\ \widetilde{\mathcal{E}}(u,v) &= \mathcal{E}(u,v)=\frac{1}{2}\int_{E\setminus K}\nabla u(x)\cdot\nabla v(x)dx=\mathcal{E}^*(u,v) \qquad \text{for } u,v\in\widetilde{\mathcal{F}}.\end{aligned}$$

Here $(\mathcal{E}^*,\mathcal{F}^*)$ is the Dirichlet form for BMD X^* on $(E\setminus K)^*$. This proves that Z has the same distribution as BMD X^* on $(E\setminus K)^*$. □

One can also prove the above two theorems just by using the definition of BMD on D^*.

1.4 Conformal Invariance of Planar BMD

In this section, we assume the dimension $d=2$, E is a domain in $\mathbb{R}^2$ and $K_1,\dots,K_N$ are disjoint non-polar compact subsets of E. Let $K=\cup_{j=1}^N K_j$ and $D=E\setminus K$ and X^* be BMD in $D^*=D\cup\{a_1^*,\dots,a_N^*\}$.

Theorem 1.4.1 *Let $\widehat{K}=\cup_{i=1}^N\widehat{K}_i$, where $\{\widehat{K}_1,\dots,\widehat{K}_N\}$ is a second set of disjoint non-polar compact subsets of a domain $\widehat{E}$ in $\mathbb{R}^2$. Suppose that ϕ is a conformal map from $E\setminus K$ onto $\widehat{E}\setminus\widehat{K}$ that, for each $i\geq 1$, ϕ maps the $E\setminus K$-portion of any neighborhood of K_i into the $\widehat{E}\setminus\widehat{K}$-portion of a neighborhood of $\widehat{K}_i$, and vice versa. Identify the compact set $\widehat{K}_i$ with a single point $\widehat{a}_i^*$ and equip $\widehat{D}^*:=(\widehat{E}\setminus\widehat{K})\cup\{\widehat{a}_1^*,\dots,\widehat{a}_N^*\}$ the topology induced from $\widehat{E}$ by identifying each set $\widehat{K}_i$ into one point $\widehat{a}_i^*$. Define $\phi(a_i^*)=\widehat{a}_i^*$, $1\leq i\leq N$. Then ϕ is a topological homeomorphism from D^* onto $\widehat{D}^*$. Moreover, $\phi(X^*)$ is, up to a time change, BMD on $\widehat{D}^*$.*

Proof. In view of Theorem 1.3.1, we may assume that the domain E is bounded with smooth boundary and that ϕ extends continuously to ∂E to be a homeomorphism from ∂E to $\partial\widehat{E}$. Let $\widehat{m}$ be the Lebesgue measure on $\widehat{D}:=\widehat{E}\setminus\widehat{K}$ extended to $\widehat{D}^*$ by setting $\widehat{m}(\{\widehat{a}_i^*\})=0$ for $i=1,\dots,N$. BMD $X^*=(X_t^*,\mathbb{P}_z^*)$ on D^* is an extension of the absorbing Brownian motion in D to D^* and is m-symmetric. By Theorem 1.2.5, the extended Dirichlet space $(\mathcal{F}_e^*,\mathcal{E}^*)$ of X^* is given by

$$\begin{cases}\mathcal{F}_e^*=\left\{f+\sum_{i=1}^N c_i\varphi_i|_D: f\in W_{0,e}^{1,2}(D),\ c_i\in\mathbb{R}\right\},\\ \mathcal{E}^*(u,v)=\frac{1}{2}\int_D\nabla u(x)\cdot\nabla v(x)dx \quad \text{for } u,v\in\mathcal{F}_e^*,\end{cases}$$

where $\varphi_i(x) := \mathbb{P}_x(X^E_{\sigma_K} \in K_i)$ for $x \in D$.

We define a Markov process $Y = (Y_t, \mathbb{P}^Y_w)_{w \in \widehat{D}^*}$ on $\widehat{D}^*$ by

$$Y_t = \phi(X^*_t), \quad \mathbb{P}^Y_w = \mathbb{P}_{\phi^{-1}(w)}, \quad w \in \widehat{D}^*. \tag{1.4.1}$$

Y is clearly a diffusion process on $\widehat{D}^*$. We claim that Y, after a time change, is actually a BMD on $\widehat{D}^*$.

Denote by $\{P_t, t > 0\}$ and $\{P^Y_t, t > 0\}$ the transition function of X^* and Y, respectively. It then hold that $P^Y_t f(w) = P_t(f \circ \phi)(\phi^{-1}(z))$ for $w \in \widehat{D}^*$. Let $\psi = \phi^{-1}$ be the inverse map from $\widehat{D}^*$ to D^* and let $\mu(dw) = |\psi'(w)|^2 1_{\widehat{D}}(w)dw$, which is extended to $\widehat{D}^*$ by setting $\mu(\widehat{K}^*) = 0$. Recall the change-of-variables formula that for any function $u \geq 0$ defined on D,

$$\int_{\widehat{D}} u(\psi(w))\mu(dw) = \int_D u(z)dz,$$

The above in particular implies that $\mu(\widehat{D}) = |D|$ is finite. From the change-of-variable formula, we immediately obtain $\|P^Y_t f\|_{L^2(\widehat{D};\mu)} = \|P_t(f \circ \phi)\|_{L^2(D;m)}$ and

$$(P^Y_t f, g)_{L^2(\widehat{D};\mu)} = (P^X_t(f \circ \phi), g \circ \phi)_{L^2(D;m)},$$

from which the μ-symmetry of Y follows. Let $(\mathcal{E}^Y, \mathcal{F}^Y)$ be the Dirichlet form of Y on $L^2(\widehat{D}^*; \mu)$. For $f \in L^2(\widehat{D}; \mu)$, we let $t \downarrow 0$ in the equality

$$t^{-1}(f - P^Y_t f, f)_{L^2(\widehat{D}^*;\mu)} = t^{-1}(f \circ \phi - P^X_t(f \circ \phi), f \circ \phi)_{L^2(D^*;m)}$$

to see that $f \in \mathcal{F}^Y$ if and only if $f \circ \phi \in \mathcal{F}^*$, and in this case,

$$\begin{aligned} \mathcal{E}^Y(f,f) &= \frac{1}{2}\int_D |\nabla(f \circ \phi)|^2(z)dx \\ &= \frac{1}{2}\int_D |\nabla f|^2(\phi(z))|\phi'(z)|^2 dz = \frac{1}{2}\int_{\widehat{D}} |\nabla f(w)|^2 dw. \end{aligned}$$

The above identity also implies that $f \in \mathcal{F}^Y_e$ if and only if $f \circ \phi \in \mathcal{F}^*_e$, and

$$\mathcal{E}^Y(f,f) = \mathcal{E}^*(f \circ \phi, f \circ \phi) = \frac{1}{2}\int_{\widehat{D}} |\nabla f(w)|^2 dw \quad \text{for } f \in \mathcal{F}^Y_e.$$

Let $(\widehat{\mathcal{E}}^*, \widehat{\mathcal{F}}^*)$ and $\widehat{\mathcal{F}}^*_e$ denote the Dirichlet form and extended Dirichlet space of BMD on $\widehat{D}^*$. We then conclude from Theorem 1.2.5 that $\mathcal{F}^Y_e = \widehat{\mathcal{F}}^*_e$. Since the finite measure $\mu(dz)$ on D^* is mutually absolutely continuous with respect to $\widehat{m}$ on D^*, we have by [2, Theorem 5.2.7] that Y is a time-change of BMD $\widehat{X}^*$ on D^* (and vice verse). $\square$

1.5 Zero Flux Characterization of Generator

The L^2-generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ of $(\mathcal{E}^*, \mathcal{F}^*)$ is defined as follows: $u \in \mathcal{D}(\mathcal{L})$ if and only if $u \in \mathcal{F}^*$ and there is some $f \in L^2(D; dx) = L^2(D^*; m)$ so that

$$\mathcal{E}^*(u, v) = -\int_D f(x)v(x)dx \qquad \text{for every } v \in \mathcal{F}^*. \tag{1.5.1}$$

We denote the above f as $\mathcal{L}u$. In view of (1.2.1), condition (1.5.1) is equivalent to

$$\frac{1}{2}\int_D \nabla u(x) \cdot \nabla v(x)dx = -\int_D f(x)v(x)dx \qquad \text{for every } v \in C_c^\infty(D) \tag{1.5.2}$$

and

$$\frac{1}{2}\int_D \nabla u(x) \cdot \nabla u_j(x)dx = -\int_D f(x)u_j(x)dx \qquad \text{for every } j = 1, \ldots, N. \tag{1.5.3}$$

(1.5.2) says that Δu exists on D in the distribution sense and $f = \frac{1}{2}\Delta u \in L^2(D; dx)$. Let us define the flux $\mathcal{N}(u)(a_j^*)$ of u at a_j^* by

$$\mathcal{N}(u)(a_j^*) = \int_D \nabla u(x) \cdot \nabla u_j(x)dx + \int_D \Delta u(x)u_j(x)dx. \tag{1.5.4}$$

Then (1.5.3) is equivalent to

$$\mathcal{N}(u)(a_j^*) = 0. \qquad \text{for every } j = 1, \ldots, N. \tag{1.5.5}$$

Hence we have established the following.

Theorem 1.5.1 *A function $u \in \mathcal{F}^*$ is in $\mathcal{D}(\mathcal{L})$ if and only if the distributional Laplacian Δu of u exists as an L^2-integrable function on D and u has zero flux at every a_j^*. Moreover, for $u \in \mathcal{D}(\mathcal{L})$, $\mathcal{L}u = \frac{1}{2}\Delta u$ on D.*

Note that when ∂K_j is smooth for $j = 1, \ldots, N$, the by Green-Gauss formula, we have

$$\mathcal{N}(u)(a_j^*) = \int_{\partial K} \frac{\partial u(x)}{\partial \mathbf{n}} u_j(x)\sigma(dx),$$

where $\mathbf{n}$ is the unit outward normal vector field of D on ∂D and σ is the surface measure on ∂D. Since $u_j(x) = 1$ on K_j and $u_j(x) = 0$ on K_i with $i \neq j$,

$$\mathcal{N}(u)(a_j^*) = \int_{\partial K_j} \frac{\partial u(x)}{\partial \mathbf{n}} \sigma(dx). \tag{1.5.6}$$

Fix some $f_j \in \mathcal{F}^*$ so that $f_j(a_j^*) = 1$ and $f_j(a_i^*) = 0$ for $i \neq j$; that is, $f_j \in W_0^{1,2}(E)$ so that $f_j = 1$ $\mathbf{D}$-q.e. on K_j and $f_j = 0$ $\mathbf{D}$-q.e. on K_i for $i \neq j$.

Theorem 1.5.2 *Suppose that $u \in W^{1,2}(D)$ with $\Delta u \in L^2(D; dx)$ in the distributional sense. Then*

$$\mathcal{N}(u)(a_j) = \int_D \nabla u(x) \cdot \nabla f_j(x) dx + \int_D \Delta u(x) f_j(x) dx. \tag{1.5.7}$$

Proof. Assume that the distributional Δu of u exists and is in $L^2(D; dx)$. Since $f_j - u_j \in W_0^{1,2}(D)$, one has

$$\int_D \nabla (f_j(x) - u_j(x)) \cdot \nabla u(x) dx + \int_D (f_j(x) - u_j(x)) \Delta u(x) dx = 0,$$

which establishes (1.5.7). □

Suppose that E is bounded. Then it is well known that the first eigenvalue of the Dirichlet Laplacian in E is strictly positive; that is, there is $\lambda_1 > 0$ so that

$$\mathbf{D}(f, f) \geq \lambda_1 \int_E f(x)^2 dx \qquad \text{for } f \in W_0^{1,2}(E).$$

In view of Theorem 1.2.1 and (1.2.1), this in particular implies that

$$\mathcal{E}^*(u, u) \geq \lambda_1 \int_D u(x)^2 dx \qquad \text{for } u \in \mathcal{F}^*.$$

It follows that $(\mathcal{E}^*, \mathcal{F}^*)$ is transient and for every $f \in L^2(D; dx)$, there is $u \in \mathcal{F}^*$ so that $\mathcal{E}^*(u, v) = -\int_D f(x) v(x) dx$ for every $v \in \mathcal{F}^*$. We denote this u by $G^* f$. It is easy to see (cf. [2]) that G^* is the 0-order resolvent of X^* and $G^* f(x) = \mathbb{E}_x \left[\int_0^\infty f(X_s^*) ds \right]$ on D^*.

Theorem 1.5.3 *If E is bounded, then for every $f \in L^\infty(D) (= L^\infty(D; m))$, $G^* f \in \mathcal{D}(\mathcal{L})$ with $\mathcal{L} G^* f = -f$.*

Proof. For $f \in L^2(D; dx)$, by the strong Markov property of X^*, we have for $x \in D$,

$$\begin{aligned} G^* f(x) &= G_D f(x) + \mathbb{E}_x \left[\int_{\sigma_{K^*}}^\infty f(X_s^*) ds \right] = G_D f(x) + \sum_{j=1}^N G^* f(a_j^*) \mathbb{E}_x \left[X^*_{\sigma_{K^*}} = a_j^* \right] \\ &= G_D f(x) + \sum_{j=1}^N G^* f(a_j^*) \varphi_j(x). \end{aligned} \tag{1.5.8}$$

Since $D = E \setminus K$ is bounded, $G_D(L^\infty(D)) \subset L^\infty(D) \subset L^2(D)$. In view of (1.5.8), G^* has the same property. Hence the resolvent equation $G^* f = G_1^* f + G_1^*(G^* f)$ yields that $G^* f \in \mathcal{D}(\mathcal{L})$ with $\mathcal{L} G^* f = -f$. □

1.6 Harmonic Functions and Zero Period Property

Definition 1.6.1 A function u defined on a connected open subset O of D^* is said to be *X^*-harmonic* or *BMD-harmonic* on O if for every relatively compact open subset O_1 of O,

$$\mathbb{E}_x\left[|u(X^*_{\tau_{O_1}})|\right] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x\left[u(X^*_{\tau_{O_1}})\right] \qquad \text{for every } x \in O_1. \tag{1.6.1}$$

Here $\tau_{O_1} := \inf\{t \geq 0 : X^*_t \notin O_1\}$.

Clearly, the restriction to $O \cap D$ of any X^*-harmonic function on O is harmonic there in the classical sense (i.e. with respect to Brownian motion) and so u is continuous in $O \cap D$. It follows that X^*-harmonic functions in O_1 is locally bounded.

Proposition 1.6.2 *If u is X^*-harmonic in a connected open subset O of D^*, then u is $\mathcal{E}^*$-quasi-continuous on O. In fact, for every relatively compact open subset O_1 of O, there is some function $f \in \mathcal{F}^*$ so that $u = f$ m-a.e. in O_1.*

Proof. Without loss of generality, we may assume that $\partial O_1 \subset D$. Since u is harmonic in $O \cap D$, u is C^∞-smooth in $O \cap D$. Let $\varphi \in C_c^\infty(D)$ so that $\varphi = 1$ in a neighborhood of ∂O_1. Note that $u\varphi \in C_c^\infty(D) \subset \mathcal{F}^*$ and

$$u(x) = \mathbb{E}_x\left[(u\varphi)(X^*_{\tau_{O_1}})\right] = \mathbf{H}_{D^* \setminus O_1}(u\varphi)(x) \qquad \text{for } x \in O_1.$$

Since $u\varphi$ is bounded and compactly supported in D, $\mathbf{H}_{D^* \setminus O_1}(u\varphi) \in \mathcal{F}^*_e \cap L^2(D^*; m) = \mathcal{F}^*$. It follows that u is $\mathcal{E}^*$-quasi-continuous in O. □

Lemma 1.6.3 *Suppose that u is X^*-harmonic in a connected open subset O of D^*. Then* $\lim_{O \cap D \ni x \to z} u(x) = u(a_j^*)$ *for* $\mathbf{D}$*-q.e.* $z \in K_j \cap \partial(O \cap D)$ *whenever* $a_j^* \in O$.

Proof. Suppose that $a_j^* \in O$. Let O_1 be a relatively compact connected open subset of O so that $O \cap K^* = \{a_j^*\}$. By Proposition 1.6.2, there is a function $f \in \mathcal{F}^*$ so that $u = f$ m-a.e. in a neighborhood of $\overline{O}_1$. By Theorem 1.2.2, f is the restriction to D of a function $\widetilde{f} \in W_0^{1,2}(E)$ that takes constant value $\mathbf{D}$-q.e. on each K_i. Let $\{D_k; k \geq 1\}$ be an increasing sequence of smooth subdomains of $D \cap O_1$ so that $\overline{D}_k \subset D_{k+1}$ and $\cup_{k \geq 1} D_k = O_1 \cap D$. Since f is harmonic in $O \cap D$, we have for $x \in O_1 \cap D$,

$$\begin{aligned} u(x) &= f(x) = \lim_{k\to\infty} \mathbb{E}_x\left[f(X_{\tau_{D_k}})\right] = \mathbb{E}_x\left[f(X_{\tau_{O_1 \cap D}})\right] \\ &= \mathbb{E}_x\left[f(X_{\tau_{O_1 \cap D}}); X_{\tau_{O_1 \cap D}} \in D \cap \partial O_1\right] + u(a_j^*)\mathbb{P}_x\left(X_{\tau_{O_1 \cap D}} \in K_j\right). \end{aligned}$$

Since $K_j \setminus K_j^r$ is semipolar, we conclude $\lim_{O \cap D \ni x \to z} u(x) = u(a_j^*)$ for $\mathbf{D}$-q.e. $z \in K_j \cap \partial(O \cap D)$. □

If $K_j^r \subset K_j$ for every $a_j^* \in O$, then every function u that is X^*-harmonic in a connected open subset O of D^* is continuous in O. In particular, such u is a harmonic function in $O \cap D$, taking boundary value $u(a_j^*)$ on each K_j whenever $a_j^* \in O$.

Theorem 1.6.4 *Suppose that D_1 and D_2 are two connected subsets of D^* and that $D_1 \cap D_2 \neq \emptyset$. If u is X^*-harmonic in D_i for $i = 1, 2$, then u is X^*-harmonic in $D_1 \cup D_2$.*

Proof. Let O be a relatively compact open subset of $D_1 \cup D_2$. Let $\{U_k^{(i)}; k \geq 1\}$ be an increasing sequence of relatively compact open subsets whose union is D_i and $\partial U_k^{(i)}$ is a smooth subset in D for $i = 1, 2$. Since $\{U_k^{(1)} \cup U_k^{(2)}; k \geq 1\}$ forms an open cover for $\overline{O}$, there is some $k_0 \geq 1$ so that $\overline{O} \subset U_{k_0}^{(1)} \cup U_{k_0}^{(2)}$. For notational simplicity, denote $U_{k_0}^{(i)}$ by U_i for $i = 1, 2$. Note that $O_i := O \cap U_i$ is a relatively compact open subset of D_i, $i = 1, 2$. We claim that for every $x \in O$, $u(x) = \mathbb{E}_x \left[u(X^*_{\tau_O}) \right]$. In the following we show that the above holds for every $x \in O_1$. The case for $x \in O_2$ is analogous.

Let $\{\theta_t; t \geq 0\}$ be the shift operator for BMD X^* on D^*. We use $\{\mathcal{F}_t; t \geq 0\}$ to denote the minimal augmented natural filtration generated by X^*. Define a sequence of stopping times as follows. $T_1 := \tau_{O_1}$, $T_2 := \tau_{O_2}$, and for $k \geq 1$,

$$T_{2k+1} := T_{2k} + \tau_{O_1} \circ \theta_{T_{2k}} \quad \text{and} \quad T_{2k+2} := T_{2k+1} + \tau_{O_2} \circ \theta_{T_{2k+1}}.$$

In view of (1.5.8), $\mathbb{E}_x [\tau_O]$ is a bounded function on O and so $\tau_O < \infty$ $\mathbb{P}_x$-a.s. for every $x \in O$. Note that $T_k \leq \tau_O$ for every $k \geq 1$. Since u is X^*-harmonic in both D_1 and D_2, we have for $x \in O_1$, $\mathbb{P}_x$-a.s.

$$u(X^*_{T_k}) = \mathbb{E}_{X^*_{T_{k+1}}} \left[u(X^*_{T_{k+1}}) | \mathcal{F}_{T_k} \right] \qquad \text{for every } k \geq 1.$$

In other words, $\{u(X^*_{T_k}); k \geq 1\}$ is an $\{\mathcal{F}_{T_k}\}_{k \geq 1}$-martingale under $\mathbb{P}_x$ for every $x \in O_1$. Let $T := \lim_{k \to \infty} T_k$. Since u is bounded and $\mathcal{E}^*$-quasi-continuous on $\overline{O}$, we have

$$u(x) = \lim_{k \to \infty} \mathbb{E}_x \left[u(X^*_{T_k}) \right] = \mathbb{E}_x \left[u(X^*_T) \right].$$

We next show that $T = \tau_O$. Clearly $T \leq \tau_O$ $\mathbb{P}_x$-a.s.. On $\{T < \tau_O\}$, $X^*_T(\omega) \in O = O_1 \cup O_2$, say, $X^*_T(\omega) \in O_2$. There is some large $k_0 = k_0(\omega)$ so that $X^*_{T_k}(\omega) \in O_2$ for all $k \geq k_0$. This is impossible as for even $k \geq k_0$, $X^*_{T_k} \notin O_2$. So we must have $T = \tau_O$ $\mathbb{P}_x$-a.s. and consequently, $u(x) = \mathbb{E}_x \left[u(X^*_{\tau_O}) \right]$ for every $x \in O_1$. This shows that u is X^*-harmonic in O for every relatively compact subdomain O of $D_1 \cup D_2$ and so u is X^*-harmonic in $D_1 \cup D_2$. $\square$

Let O be a connected open subset of E and v is a harmonic function in $O \cap D$. Suppose that $K_j \in O$. Let U be any relatively compact C^1-smooth subdomain of O that contains K_j and that $K_i \cap \overline{U} = \emptyset$ for any $i \neq j$. We define

$$\text{the } \textit{period} \text{ of } v \text{ at } a_j^* \text{ (or around the compact set } K_j) := \int_{\partial U} \frac{\partial v(x)}{\partial \mathbf{n}} \sigma(dx),$$

where $\mathbf{n}$ is the inward normal vector field of U on ∂U and σ is the surface measure on ∂U. Note that by the Green-Gauss formula and the harmonicity of v in $O \cap D$, the value on the right hand side is independent of the choice of the subdomain U. Note that $E \setminus K_1$ may be connected or disconnected; see Example 1.1.3(i) and (ii) for these two concrete cases. The next result says that locally an X^*-harmonic function can be expressed as the Green potential of a bounded function with compact support that is supported away from that region.

Lemma 1.6.5 *Suppose that v is an X^*-harmonic function in an open subset O_1 of D^*. For any relatively compact open subset $O_2 \subset O_1$, there is a compactly supported bounded function f on D^* with* $\mathrm{supp}[f] \cap O_2 = \emptyset$ *such that $v = G^* f$ in O_2.*

Proof. Let $\Lambda_i = \{j : a_j^* \in O_i\}$ for $i = 1, 2$. There is an open subset U_1 of E and a relatively compact open subset U_2 of U_1 so that $\cup_{j\in\Lambda_i} K_j \subset U_i$ and $U_i \cap D = O_i \cap D$ for $i = 1, 2$. Take some $\psi \in C_c^\infty(U_1)$ so that $0 \le \psi \le 1$ with $\psi = 1$ on U_2. Define $f(x) = -\frac{1}{2} 1_D(x) \Delta(\psi v)(x)$. Note that $f \in L^\infty(D; dx)$ and $f = 0$ on $D \setminus (O_1 \setminus O_2)$. Hence $G^* f \in \mathcal{F}^*$ is X^*-harmonic in $(U_2 \cap D) \cup \{a_i^*, i \in \Lambda_2\}$ and so is $w := \psi v - G^* f$. On the other hand, (1.5.8) implies that w is harmonic and hence X^*-harmonic in D. Thus by Theorem 1.6.4, w is X^*-harmonic in D^*. Since both ψv and $G^* f$ vanish on $\partial E = \partial D^*$, so is w. Thus by maximum principle for the bounded X^*-harmonic function w on D^* (note that a_j^*'s are interior points of D^*), we have $w = 0$ on D^*, and in particular $v = G^* f$ in O_2. □

Theorem 1.6.6 *Let O be a connected open subset of D^*. An $\mathcal{E}^*$-quasi-continuous function v is X^*-harmonic in O if and only if v is harmonic in $D \cap O$ and the period of v at a_i^* is 0 for every i such that $a_i^* \in O$.*

Proof. The assertion trivially holds if O does not contain any a_i^*. In view of Theorem 1.3.1 and Theorem 1.6.4, without loss of generality, we may and do assume that E is bounded with smooth boundary ∂E, $D^* = O$ and that D^* contains exactly one a_1^* (that is, K consists of exactly one compact set K_1).

Since we do not assume that $K \subset K^r$, the function u_1 may not be continuous on K and hence $\{x \in E : u_1(x) > 1 - \varepsilon\}$ may not be an open set that decreases to K as $\varepsilon \downarrow 0$. We will construct a continuous function ψ_1 on E taking values in $[0, 1]$ that is 1 precisely on K, smooth in D and the open set $\{x \in E : u_1(x) > 1 - \varepsilon\}$ decreases to K as $\varepsilon \downarrow 0$. For this, we first recall a result about the regularized distance function. Let $d_K(x)$ denote the Euclidean distance between x and K. By [18, Theorem 2, p. 171], there exists a C^∞-smooth function $\delta_K(x)$ in K^c and constants $c_1 > c_2 > 0$ so that

$$c_2 d_K(x) \le \delta_K(x) \le c_1 d_K(x) \quad \text{and} \quad |\nabla \delta_K(x)| \le c_1 \quad \text{for every } x \in K^c.$$

Clearly, $\delta_K(x)$ extends to be a continuous function on $\mathbb{R}^d$ after setting $\delta_K(x) = 0$ for $x \in K$. Let U_1 and U_2 be relatively compact open subsets of E such that $K_1 \subset U_1 \subset \overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset E$ so that $\delta_K(x) < 1$ for $x \in U_2$. Take some $\psi \in C_c^\infty(U_2)$ so that $0 \le \psi \le 1$ with $\psi = 1$ on U_1. Define

$$\psi_1(x) = (1 - \delta_K(x))\psi. \tag{1.6.2}$$

Clearly, $\psi_1 \in C_c^\infty(U_2)$ with $0 \le \psi_1 \le 1$ $\psi_1(x) = 1$ if and only if $x \in K_1$.

Suppose that v is X^*-harmonic in D^*. For $\varepsilon \in (0,1)$, let η_ε be the boundary of the connected component of $\{x \in E : \psi_1(x) > 1-\varepsilon\}$ that contains K_1. By Sard's theorem (see, e.g., [15]), there is a set $\mathcal{N}_0$ having zero Lebesgue measure so that for every $\varepsilon \in (0,1) \setminus \mathcal{N}_0$, η_ε is a C^∞-smooth $(d-1)$-dimensional hypersurface. Take a decreasing sequence $\{\varepsilon_n, n \ge 1\} \in (0,1) \setminus \mathcal{N}_0$ with $\lim_{n\to\infty} \varepsilon_N = 0$. Since $\{x \in E : \psi_1(x) > 1-\varepsilon_n\}$ decreases to K_1, we may assume that each η_{ε_n} is contained inside U_1. Call the connected component of $\mathbb{R}^d \setminus \eta_\varepsilon$ that contains K_1 the interior of η_ε.

By Lemma 1.6.5, there is a bounded compactly supported function f on D^* with $\mathrm{supp}[f] \cap U_1 = \emptyset$ so that $v = G^* f$ in U_1. By the Green-Gauss formula, Theorems 1.5.1, 1.5.2 and 1.5.3, we have

$$\begin{aligned} \text{period of } v \text{ at } a_1^* &= \lim_{n\to\infty} \int_{\eta_{\varepsilon_n}} \frac{\partial G^* f(\xi)}{\partial \mathbf{n}_\xi} \sigma(d\xi) \\ &= \lim_{n\to\infty} \frac{1}{1-\varepsilon_n} \int_{\eta_{\varepsilon_n}} \frac{\partial G^* f(\xi)}{\partial \mathbf{n}_\xi} \psi_1(\xi) \sigma(d\xi) \\ &= \lim_{n\to\infty} \frac{1}{1-\varepsilon_n} \int_{D \setminus \mathrm{int}(\eta_N)} \left(\nabla \psi_1 \cdot \nabla G^* f + \psi_1 \Delta G^* f\right) dx \\ &= \int_D \nabla \psi_1(x) \cdot \nabla G^* f(x) dx + \int_D \psi_1(x) \Delta G^* f(x) dx \\ &= 2\mathcal{N}(G^* f)(a_1^*) = 0. \end{aligned}$$

Here $\mathbf{n}$ denote the unit inward normal vector field on η_{ε_n} for the interior of η_{ε_n}.

Conversely, assume that v is an $\mathcal{E}^*$-quasi-continuous function on D^* that is harmonic in D and has zero period at a_1^*. Let the relatively compact open subsets $U_1 \subset U_2$ of E, the smooth function ψ and the smooth curves η_{ε_n} be defined as above. Set $\varphi(x) = \mathbb{P}_x(\sigma_{a_1^*} < \infty)$. Observe that $\varphi \in W^{1,2}(D)$ and the function $w := \psi v - v(a_1^*)\varphi$ is smooth in D, vanishing $\mathbf{D}$-q.e. on ∂D. So $w = G_D f \in W_0^{1,2}(D)$, where $f = -1_D(x)\frac{1}{2}\Delta w(x)$. We have therefore

$$\psi v = w + v(a_1^*)\varphi \in \mathcal{F}^* \quad \text{with} \quad \Delta(\psi v) \in L^2(D; dx).$$

Since v has zero period at a_1^*, we have by the Green-Gauss formula that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_{\eta_{\varepsilon_n}} \frac{\partial v(\xi)}{\partial \mathbf{n}_\xi} \sigma(d\xi) = \lim_{n\to\infty} \int_{\eta_{\varepsilon_n}} \frac{\partial (\psi v)(\xi)}{\partial \mathbf{n}_\xi} \sigma(d\xi) \\ &= \lim_{n\to\infty} \frac{1}{1-\varepsilon_n} \int_{\eta_{\varepsilon_n}} \frac{\partial (\psi v)(\xi)}{\partial \mathbf{n}_\xi} \psi_1(\xi) \sigma(d\xi) \\ &= \lim_{n\to\infty} \frac{1}{1-\varepsilon_n} \int_{D \setminus \mathrm{int}(\eta_N)} \left(\nabla \psi_1 \cdot \nabla(\psi v) + \psi_1 \Delta(\psi v)\right) dx \\ &= \int_D \nabla \psi_1(x) \cdot \nabla(\psi v)(x) dx + \int_D \psi_1(x) \Delta(\psi v)(x) dx \\ &= 2\mathcal{N}(\psi v)(a_1^*), \end{aligned}$$

where the last equality is due to Theorem 1.5.2. Hence we conclude by Theorem 1.5.1 that $\psi v \in \mathcal{D}(\mathcal{L})$.

Let $g := -1_D(x)\frac{1}{2}\Delta(\psi v)(x)$, which is smooth and compactly supported. Define $w_1 = G^* g$, which by Theorem 1.5.3, is in $\mathcal{D}(\mathcal{L}) \subset \mathcal{F}^*$ with $\mathcal{L}w_1 = -g$. Since

$$\mathcal{E}^*(\psi v - w_1, u) = -(\mathcal{L}(\psi v - w_1), u) = -\frac{1}{2}(\Delta(\psi v - w_1) = 0 \qquad \text{for every } u \in \mathcal{F}^*,$$

and that $(\mathcal{E}^*, \mathcal{F}^*)$ is transient, we have $\psi v = w = G^* g$. Since $g = 0$ and $v = \psi v = G^* g$ on U_1, v is X^*-harmonic in U_1. This together with Theorem 1.6.4 implies that v is X^* harmonic in D^*. □

Remark 1.6.7 Let Y be Brownian motion in E reflected on compact sets K_j, $j = 1, \ldots, N$. Then harmonic functions of Y in $D = E \setminus K$ have zero normal derivatives at $K_j \cap \partial D$ and hence zero period around each K_j. However these harmonic functions typically do not take constant values on $K_j \cap \partial D$. BMD-Harmonic functions in $O \subset D^*$ takes constant values on K_j whenever $a_j^* \in O$. This property is important for the Riemann mapping theorem in multiply connected domains in $\mathbb{C} \cong \mathbb{R}^2$; see Section 1.10.

1.7 Harmonic Conjugate

Throughout this section, the dimension $d = 2$. The next theorem is a consequence of Theorem 1.6.6. Note that in multiply connected planar domains, classical harmonic functions (i.e. with respect to Brownian motion) in D can only *locally* be realized as the imaginary (or real) part of an analytic function in D. Theorem 1.7.1 shows that BMD is the right tool to study complex analysis in multiply connected domains in $\mathbb{R}^2$.

Theorem 1.7.1 *Suppose that $D := E \setminus K$ is connected. If v is X^*-harmonic on D^*, then $-v\big|_D$ admits a harmonic conjugate u on D uniquely up to an additive real constant in D so that $f(z) = u(z) + iv(z)$, $z \in D$, is an analytic function in D.*

Proof. Fix some $z_0 \in D$ and the value $u(z_0)$. For any $z \in D$, define

$$u(z) = u(z_0) + \int_\gamma \frac{\partial v(\xi)}{\partial \mathbf{n}_\xi} \sigma(d\xi), \tag{1.7.1}$$

where γ is a C^2-smooth simple curve in D that connects z_0 to z, $\sigma(d\xi)$ is the arc-length measure along γ and $\mathbf{n}$ the unit normal vector field along γ in the counter-clockwise direction (that is, if γ is parameterized by $(x(t), y(t))$, then $\mathbf{n}$ is the unit vector pointing to the same direction as $(-y'(t), x'(t))$). By the zero period property of v, the value of $v(x)$ is independent of the choice of the smooth C^2 simple curve γ that joins z_0 to z and hence well defined. One checks easily that (u, v) satisfies the Cauchy-Riemann equation and hence $f(z) := u(z)+iv(z)$ is an analytic function in D. □

1.8 Boundary Process

Let μ be the counting measure on $K^* = \{a_1^*, \ldots, a_N^*\}$. Since each a_j^* has positive capacity with respect to the Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$, μ is a smooth measure with respect to the BMD X^*. Let A^μ be the positive continuous additive functional (PCAF in abbreviation) of X^* having μ as its Revuz measure. Define its inverse

$$\tau_t = \inf\{s > 0 : A_s^\mu > t\}.$$

The time changed process $Y_t := X^*_{\tau_t}$ is the trace (boundary) process of X^* on K^*. It is a μ-symmetric continuous-time finite state Markov chain on K^*. Let $(\check{\mathcal{E}}^*, \check{\mathcal{F}}^*)$ be the Dirichlet form of Y on K^*. It is known that $\check{\mathcal{F}}_e^* = \mathcal{F}_e^*|_{K^*}$, $\check{\mathcal{F}}^* = \check{\mathcal{F}}_e \cap L^2(K^*; \mu)$, which is just $L^2(K^*, \mu)$ as K^* is finite, and

$$\check{\mathcal{E}}^*(u, v) = \mathcal{E}^*(\mathbf{H}_{K^*} u, \mathbf{H}_{K^*} v) = \sum_{i,j=1}^N u(a_i^*) u(a_j^*) \mathcal{E}^*(\varphi_i, \varphi_j) \qquad \text{for } u, v \in \check{\mathcal{F}}^*.$$

It follows that Y has infinitesimal generator $\mathcal{L}^Y$ in $L^2(K^*; \mu)$

$$\mathcal{L}^Y v(k) = -\sum_{j=1}^N \mathcal{E}^*(\varphi_i, \varphi_j) v(j) \qquad \text{for } v \in \mathbb{R}^N.$$

In other words, $(q_{ij} := -\mathcal{E}^*(\varphi_i, \varphi_j))_{1 \le i,j \le N}$ is the Q-matrix for the finite-state Markov chain Y, which in particular implies that $q_{kj} \ge 0$ for every pair $k \ne j$ and $\sum_{j=1}^N q_{kj} \le 0$ for every $1 \le k \le N$. We can also check the above property directly. Note that for $i \ne j$,

$$\begin{aligned}
q_{ij} &= -\check{\mathcal{E}}^*(1_{\{a_i^*\}}, 1_{\{a_j^*\}}) \\
&= \frac{1}{4}\Big(\mathcal{E}^*(1_{\{a_i^*\}} - 1_{\{a_j^*\}}, 1_{\{a_i^*\}} - 1_{\{a_j^*\}}) - \check{\mathcal{E}}^*(1_{\{a_i^*\}} + 1_{\{a_j^*\}}, 1_{\{a_i^*\}} + 1_{\{a_j^*\}})\Big) \\
&\ge \frac{1}{4}\Big(\mathcal{E}^*(|1_{\{a_i^*\}} - 1_{\{a_j^*\}}|, |1_{\{a_i^*\}} - 1_{\{a_j^*\}}|) - \check{\mathcal{E}}^*(1_{\{a_i^*\}} + 1_{\{a_j^*\}}, 1_{\{a_i^*\}} + 1_{\{a_j^*\}})\Big) \\
&= \frac{1}{4}\Big(\mathcal{E}^*(1_{\{a_i^*\}} + 1_{\{a_j^*\}}, 1_{\{a_i^*\}} + 1_{\{a_j^*\}}) - \check{\mathcal{E}}^*(1_{\{a_i^*\}} + 1_{\{a_j^*\}}, 1_{\{a_i^*\}} + 1_{\{a_j^*\}})\Big) \\
&= 0,
\end{aligned}$$

while

$$\begin{aligned}
\sum_{j=1}^N q_{ij} &= -\sum_{k=1}^N \check{\mathcal{E}}^*(1_{\{a_i^*\}}, 1_{\{a_k^*\}}) \\
&= -\sum_{k=1}^N \mathcal{E}^*(\mathbf{H}_{K^*} 1_{\{a_i^*\}}, \mathbf{H}_{K^*} 1_{\{a_k^*\}}) = \mathcal{E}^*(\mathbf{H}_{K^*} 1_{\{a_i^*\}}, \mathbf{H}_{K^*} 1_{K^*}) \\
&= -\mathbf{D}(\varphi_i, \mathbf{H}_K 1_K) \le 0
\end{aligned}$$

as $\mathbf{H}_K 1_K(x) = \mathbb{P}_x(\sigma_K < \infty)$ is the zero-order equilibrium potential of K in E and $\varphi_i \geq 0$. Let

$$\kappa_i = \sum_{j=1}^N \mathcal{E}^*(1_{\{a_i^*\}}, 1_{\{a_k^*\}}) = -\sum_{k=1}^N q_{ik}.$$

Then for $u \in \check{\mathcal{F}}^* = L^2(K^*; \mu)$,

$$\check{\mathcal{E}}^*(u,u) = \frac{1}{2}(u(a_i^*) - u(a_j^*))^2 q_{ij} + \sum_{i=1}^N u(a_i^*)^2 \kappa_i. \tag{1.8.1}$$

Hence we have the following.

Theorem 1.8.1 *The boundary process $\check{X}^*$ on K^* is a Markov chain on K^* with Q-matrix (q_{ij}); that is, $\check{X}^*$ is a continuous-time symmetric Markov chain on K^* with jumping intensities q_{ij} for $i \neq j$ and killing rates κ_i.*

1.9 Green Function and Poisson Kernel

Recall that G^* is the 0-resolvent of BMD X^* in D^* and G_D is the Green function of Brownian motion X^D in D. The next theorem gives the explicit expression for the Green function $G^*(x,y)$ of X^*.

Theorem 1.9.1 *Let $\Phi(z) = (\varphi_1(x), \dots, \varphi_N(z))$ and $\mathcal{A}$ an $N \times N$-matrix whose (i,j)-component p_{ij} is the period of φ_j around the compact set K_i. Then $\mathcal{A}$ is symmetric and invertible. For any Borel measurable function $f \geq 0$ on D^*,*

$$G^* f(x) = \int_D G^*(x,y) f(y) m(dy),$$

where

$$G^*(x,y) = G_D(x,y) + 2\Phi(x)\mathcal{A}^{-1} \cdot \Phi(y) \quad \text{for } x \in D^* \text{ and } y \in D. \tag{1.9.1}$$

Proof. For any $f \in C_c(D)$, by Theorem 1.5.3 and (1.5.8), $G^* f$ is X^*-harmonic in $O := D^* \setminus \mathrm{supp}[f]$ and that

$$G^* f(x) = G_D f(x) + \sum_{j=1}^N G^* f(a_i^*) \varphi_j(z). \tag{1.9.2}$$

By the same reasoning for the construction of the function ψ_1 in (1.6.2), for each $i \in \{1, \dots, N\}$, there is a $\psi_i \in C_c(E)$ so that $0 \leq \psi_1 \leq 1$, $\psi_i \in C_c^\infty(D)$, $\psi_i(x) = 1$ if and only if $x \in K_i$, and that ψ_i and ψ_j have disjoint support for $i \neq j$. Now fix $i \in \{1, \dots, N\}$. For $\varepsilon \in (0,1)$, let η_ε be the boundary of the connected component of $\{x \in E : \psi_i(x) > 1 - \varepsilon\}$ that contains K_i. Again by Sard's theorem, there is a set $\mathcal{N}_i$ having zero Lebesgue measure so that for every $\varepsilon \in (0,1)\setminus\mathcal{N}_i$, η_ε is a C^∞-smooth $(d-1)$-dimensional hypersurface. Take a decreasing sequence $\{\varepsilon_n, n\geq 1\} \in (0,1)\setminus\mathcal{N}_i$ with $\lim_{n\to\infty}\varepsilon_N = 0$. Since $\{x\in E : \varphi_i(x) > 1-\varepsilon_n\}$ decreases to K_j, we may assume that each η_{ε_n} is contained inside O. Let us call the connected component of $\mathbb{R}^d\setminus\eta_\varepsilon$ that contains K_i the interior of η_ε. Since $f\in C_c(D)$, G^*f is X^*-harmonic in a neighborhood of a_i^* and so it has zero period at a_i^* by Theorem 1.6.6. Moreover,

$$G_D f \in W^{1,2}_{0,e}(D) \quad \text{with} \quad \Delta G_D f = -2f. \tag{1.9.3}$$

By computing the period of both side of (1.9.2) at a_i^*, we deduce from the Green-Gauss formula that

$$\begin{aligned}
\sum_{i=1}^N p_{ij} G^* f(a_j^*) &= -\lim_{n\to\infty}\int_{\eta_{\varepsilon_n}} \frac{\partial G_D f(y)}{\partial \mathbf{n}}\sigma(dy) \\
&= -\lim_{n\to\infty}\frac{1}{1-\varepsilon_n}\int_{\eta_{\varepsilon_n}} \frac{\partial G_D f(y)}{\partial \mathbf{n}}\psi_i(y)\sigma(dy) \\
&= -\lim_{n\to\infty}\frac{1}{1-\varepsilon_n}\int_{D\setminus \mathrm{int}(\eta_{\varepsilon_n})} \left(\psi_i(y)\Delta G_D f(y) + \nabla\psi_i(y)\cdot\nabla G_D f(y)\right) dy \\
&= 2\int_D \psi_i(y) f(y)dy - \int_D \nabla\psi_i(y)\cdot\nabla G_D f(y)dy.
\end{aligned}\tag{1.9.4}$$

Since $\psi_i - \varphi_i$ is a bounded function in $W^{1,2}_{0,e}(D)$, by (1.9.3),

$$\int_D \nabla(\psi_i - \varphi_i)(y)\cdot\nabla G_D f(y)dy = 2\int_D (\psi_i(y)-\varphi_i(y))f(y)dy.$$

Thus we have from (1.9.4) that

$$\begin{aligned}
\sum_{i=1}^N p_{ij} G^* f(a_j^*) &= 2\int_D \varphi_i(y) f(y)dy - \int_D \nabla\varphi_i(y)\cdot\nabla G_D f(y)dy \\
&= 2\int_D \varphi_i(y) f(y)dy.
\end{aligned}$$

In the last equality we used the fact that $G_D f\in W^{1,2}_{0,e}(D)$ and φ_i is $\mathbf{D}$-orthogonal to $W^{1,2}_{0,e}(D)$. Since $\{\varphi_i; 1\leq i\leq N\}$ are linearly independent as functions on D and a_i^*'s are non-polar, the above identity implies that $\mathcal{A}$ is invertible and

$$(Gf^*(a_1^*),\ldots,G^*f(a_N^*))^{tr} = 2\mathcal{A}^{-1}\int_D \Phi(y)^{tr} f(y)dy.$$

Here the superscript "tr" stands for vector transpose. This together with (1.9.2) establishes (1.9.1). Since $G^*(z,\zeta)$ is symmetric in x and ζ, it follows from (1.9.1) that $\mathcal{A}^{-1}$ is symmetric and so is $\mathcal{A}$. This completes the proof of the theorem. □

We call the kernel $G^*(x,y)$ the Green function of X^* in D^*.

Lemma 1.9.2 *For each $x \in D^*$, $y \to G^*(x,y)$ extends to be an $\mathcal{E}^*$-quasi-continuous X^*-harmonic function on $D^* \setminus \{x\}$. If $K_j \subset K_j^r$ for each j, then $y \to G^*(x,y)$ extends to be a continuous X^*-harmonic function on $D^* \setminus \{x\}$.*

Corollary 1.9.3 *For each $j \in \{1,\ldots,N\}$, the period of $y \mapsto G_D(x,y)$ and $y \mapsto G^*(a_i^*,y)$ around K_j are $-2\varphi_j(x)$ and $2\delta_{ij}$, respectively.*

Proof. Computing the period around K_j on both sides of (1.9.1), we have by Theorem 1.6.6 and Lemma 1.9.2 that for $x \in D$, the period of $y \mapsto G_D(x,y)$ around K_j equals

$$-2\Phi(x)\mathcal{A}^{-1} \cdot (p_{j1},\ldots,p_{jn}) = -2\Phi(x) \cdot e_j = -2\varphi_j(x).$$

Here e_j denotes the unit vector in the positive direction of the x_j-axis. Since $G^*(a_j^*,y) = 2\Phi(a_j^*)\mathcal{A}^{-1} \cdot \Phi(y)$, its period around K_j is

$$2\Phi(a_i^*)\mathcal{A}^{-1} \cdot (p_{j1},\ldots,p_{jn}) = 2\Phi(a_i^*) \cdot e_j = 2\varphi_j(a_i^*) = 2\delta_{ij}.$$

□

Without loss of generality, we may and do assume that ∂E is smooth. We use σ to denote the Lebesgue surface measure on ∂E. Define

$$K^*(x,z) := \frac{1}{2}\frac{\partial G^*(x,z)}{\partial \mathbf{n}_z} \qquad \text{for } x \in D^* \text{ and } z \in \partial E.$$

Here $\mathbf{n}_z$ denotes the inward normal vector field for E on ∂E. Since $y \mapsto G^*(x,y)$ vanishes continuously on ∂E, $K^*(x,z) \geq 0$ for $x \in D^*$ and $z \in \partial E$. Note that for each fixed $z \in \partial E$, $x \mapsto K^*(x,z)$ is an X^*-harmonic function in D^*. We call K^* the Poisson kernel of X^*. For each $z \in \partial D$, define

$$K_D(x,z) = \begin{cases} \frac{1}{2}\frac{\partial G_D(x,z)}{\partial \mathbf{n}_z} & \text{for } x \in D, \\ 0 & \text{for } x \in K^*, \end{cases}$$

which is the classical Poisson kernel for Brownian motion in D (more precisely, on the part of $\partial E \subset \partial E$). By (1.9.1), we have for $x \in D^*$ and $z \in \partial E$,

$$K^*(x,z) = K_D(x,z) + \Phi(x)\mathcal{A}^{-1} \cdot \frac{\partial \Phi(z)}{\partial \mathbf{n}_z}. \tag{1.9.5}$$

Recall that X is Brownian motion in $\mathbb{R}^d$.

Lemma 1.9.4 *For every bounded continuous function f on ∂E,*

$$\mathbb{E}_x[f(X_{\tau_D}); X_{\tau_D} \in \partial E] = \int_{\partial E} K_D(x,z) f(z)\sigma(dz) \qquad \textit{for } x \in D.$$

Proof. When D is a bounded smooth domain, this is a classical result. So the main point of the proof is to take care of the case when ∂K_j may be non-smooth and that E may be possibly unbounded. We first assume that $f \in C_b(\partial E)$ is nonnegative. Let D_k be an increasing sequence of bounded smooth subdomains of D so that $\cup_{k\geq 1} D_k = D$, $\overline{D}_k \cap K_j = \emptyset$, the relative interior of $\partial E \cap \partial D_{k+1}$ contains $\partial E \cap \partial D_k$ and $\partial E \subset \cup_{k\geq 1} \partial D_k$. Clearly $G_{D_k}(x,y) \leq G_D(x,y)$ and $\lim_{k\to\infty} G_{D_k}(x,y) = G_D(x,y)$ for $x, y \in D$. For $x \in D_k$ and $z \in \partial D_k$, define

$$K_{D_k}(x,z) = \frac{\partial G_{D_k}(x,z)}{\partial \mathbf{n}_z^{(k)}},$$

where $\mathbf{n}_z^{(k)}$ is the unit inward normal vector field of D_k on ∂D_k. It is well known (cf. [16]) that

$$\mathbb{E}_x[f(X_{\tau_{D_k}}); X_{\tau_{D_k}} \in \partial E] = \int_{\partial E\cap\partial D_k} K_{D_k}(x,z) f(z)\sigma(dz) \qquad \text{for every } x \in D_k. \tag{1.9.6}$$

By the strong Markov property of X, one has

$$G_D(x,y) = G_{D_k}(x,y) + \mathbb{E}_x\left[G_D(X_{\tau_{D_k}}, y); X_{\tau_{D_k}} \in D\right] \qquad \text{for } x \in D_k \text{ and } y \in D. \tag{1.9.7}$$

For each $z \in \partial E$, let $\varepsilon > 0$ so that $B(z, 2\varepsilon) \cap D^c = \emptyset$. Fix some $y_0 \in B(z,\varepsilon) \cap D$. By the boundary Harnack principle for Brownian motion, there is a constant $c \geq 1$ so that

$$\frac{G_D(x,y)}{G_D(x,y_0)} \leq c \frac{\delta_{\partial E}(y)}{\delta_{\partial E}(y_0)} \qquad \text{for every } x \in B(z,2\varepsilon)^c \cap D \text{ and } y \in B(z,\varepsilon).$$

Here $\delta_{\partial E}(y)$ denotes the Euclidean distance between y and ∂E. Taking $y \to z$ along the normal direction at z gives

$$\frac{K_D(x,z)}{G_D(x,y_0)} \leq \frac{c}{\delta_{\partial E}(y_0)} \qquad \text{for every } x \in B(z,2\varepsilon)^c \cap D. \tag{1.9.8}$$

It follows from (1.9.7)

$$K_D(x,z) = K_{D_k}(x,z) + \mathbb{E}_x\left[K_D(X_{\tau_{D_k}}, z); X_{\tau_{D_k}} \in D\right] \qquad \text{for } x \in D \text{ and } z \in \partial E \cap \partial D_k.$$

Similarly, for $x \in D_k$ and $z \in \partial E \cap \partial D_k$,

$$K_{D_{k+1}}(x,z) = K_{D_k}(x,z) + \mathbb{E}_x\left[K_{D_{k+1}}(X_{\tau_{D_k}}, z); X_{\tau_{D_k}} \in D_{k+1}\right] \geq K_{D_k}(x,z).$$

Thus in view of (1.9.8), we have

$$K_D(x,z) = \uparrow \lim_{k\to\infty} K_{D_k}(x,z) \qquad \text{for } x \in D \text{ and } z \in \partial E.$$

Now taking $k \to \infty$ in (1.9.6), we have by the monotone convergence theorem that the theorem holds for nonnegative $f \in C_b(\partial E)$ and hence for general $f \in C_b(\partial E)$. □

Theorem 1.9.5 (i) *For each* $x \in D^*$, $\int_{\partial E} K^*(x,z)\sigma(dz) \leq 1$; *the equality holds if* E *is bounded.*

(ii) *For every bounded measurable function* f *on* ∂E, *the function*

$$\mathbf{H}^* f(x) := \int_{\partial E} K^*(x,z) f(z)\sigma(dz), \quad z \in D^*$$

is well defined and is a bounded X^**-harmonic function in* D^*. *Moreover, for any point* $z \in \partial E$ *at which* f *is continuous,*

$$\lim_{x \to z, x \in D} \mathbf{H}^* f(x) = f(z). \tag{1.9.9}$$

(iii) *For every bounded continuous function* f *on* ∂E,

$$\mathbb{E}_x[f(X^*_{\zeta-}); X^*_{\zeta-} \in \partial E] = \int_{\partial E} K^*(x,z) f(z)\sigma(dz) \qquad \text{for every } x \in D^*. \tag{1.9.10}$$

Proof. (i) Let U_j be relatively compact smooth sub-domains of E so that $K_j \subset U_j$, $\overline{U}_i \cap \overline{U}_j = \emptyset$ for $i \neq j$. When E is bounded, it follows from (1.9.5) and the Green-Gauss formula that for $x \in D^*$

$$\begin{aligned}
\int_{\partial E} K^*(x,z)\sigma(dz) &= \int_{\partial E} K_D(x,z)\sigma(dz) + \Phi(x)\mathcal{A}^{-1} \cdot \int_{\partial E} \frac{\partial \Phi(z)}{\partial \mathbf{n}_z}\sigma(dz) \\
&= \int_{\partial E} K_D(x,z)\sigma(dz) + \Phi(x)\mathcal{A}^{-1} \cdot \sum_{j=1}^N \int_{\partial U_j} \frac{\partial \Phi(z)}{\partial \mathbf{n}_z}\sigma(dz) \\
&= \int_{\partial E} K_D(x,z)\sigma(dz) + \Phi(x)\mathcal{A}^{-1} \cdot \mathcal{A}1 \\
&= \int_{\partial E} K_D(x,z)\sigma(dz) + \sum_{i=1}^N \varphi_i(x) \\
&= \mathbb{P}_x(X_{\tau_D} \in \partial E) + \sum_{i=1}^N \mathbb{P}_x(X_{\tau_D} \in K_i) \\
&= 1.
\end{aligned}$$

We used Lemma 1.9.4 for the second to the last equality.

When E is unbounded, let $\{E_k, k \geq 1\}$ be an increasing sequence of bounded smooth subdomains of E so that $\cup_{k\geq 1} E_k = E$, $\cup_{k \geq 1} \partial E_k = \partial E$, the relative interior of $\partial D_{k+1} \cap \partial E$ contains $\partial D_k \cap \partial E$, and that $U_j \subset E_1$ for $j = 1, \ldots, N$. Let $D_k := E_k \setminus K$, $D_k^* = D_k \cup \{a_1^*, \ldots, a_N^*\}$. The Green function of Brownian motion X in D_k is denoted by G_{D_k} and the Green function of the BMD in D_k^* is denoted as $G^*_{D_k}$. Similar notations applies to the Poisson kernels K_{D_k} and $K^*_{D_k}$. Recall from Theorem 1.3.1 that the part process X^{*,D_k^*} of

BMD X^* in D^* killed upon leaving D^*_k is the BMD in D^*_k. By the same argument as that for (1.9.7)-(1.9.8), we have for every $z \in \partial E$, there is $\varepsilon > 0$ and $y_0 \in D \cap B(z, \varepsilon)$ so that

$$\frac{K^*(x,z)}{G^*(x,y_0)} \leq \frac{c}{\delta_{\partial E}(y_0)} \qquad \text{for every } x \in B(z, 2\varepsilon)^c \cap D \tag{1.9.11}$$

and that

$$K^*(x,z) = K^*_{D_k}(x,z) + \mathbb{E}_x \left[K^*(X^*_{\tau_{D^*_k}}, z); X^*_{\tau_{D^*_k}} \in D^* \right] \quad \text{for } x \in D^*_k \text{ and } z \in \partial E \cap \partial D_k.$$

Similar relation holds with $K^*_{D_{k+1}}$ and D^*_{k+1} in place of K^* and D^* and thus we have

$$K^*_{D_{k+1}}(x,z) \geq K^*_{D_k}(x,z) \quad \text{for } x \in D^*_k \text{ and } z \in \partial E \cap \partial D_k.$$

It follows from the above two displays that

$$K^*(x,z) = \uparrow \lim_{k \to \infty} K^*_{D_k}(x,z) \quad \text{for } x \in D^* \text{ and } z \in \partial E. \tag{1.9.12}$$

By Fauto's lemma, for every $z \in D^*$,

$$\int_{\partial E} K^*(x,z)\sigma(dx) \leq \lim_{k\to\infty} \int_{\partial E \cap \partial E_k} K^*_{D_k}(x,z)\sigma(dx) \leq \lim_{k\to\infty} \int_{\partial E_k} K^*_{D_k}(x,z)\sigma(dx) = 1.$$

This establishes (i).

(ii) The first part follows from the fact that for each $z \in \partial E$, $x \mapsto K^*(x,z)$ is X^*-harmonic in D^*, (i) and Fubini's theorem. It follows from (1.9.5) that

$$\mathbf{H}f(x) = \int_{\partial E} K_D(x,z) f(z)\sigma(dz) + \sum_{j=1}^N c_j \varphi_j(x),$$

for some constants $c_1, \ldots, c_N$. Defining $f(z) = c_j$ for $z \in K_j$, we then have by Lemma 1.9.4 that $\mathbf{H}f(x) = \mathbb{E}_x \left[f(X_{\tau_D}) \right]$ for $x \in D$. Property (1.9.9) now follows from the corresponding result for Brownian motion.

(iii) Clearly by the strong Markov property of X^*, $h(x) := \mathbb{E}_x[f(X^*_{\zeta-})]$ is a bounded X^*-harmonic function in D^*. Since the part process $X^{*,D}$ of X^* in D is just the Brownian motion killed upon leaving D, we conclude from the corresponding classical result for Brownian motion that h is continuous up to the boundary $\partial D^* = \partial E$ with boundary value f. On the other hand, we know from (ii) that $\mathbf{H}f$ is also a bounded X^*-harmonic function in D^* that is continuous up to the boundary $\partial D^* = \partial E$ with the same boundary value f. Thus when E is bounded, by the maximum principle, we must have $h = \mathbf{H}f$. When E is unbounded, let E_k be an increasing sequence of bounded smooth domains approximating E as in the proof of (i) above, $D_k = E_k \setminus K$ and $D^*_k = D \cup \{a^*_1, \ldots, a^*_N\}$. Define $\tau^*_k := \inf\{t > 0 : X^*_t \notin D^*_k\}$. Since

$$\mathbb{E}_x[f(X^*_{\tau^*_k -}); X^*_{\tau^*_k -} \in \partial E] = \int_{\partial E} K^*_{D_k}(x,z) f(z)\sigma(dz) \qquad \text{for every } x \in D^*,$$

and $\lim_{k\to\infty} f(X^*_{\tau^*_k -}) 1_{\{X^*_{\tau^*_k -} \in \partial E\}} = f(X^*_{\zeta-}) 1_{\{X^*_{\zeta-} \in \partial E\}}$ $\mathbb{P}_x$-a.s., we have by (1.9.12) and the monotone convergence theorem that (1.9.10) holds for nonnegative $f \in C_b(\partial E)$ and hence for general $f \in C_b(\partial E)$. The proof of the theorem is now complete. $\square$

1.10 Applications to Conformal Mappings

The classical Riemann mapping theorem asserts that any simply connected planar domain can be conformally mapped onto the upper half space $\mathbb{H}$. The Riemann mapping theorem also holds for multiply connected domains. BMD can be used to give an "explicit" comformal mapping that maps multiply connected planar domains into the canonical slit domains.

In this section, let $d = 2$. Denote by $\mathbb{H}$ the upper half plane in $\mathbb{C} \cong \mathbb{R}^2$. We consider the set

$$D = \mathbb{H} \setminus K, \quad \text{where} \quad K = \bigcup_{j=1}^{N} K_j, \tag{1.10.1}$$

for mutually disjoint compact continua $K_1, \cdots, K_N$ contained in $\mathbb{H}$ such that for $\mathbb{H} \setminus K_j$ is connected for each j. Let $K^* = \{a_1^*, \cdots, a_N^*\}$ obtained from $\mathbb{H}$ by regarding each continuum K_j as a one point a_j^*. Denote by $Z^{\mathbb{H}} = (Z_t^{\mathbb{H}}, \mathbb{P}_z^{\mathbb{H}})$ the absorbing Brownian motion on $\mathbb{H}$ and by $Z^* = (Z_t^*, \mathbb{P}_z^*)$ the BMD on $D^* = D \cup K^*$.

For $r > 0$, let $\Gamma_r = \{z = x + iy : y = r\}$ and

$$v^*(z) := \lim_{r\to\infty} r \cdot \mathbb{P}_z^*(\sigma_{\Gamma_r} < \infty), \quad z \in D^*. \tag{1.10.2}$$

Theorem 1.10.1 (i) *The function v^* on D^* is well defined and is Z^*-harmonic on D^*.*

(ii) $v^*\big|_D$ *admits a unique harmonic conjugate u^* such that $f(z) = u^*(z) + iv^*(z)$, $z \in D$, is analytic on D and*

$$f(z) = z + \frac{a}{z} + o(\frac{1}{z}), \qquad z \to \infty \tag{1.10.3}$$

for some positive constant a.

(iii) *Suppose that each ∂K_i is a piecewise Lipschitz curve. Then the analytic function f is a conformal mapping from $\mathbb{H} \setminus \bigcup_{i=1}^N K_i$ onto $\mathbb{H} \setminus \bigcup_{i=1}^N \widetilde{C}_i$, where $\widetilde{C}_i$, $1 \le i \le N$, are mutually disjoint horizontal line segments in $\mathbb{H}$.*

We refer the reader to [7] for a proof of the above theorem. We remark here that the way of constructing v^* in the above theorem is due to G. Lawler [14], where the excursion reflected Brownian motion on the N-connected domain is utilized in place of BMD. The condition "each ∂K_i is a piecewise Lipschitz curve" imposed in Theorem 1.10.1(iii) is a technical assumption. It can be dropped with some extra work.

The complex Poisson kernel $K_D^*(x, z)$ presented in the previous section plays an important role for the chordal Komatu-Loewner equation in multiply connected domains. See [7] for details.

Chapter 2

Notes

Chapter 1 is a self-contained introduction to Brownian motion with darning (BMD) and its basic properties. BMD is a particular case of symmetric Markov processes with darning presented in Chapter 7 of Chen and Fukushima [2]. Some material presented in sections §1.1, §1.2, §1.4 and §1.5 can be derived from the more general results in [2, Chapter 7]. But the presentation here (including some of the proofs) is new and more direct. We took the view point that BMD is obtained from Brownian motion by "shorting" on each compact set K_j, in spirit with "shorting" in electric network or excursion-reflected random walk described in the first paragraph of [14, Section 5.1]. Theorem 1.2.1 is taken from [2, Theorem 4.3.8], which is an extension of Theorems I.5.2.3 and I.7.1.1 in Bouleau-Hirsh [1]. Some of the results presented in sections §1.1, §1.2, §1.4 and §1.5 are new; for example, Theorem 1.2.2 holds for any compact sets K_j without additional regular points assumption on K_j. Most examples in §1.1 appeared here for the first time. Theorems 1.3.2 and 1.3.3 are new. Section §1.6 is new and holds for any dimension. Some of its two-dimensional version has been given in [7]. Sections §1.7 and §1.9 are based on [7], while some of its presentation here is new. Section §1.8 is new.

As mentioned in the text, most of the results covered in Sections §1.1-1.6 can be extended easily to diffusions with darns and even to Markov processes with darns. We plan to spell these out in a future expansion of this Lecture Notes.

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