Exploring the theory of shifted derivators

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What are derivators?

A derivator is a strict 2-functor \( \mathbb{D} : \text{Cat}^{op} \to \text{CAT} \) satisfying some axioms (Der1)-(Der4)

**Der1:** \( \mathbb{D} : \text{Cat}^{op} \to \text{CAT} \) takes coproducts to products, i.e.

\[
\mathbb{D}(\bigsqcup_{i \in I} J_i) \cong \prod_{i \in I} \mathbb{D}(J_i).
\]

**Der2:** For any \( A \in \text{Cat} \), a morphism \( f : X \to Y \) is an isomorphism in \( \mathbb{D}(A) \) if and only if the morphisms

\[
a^*f : a^*X \to a^*Y
\]

are isomorphisms in \( \mathbb{D}(e) \) for all \( a \in A \).

**Der3:** For each functor \( u : A \to B \), \( u^* : \mathbb{D}(B) \to \mathbb{D}(A) \) has a left adjoint \( u_! \) and a right adjoint \( u_* \).

**Der4:** Pointwise computation of homotopy Kan extensions \( u_! \), \( u_* \)
Some examples of derivators

1. If $\mathcal{C}$ is a complete and cocomplete category,

$$I \mapsto \mathcal{C}^I$$

is a derivator.

2. If $\mathcal{M}$ is a model category and we let $\mathcal{W}$ denote the weak equivalences in $\mathcal{M}$, then

$$I \mapsto \mathcal{M}^I[(\mathcal{W}^I)^{-1}]$$

is a derivator.

3. If $\mathcal{A}$ is a Grothendieck abelian category,

$$\mathbb{D}_{\mathcal{A}} : I \mapsto \mathcal{D}(\mathcal{A}^I)$$

is a derivator, where $\mathcal{D}$ denotes the derived category. Specifically we may consider the cases $R$-Mod or $\text{Qcoh}(X)$ for a commutative ring $R$ or a reasonable scheme $X$. 
Shifted derivators

**Theorem (Groth)**

For any small category $J$, $\mathbb{D}^J(I) := \mathbb{D}(J \times I)$ defines another derivator.

This is our main tool in creating new derivators out of pre-existing derivators.
Affine lines

**Theorem (Balmer-Z)**

Let $\mathbb{N}$ denote the category with one object and endomorphism monoid $(\mathbb{N}, +)$ and $\mathbb{D}$ be any derivator. Then we call the shifted derivator $\mathbb{D}^\mathbb{N}$ the affine line associated to $\mathbb{D}$. If $\mathbb{D}$ is the derivator associated to a scheme, then $\mathbb{D}^\mathbb{N}$ is the derivator associated to $\mathbb{A}^1$ of that scheme.

If we replace $\mathbb{N}$ by $\mathbb{N}^n$ we similarly have that $\mathbb{D}^{\mathbb{N}^n}$ is akin to $\mathbb{A}^n$ of a derivator.
Theorem (Z)

Let \( \mathbb{Z} \) denote the category with one object and endomorphism monoid \((\mathbb{Z}, +)\) and \( D \) be any derivator. Then we call the shifted derivator \( D^\mathbb{Z} \) the “\( \mathbb{G}_m \)” associated to \( D \). If \( D \) is the derivator associated to a scheme, then \( D^\mathbb{Z} \) is the derivator associated to \( \mathbb{G}_m \) of that scheme.

Further, in the case of a scheme the left Kan extension morphism

\[
(i_N)! : D^N \to D^\mathbb{Z}
\]

is the inverse image functor along the inclusion \( \mathbb{G}_m \hookrightarrow \mathbb{A}^1 \).
Projective space

**Theorem (Z)**

Let $Q_n$ be the category with objects indexed by $\mathbb{N}$, arrows $\{x_0, \cdots, x_n\}$ between $k$ and $k + 1$ such that $x_ix_j = x_jx_i$ and $\mathbb{D}$ be a triangulated derivator. There is a Verdier localization, $\mathbb{D}^{Q_n}/\mathcal{C}$ of the derivator $\mathbb{D}^{Q_n}$ that gives us a construction of projective space over a derivator, i.e. if $\mathbb{D}$ is the derivator associated to a scheme $X$, then $\mathbb{D}^{Q_n}/\mathcal{C}$ is the derivator associated to $\mathbb{P}_X^n$.

We also note that there are canonical morphisms from this derivator to the $\mathbb{A}^n$ described above that model restriction onto one of the copies of $\mathbb{A}^n$ that cover $\mathbb{P}^n$. 
Future inquiries

1. What does shifting by other categories “mean” (independent of derivator)?
2. How can we incorporate the “geometry” of the derivator (e.g. tensor-triangular spectrum)?
3. What are some interesting definitions/theorems that can be studied purely via the formalism of derivators?
Thank you very much!