Fibrations and Comodule Categories

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If $p : E \to B$ is a covering map:

\[ \pi_1(B, b) \text{ acts on } E \]

\[ \pi_1(B) \text{ acts on fibers } \]

\[ \to \text{Set} \]
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- \( \pi_1(B) \) acts on fibers

\[
\pi_1(B)^{op} \rightarrow \text{Set}
\]
Covering maps and groupoid actions

\[ \begin{array}{c}
E \\
\downarrow^p \\
B
\end{array} \quad \Leftrightarrow \quad \pi_1(B)^{op} \to \text{Set} \]

Covering maps over \( B \) \hspace{1cm} \text{Groupoid actions of} \ \pi_1(B)
Étale spaces and (pre)sheaves

\[ E \xrightarrow{p} X \]

\[ \text{Open}(X)^{op} \to \text{Set} \]

Étale spaces over $X$ \quad \leftrightarrow \quad \text{Sheaves on } X
In both these situations, there is a duality between spaces varying nicely over $X$ and sets indexed ‘by $X$’.
Theorem (Grothendieck, 1964)

Let $B$ be a category. There is a 2-equivalence

$$\text{Fib}(B) \cong 2\text{-Fun}(B^{op}, \text{Cat})$$

Fibrations over $B$ \leftrightarrow Categories indexed by $B$
Suppose the unit $1 \in \mathcal{V}$ is terminal and pullbacks preserve coproducts. Let $B$ be a category. Then

$$\mathcal{V}\text{-Fib}(B_{\mathcal{V}}) \cong 2\text{-Fun}(B^{\text{op}}, \mathcal{V}\text{-Cat}).$$

$\mathcal{V}$-fibrations over $B_{\mathcal{V}}$ $\leftrightarrow$ $\mathcal{V}$-categories indexed by $B$
Theorem (Beardsley-W.)

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$$\mathcal{V}\text{-Fib}(B_\mathcal{V}) \cong 2\text{-Fun}(B^{op}, \mathcal{V}\text{-Cat}).$$

What if $1$ is not terminal? e.g. $k$ in $\textbf{Vect}_k$
We may not have maps $V \rightarrow 1$, but every $V$ has a coaction by $1$:

$$V \xrightarrow{\sim} V \otimes 1.$$
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More generally, instead of maps $V \to C$, where $C$ is a comonoid, we can ask for coactions

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We may not have maps $V \to \mathbf{1}$, but every $V$ has a coaction by $\mathbf{1}$:

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When $\otimes = \times$, coactions correspond to maps $V \to C$, so coactions are ‘generalized maps’.
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When $\otimes = \times$, coactions correspond to maps $V \to C$, so coactions are ‘generalized maps’. ← Can’t always be composed!
The comodule bifibration

- Arbitrary coactions can’t be composed
- Coactions arising from comonoid maps can be composed
- Comodule maps can be composed, but are not coactions
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What framework handles all these? The comodule bifibration!

\[(M, C) \xrightarrow{\cdot} (N, D)\]

\[\text{Comod}(\mathcal{V})\]

\[\Downarrow\]

\[C \xrightarrow{\cdot} D\]

\[\text{Comon}(\mathcal{V})\]
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\[(M, C) \rightarrow (N, D)\]

\[
\begin{array}{c}
\text{Comod}(\mathcal{V}) \\
\downarrow \\
\text{Comon}(\mathcal{V})
\end{array}
\]

Cotensoring acts like pullback against a coaction, so this behaves like a category with pullbacks.
Fibrations ‘across’ a 2-functor

Going up a dimension, we get:

\[ P : \text{Comod}(\mathcal{V}\text{-Cat}) \to \text{Comon}(\mathcal{V}\text{-Cat}) \]

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**Proposition (W.)**

For suitable \( \mathcal{V} \), there are 2-functors

\[ P\text{-Fib}(B_{\mathcal{V}}) \leftrightarrow 2\text{-Fun}(B^{op}, \mathcal{V}\text{-Cat}). \]
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**Proposition (W.)**

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$$P\text{-Fib}(B_{\mathcal{V}}) \leftrightarrow 2\text{-Fun}(B^{\text{op}}, \mathcal{V}\text{-Cat}).$$

Can we get an equivalence?