Invariant Theory of Artin-Schelter Regular Algebras: The Shephard-Todd-Chevalley Theorem

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Goal and Rationale:

Extend “Classical Invariant Theory” to an appropriate noncommutative context.

“Classical Invariant Theory”: Group $G$ acts on $k[x_1, \cdots, x_n]$. $f$ is \textit{invariant} under $G$ if $g \cdot f = f$ for all $g$ in $G$.

Invariant theory important in the theory of commutative rings.

Productive context for using homological techniques.

Further the study of Artin-Schelter Regular Algebras $A$ and other non-commutative algebras.

Extend from group $G$ action to Hopf algebra $H$ action.
Linear Group Actions on $k[x_1, \cdots, x_n]$

Let $G$ be a finite group of $n \times n$ matrices acting on $k[x_1, \cdots, x_n]$

$$g = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}$$

$$g \cdot x_j = \sum_{i=1}^{n} a_{ij}x_i$$

Extend to an automorphism of $k[x_1, \cdots, x_n]$. 
Invariants under $S_n$
Permutations of $x_1, \cdots, x_n$. 

(Painter: Christian Albrecht Jensen) (Wikipedia)
The subring of invariants under $S_n$ is a polynomial ring

$$k[x_1, \cdots, x_n]^{S_n} = k[\sigma_1, \cdots, \sigma_n]$$

where $\sigma_k$ are the $n$ elementary symmetric functions for $k = 1, \ldots, n$:

$$\sigma_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}x_{i_2} \cdots x_{i_k} = \mathcal{O}_{S_n}(x_1x_2 \cdots x_k)$$

or the $n$ power functions:

$$P_k = \sum x_i^k = \mathcal{O}_{S_n}(x_1^k).$$

**Question:** When is $k[x_1, \cdots, x_n]^G$ a polynomial ring?
Shephard-Todd-Chevalley Theorem

Let \( k \) be a field of characteristic zero.

**Theorem (1954).** The ring of invariants \( k[x_1, \cdots, x_n]^G \) under a finite group \( G \) is a polynomial ring if and only if \( G \) is generated by reflections.

A linear map \( g \) on \( V \) is called a **reflection** of \( V \) if all but one of the eigenvalues of \( g \) are 1, i.e. \( \dim V^g = \dim V - 1 \).

**Example:** Transposition permutation matrices are reflections, and \( S_n \) is generated by reflections.
Noncommutative Generalizations?

Replace $k[x_1, \cdots, x_n]$ by a “polynomial-like” noncommutative algebra $A$.

Let $A$ be Artin-Schelter regular algebra. A commutative Artin-Schelter regular ring is a commutative polynomial ring.

Consider groups $G$ of graded automorphisms acting on $A$. Note that not all linear maps act on $A$.

More generally, consider finite dimensional semi-simple Hopf algebras $H$ acting on $A$. 
Artin-Schelter Gorenstein/Regular

Noetherian connected graded algebra $A$ is Artin-Schelter Gorenstein if:

- $A$ has graded injective dimension $d < \infty$ on the left and on the right,
- $\text{Ext}^i_A(k, A) = \text{Ext}^i_{A^{op}}(k, A) = 0$ for all $i \neq d$, and
- $\text{Ext}^d_A(k, A) \cong \text{Ext}^d_{A^{op}}(k, A) \cong k(\ell)$ for some $\ell$.

If in addition,

- $A$ has finite (graded) global dimension, and
- $A$ has finite Gelfand-Kirillov dimension,

then $A$ is called Artin-Schelter regular of dimension $d$.

An Artin-Schelter regular graded domain $A$ is called a quantum polynomial ring of dimension $n$ if $H_A(t) = (1 - t)^{-n}$. 
Linear automorphisms of $\mathbb{C}_q[x, y]$

If $q \neq \pm 1$ there are only diagonal automorphisms:

$$g = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$ 

When $q = \pm 1$ there also are automorphisms of the form:

$$g = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}:$$

$$yx = qxy$$

$$g(yx) = g(qxy)$$

$$axby = qbyax$$

$$abxy = q^2 abxy$$

$$q^2 = 1.$$
Noncommutative
Shephard-Todd-Chevalley Theorem

1. $A^G$ is a polynomial ring $\sim ??? A^G \cong A??$

Example (a): Let

$$g = \begin{pmatrix} \epsilon_n & 0 \\ 0 & 1 \end{pmatrix}$$

act on $A = \mathbb{C}_{-1}[x, y]$. Then $A^G = \mathbb{C}\langle x^n, y \rangle$.

When $n$ odd, $A^G \cong A$. When $n$ even $A^G \cong \mathbb{C}[x, y]$.

Replace “$A^G$ is a polynomial ring” with “$A^G$ is AS-regular”.

When $A$ commutative $A^G \cong A$ equivalent to $A^G$ AS-regular.
1. $A^G$ is a polynomial ring $\iff A^G$ is AS-regular.

2. Definition of “reflection”:

All but one eigenvalue of $g$ is 1 $\iff$ ???
Examples $G = \langle g \rangle$ on $A = \mathbb{C}_{-1}[x, y]$ ($yx = -xy$):

Example (b): $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $A^{S_2}$ is generated by

$$P_1 = x + y \text{ and } P_2 = x^3 + y^3$$

($x^2 + y^2 = (x + y)^2$ and $g \cdot xy = yx = -xy$ so no generators in degree 2); alternatively, generators are

$$\sigma_1 = x + y \text{ and } \sigma_2 = x^2y + xy^2.$$ 

The generators are NOT algebraically independent. $A^{S_2}$ is AS-regular (but it is a hyperplane in an AS-regular algebra). The transposition $(1, 2)$ is NOT a “reflection”.
Examples $G = \langle g \rangle$ on $A = \mathbb{C}_{-1}[x, y]$
($yx = -xy$):

Example (c): $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Now $\sigma_1 = x^2 + y^2$ and $\sigma_2 = xy$ are invariant and

$A^g \cong \mathbb{C}[\sigma_1, \sigma_2]$ is AS-regular.

$g$ is a “mystic reflection”.
2. Definition of “reflection”:

All but one eigenvalue of $g$ is 1 $\Rightarrow$

The trace function of $g$ acting on $A$ of dimension $n$ has a pole of order $n - 1$ at $t = 1$, where

$$\text{Tr}_A(g, t) = \sum_{k=0}^{\infty} \text{trace}(g|A_k)t^k = \frac{1}{(t - 1)^{n-1}q(t)} \text{ for } q(1) \neq 0.$$
Examples $G = \langle g \rangle$ on $A = \mathbb{C}_{-1}[x, y]$ ($yx = -xy$):

(a) $g = \begin{bmatrix} \epsilon_n & 0 \\ 0 & 1 \end{bmatrix}$, $Tr(g, t) = \frac{1}{(1-t)(1-\epsilon_nt)}$, $A^g$ AS-regular.

(b) $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Tr(g, t) = \frac{1}{1 + t^2}$, $A^g$ not AS-regular.

(c) $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $Tr(g, t) = \frac{1}{(1-t)(1+t)}$, $A^g$ AS-regular.

For $A = \mathbb{C}_{q_{ij}}[x_1, \cdots, x_n]$ the groups generated by “reflections” are exactly the groups whose fixed rings are AS-regular rings.
Noncommutative Shephard-Todd-Chevalley Theorem

If \( G \) is a finite group of graded automorphisms of an AS-regular algebra \( A \) of dimension \( n \) then \( A^G \) is AS-regular if and only if \( G \) is generated by elements whose trace function

\[
Tr_A(g, t) = \sum_{k=0}^{\infty} \text{trace}(g|A_k)t^k = \frac{1}{(t-1)^{n-1}q(t)},
\]

i.e. has a pole of order \( n - 1 \) at \( t = 1 \).

Proven for cases:
1. \( G \) abelian and \( A \) a “quantum polynomial algebra”.
2. \( A = \mathbb{C}_{q_{ij}}[x_1 \cdots, x_n] \), skew polynomial ring.
3. \( A \) is an AS-regular graded Clifford algebra.
Molien’s Theorem:
Using trace functions

\[ H_{AG}(t) = \frac{1}{|G|} \sum_{g \in G} \text{Tr}_A(g, t) \]

\[ H_{AG}(t) = \frac{1}{4(1-t)^2} + \frac{2}{4(1-t^2)} + \frac{1}{4(1+t)^2} = \frac{1}{(1-t^2)^2}. \]

Example (c) \( A = \mathbb{C}_{-1}[x, y] \) and \( g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \)
\( \sigma_1 = x^2 + y^2, \sigma_2 = xy \) and \( A^g \cong \mathbb{C}[\sigma_1, \sigma_2]. \)
Bounds on Degrees of Generators:
Commutative Polynomial Algebras

Noether’s Bound (1916):
For $k$ of characteristic zero, generators of $k[x_1, \cdots, x_n]^{G}$ can be chosen of degree $\leq |G|$.

Göbel’s Bound (1995):
For subgroups $G$ of permutations in $S_n$, generators of $k[x_1, \cdots, x_n]^{G}$ can be chosen of degree $\leq \max\{n, \binom{n}{2}\}$.
Invariants of $A = \mathbb{C}_{-1}[x_1, \ldots, x_n]$ under the full Symmetric Group $S_n$

Example (b): $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ acts on $A$.

Both bounds fail for $A^{S_2}$, which required generators of degree $3 > |S_2| = 2 = \max\{2, \binom{2}{2}\}$: Generating sets

$$P_1 = x + y = O_{S_2}(x) \text{ and } P_2 = x^3 + y^3 = O_{S_2}(x^3)$$

or

$$\sigma_1 = x + y = O_{S_2}(x) \text{ and } \sigma_2 = x^2 y + xy^2 = O_{S_2}(x^2 y).$$
Invariants of $A = \mathbb{C}_{-1}[x_1, \ldots, x_n]$ under the full Symmetric Group $S_n$

Invariants are generated by sums over $S_n$-orbits
$O_{S_n}(X^I) = \text{the sum of the } S_n\text{-orbit of a monomial } X^I.$
$O_{S_n}(X^I)$ can be represented by $X^I$, where $I$ is a partition:

$$X^{(i_1, \ldots, i_n)} \text{ where } i_1 \geq i_2 \geq \ldots \geq i_n$$

$O_{S_n}(X^I) = 0$ if and only if $I$ is a partition with repeated odd parts (e.g. $O_{S_n}(x_1^5x_2^3x_3^3) = 0$ it corresponds to the partition $5 + 3 + 3$).
$A^S_n$ is generated by the $n$ odd power sums

$$P_k = \sum x_i^{2k-1}$$

or the $n$ invariants

$$\sigma_k = \mathcal{O}_{S_n}(x_1^2 \ldots x_{k-1}^2 x_k)$$

for $k = 1, \ldots, n$.

Bound on degrees of generators of $A^S_n$ is $2n - 1$. 
Invariants under the Alternating Group $A_n$: Commutative Case

$\mathbb{C}[x_1, \ldots, x_n]^{A_n}$ is generated by the symmetric polynomials (or power functions) and

$$D = \prod_{i<j} (x_i - x_j),$$

which has degree $\binom{n}{2}$. The Göbel bound is sharp.
Invariants of $A = \mathbb{C}_{-1}[x_1, \ldots, x_n]$ under the Alternating Group:

$A^{A_n}$ is generated by $\mathcal{O}_{A_n}(x_1x_2 \cdots x_{n-1})$,

and the $n-1$ polynomials $\sigma_1, \ldots, \sigma_{n-1}$

(or the power functions $P_1, \ldots, P_{n-1}$),

An upper bound on the degrees of generators of $A^{A_n}$ is $2n - 3$. 
Questions

For $A$ an Artin-Schelter regular algebra, find an upper bound on the degrees of generators of $A^G$.

Find an analogue of Göbel bound
(for $A = \mathbb{C}_{-1}[x_1, \cdots, x_n]$ we proved $n^2$, but probably not sharp).

Find an analogue of Noether bound
(consider cyclic groups?).
What are the “reflection groups”? 

Shephard-Todd classified the reflection groups (finite groups $G$ where $\mathbb{C}[x_1, \cdots, x_n]^G$ is a polynomial ring) – 3 infinite families and 34 exceptional groups.

If $A$ is a quantum polynomial ring, a “reflection” of $A$ must be a classical reflection, or a mystic reflection $\tau_{i, j, \lambda}$ where

$$
\tau_{s, t, \lambda}(x_i) = \begin{cases} 
  x_i & i \neq s, t \\
  \lambda x_t & i = s \\
  -\lambda^{-1} x_s & i = t.
\end{cases}
$$

Question: Do other AS-regular algebras have other kinds of “reflections”?
The Groups $M(n, \alpha, \beta)$

Let $A = \mathbb{C}_{-1}[x_1, \cdots, x_n]$, $\alpha, \beta \in \mathbb{N}$ with $\alpha | \beta$ and $2 | \beta$. Let $\theta_{s, \lambda}$ be the classical reflection

$$\theta_{s, \lambda}(x_i) = \begin{cases} 
  x_i & i \neq s \\
  \lambda x_s & i = s.
\end{cases}$$

$M(n, \alpha, \beta)$ is the subgroup of graded automorphisms of $A$ generated by

$$\{ \theta_{i, \lambda} | \lambda^\alpha = 1 \} \cup \{ \tau_{i,j, \lambda} | \lambda^\beta = 1 \}.$$ 

Then $M(n, \alpha, \beta)$ is a “reflection group”.
Rotation group of cube is generated by

\[ g_1 := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad g_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \]

that act on \( A = \mathbb{C}_{-1}[x, y, z] \) as the mystic reflections \( g_1 = \tau_{1,2,1} \) and \( g_2 = \tau_{2,3,1} \), respectively, and generate \( G = M(3, 1, 2) \).
The mystic reflection groups $M(2, 1, 2\ell)$, for $\ell \gg 0$, are not isomorphic to classical reflection groups as abstract groups. They are the “dicyclic groups” of order $4\ell$ generated by

$$
\begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
$$

for $\lambda$ a primitive $2\ell$th root of unity.
Let \( A = \mathbb{C}_{q_{ij}}[x_1, \cdots, x_n] \) and \( G \) be a finite subgroup of graded automorphisms of \( A \).

If \( G \) is generated by “reflections” of \( A \), then \( G \) as an abstract group is isomorphic to a direct product of classical reflection groups and groups of the form \( M(n, \alpha, \beta) \).
Invariants under Hopf Algebra Actions

Let \((H, \Delta, \epsilon, S)\) be a Hopf algebra and \(A\) be a Hopf-module algebra so

\[
h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b) \quad \text{and} \quad h \cdot 1_A = \epsilon(h)1_A
\]

for all \(h \in H\), and all \(a, b \in A\).

The **invariants of \(H\) on \(A\)** are

\[
A^H := \{a \in A \mid h \cdot a = \epsilon(h)a \text{ for all } h \in H\}.
\]

When \(H = k[G]\) and \(\Delta(g) = g \otimes g\) then \(g \cdot (ab) = g(a)g(b)\).
Kac/Masuoka’s 8-dimensional semisimple Hopf algebra

$H_8$ is generated by $x, y, z$ with the following relations:

$x^2 = y^2 = 1, \ xy = yx, \ zx = yz,$

$zy = xz, \ z^2 = \frac{1}{2}(1 + x + y - xy).$

$\Delta(x) = x \otimes x, \ \Delta(y) = y \otimes y,$

$\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$

$\epsilon(x) = \epsilon(y) = \epsilon(z) = 1, \ S(x) = x^{-1}, \ S(y) = y^{-1}, \ S(z) = z.$
Hopf Action of $H_8$ on $A = \mathbb{C}_{-1}[u, v]$

$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$A = \mathbb{C}_{-1}[u, v]$ is a left $H_8$-module algebra.

Let $a = u^3 v - uv^3$ and $b = u^2 + v^2$, then $A^{H_8} = \mathbb{C}[a, b]$, so $H_8$ is a “reflection quantum group”.
Hopf Action of $H_8$ on $A = \mathbb{C}_i[u, v]$

$(vu = iuv)$

$x \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

$A = \mathbb{C}_i[u, v]$ is an $H_8$-module algebra

$z \cdot (uv) = -vu, \quad z \cdot (vu) = uv,$

$z \cdot (u^2) = v^2, \quad z \cdot (v^2) = u^2.$

$A^{H_8} = \mathbb{C}[u^2v^2, u^2 + v^2],$ so $H_8$ is a “reflection quantum group”.

Furthermore $A^{H_8} \neq A^G$ for any finite group $G$. 
Molien’s Theorem

When $H$ is a finite dimensional semisimple Hopf algebra acting on $A$.

Then $H_{AH}(t) = Tr(\int, t)$, where $\int$ has $\epsilon(\int) = 1$.

E.g. for $H_8$

$$\int = \frac{1 + x + y + xy + z + xz + yz + xyz}{8}.$$
Questions

When is $k[x_1, \cdots, x_n]^H$ a polynomial ring?
Must $H$ be a group algebra or the dual of a group algebra?

If $H$ is a semisimple Hopf algebra and $A = \mathbb{C}[u, v]$ then if $A$ is an inner faithful $H$-module algebra then $H$ is a group algebra (Chan-Walton-Zhang).

If $A$ is Artin-Schelter regular, when is $A^H$ regular?

What happens when $G$ (or $H$) is infinite?

What happens when $H$ is not semisimple?
$H$ not semisimple

Consider the Sweedler algebra $H(-1)$ generated by $g$ and $x$

$$g^2 = 1, \quad x^2 = 0, \quad xg = -gx$$

$$\Delta(g) = g \otimes g \quad \Delta(x) = g \otimes x + x \otimes 1,$$

$$\epsilon(g) = 1, \epsilon(x) = 0 \quad S(g) = g, \quad S(x) = -gx.$$  

Then $H(-1)$ acts on $k[u, v]$ as

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$k[u, v]^{H(-1)} = k[u, v^2].$$