The Prime Spectrum and Representation Theory of the \( 2 \times 2 \) Reflection Equation Algebra

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March 17, 2018
Fix: $k$ a field, $q \in k$ not a root of unity.
The Algebra $\mathcal{A}_q(M_2)$

Fix: $k$ a field, $q \in k$ not a root of unity.
Define the $k$-algebra $\mathcal{A}_q(M_2)$ by:

Generators: $u_{11}$, $u_{12}$, $u_{21}$, $u_{22}$

Relations:
- $u_{11} u_{22} = u_{22} u_{11}$
- $u_{11} u_{12} = u_{12} (u_{11} + (q - 2 - 1) u_{22})$
- $u_{21} u_{11} = (u_{11} + (q - 2 - 1) u_{22}) u_{21}$
- $u_{22} u_{12} = q^2 u_{12} u_{22}$
- $u_{22} u_{21} = q^2 u_{22} u_{21}$
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Relations:

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\begin{align*}
  u_{11}u_{22} &= u_{22}u_{11} \\
  u_{11}u_{12} &= u_{12}(u_{11} + (q^{-2} - 1)u_{22}) \\
  u_{21}u_{11} &= (u_{11} + (q^{-2} - 1)u_{22})u_{21} \\
  u_{22}u_{12} &= q^2u_{12}u_{22} \\
  u_{21}u_{22} &= q^2u_{22}u_{21} \\
  u_{21}u_{12} - u_{12}u_{21} &= (q^{-2} - 1)u_{22}(u_{22} - u_{11}).
\end{align*}
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Where $\mathcal{A}_q(M_2)$ Comes From

$M_n(k) \hookrightarrow \text{GL}_n(k)$
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$M_n(k) \hookrightarrow \mathbb{C} \rightarrow O_q(M_n)$

$M \overset{g \in GL_n(k)}{\longrightarrow} g^{-1} Mg$
Where $\mathcal{A}_q(M_2)$ Comes From

$M_n(k) \triangleleft \text{GL}_n(k)$

$\mathcal{O}(M_n) \rightarrow \mathcal{O}(M_n) \otimes \mathcal{O}(\text{GL}_n)$ (comodule-alg)

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$t_j^i \mapsto t_i^k \otimes S(t_k^i)t_j^l$

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$Ebrahim (UCSB)$

$2 \times 2$ Reflection Equation Algebra

March 17, 2018
Where $A_q(M_2)$ Comes From

$M_n(k) \triangleleft \text{GL}_n(k)$

$O(M_n) \rightarrow O(M_n) \otimes O(\text{GL}_n)$ \hspace{1cm} (comodule-alg)

$O_q(M_n) \rightarrow O_q(M_n) \otimes O_q(\text{GL}_n)$

$t_j^i \mapsto t_l^k \otimes S(t_k^i) t_j^l$
Where $\mathcal{A}_q(M_2)$ Comes From

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$t^i_j \mapsto t^k_i \otimes S(t^i_k) t^l_j$

$u^i_j \mapsto u^k_i \otimes S(t^i_k) t^l_j$

$A_q(M_n)$ is a noncommutative deformation of $O(M_n)$ such that

$A_q(M_n) \rightarrow A_q(M_n) \otimes O_q(GL_n)$

is a comodule-algebra.
Where $\mathcal{A}_q(M_2)$ Comes From

$M_n(k) \hookrightarrow \text{GL}_n(k)$

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$t^i_j \mapsto t^k_i \otimes S(t^i_k)t^l_j$

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$u^i_j \mapsto u^k_i \otimes S(t^i_k)t^l_j$

is a comodule-algebra.

General construction uses an $R$-matrix, with relations given by reflection equation:

$R^l_i R^p_m u^k_i u^n_p = R^k_i R^n_m u^l_i u^p_r$
What Is Known About $A_q(M_2)$

- Reflection equation first introduced by Cherednik
- REAs later emerged from Majid’s transmutation theory
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  - It has a $k$-basis consisting of monomials in the generators $u_{ij}$. 

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Factorizing Particles on a Half-Line, and Root Systems.  

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$A_q(M_2)$ is a GWA

Rename $u_{21}$ to $x$ and rename $u_{12}$ to $y$. 
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Notice how $x$ and $y$ move past $u_{11}$ and $u_{22}$:

\[
x u_{22} = (q^2 u_{22}) x \quad u_{22} y = y (q^2 u_{22})
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x u_{11} = (u_{11} + (q^{-2} - 1) u_{22}) u_{11} \quad u_{11} y = y (u_{11} + (q^{-2} - 1) u_{22})
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**Observation**

$A_q(M_2)$ has the structure of a generalized Weyl algebra over the commutative subalgebra $k[u_{11}, u_{22}, yx]$. 
For this slide, assume $k$ is algebraically closed.

**Theorem**

The simple finite dimensional left $A_q(M_2)$-modules are as follows:

1. Ones annihilated by $u^{22}$, modules over $A_q(M_2)/\langle u^{22} \rangle \sim k[u_{11}, x, y]$.

2. Fix any $u_0 \in k \times$ and $n > 0$. There is a unique $n$-dimensional highest weight $A_q(M_2)$-module for which $u^{22}$ acts on the highest weight space as $u_0$. 

**Theorem**

Finite dimensional weight $A_q(M_2)$-modules on which $u^{22}$ acts invertibly are semisimple.
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Theorem

$A_q(M_2)$ satisfies the Dixmier-Moeglin equivalence (and so we can write down its primitive spectrum).