On the adjoint representation of Hopf algebras

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Outline

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1. Background and motivation

2. The Hopf annihilator of the adjoint representation

3. Conjugacy classes

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Notation

Throughout the talk we will use the following notation.

- $\mathbb{K}$ will be a field with char $\mathbb{K} = p \geq 0$
- $(\,.\,)^* := \mathrm{Hom}_{\mathbb{K}}(\,.\,,\mathbb{K})$ will denote the $\mathbb{K}$-linear dual
- G will denote a finite group
- H will denote an arbitrary Hopf $\mathbb{K}$-algebra
- $(\,.\,)^+ := \ker \epsilon$ will denote the augmentation ideal

The adjoint representation of a group

A group G acts on its self by conjugation.

$$^{g}h = ghg^{-1} \qquad (g, h \in G)$$

Extending $\mathbb{K}$-linearly gives an action of $\mathbb{K}G$ on itself.

Definition.

The group algebra equipped with this action will be called the *adjoint representation*, denoted ${}^{\mathrm{ad}}\mathbb{K}G$.

The group picture

Theorem.

$$\begin{tikzcd}
\boxed{\text{${}^{\mathrm{ad}}\mathbb{K}G$ is completely reducible}} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \boxed{\text{G has a central Sylow p-subgroup}} \arrow[d, Rightarrow] \\
\boxed{\text{${}^{\mathrm{ad}}(\mathbb{K}G/\operatorname{rad}\mathbb{K}G)$ is completely reducible}} \arrow[r, Leftrightarrow] & \boxed{\text{G has a normal Sylow p-subgroup}}
\end{tikzcd}$$

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A module V has the *Chevalley property* if $T(V) := \bigoplus_{n\in\mathbb{N}} V^{\otimes n}$ is completely reducible.

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Components of the proof of the top implication

Sketch of proof

1. The largest Hopf ideal of $\mathbb{K}G$ that annihilates ${}^{\mathrm{ad}}\mathbb{K}G$ is:

$$\mathbb{K}G(\mathbb{K}\mathscr{Z}(G))^+$$

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Definitions and notation

A Hopf algebra H acts on its self via the adjoint action.

$$^{h}k = h_{(1)}kS(h_{(2)}) \qquad (h, k \in H)$$

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- Let $\zeta(H)$ denoted the largest Hopf subalgebra contained in the center of H.

The Hopf annihilator of the adjoint representation

Theorem 1. (J.)

Let H be a Hopf algebra that satisfies one of the following conditions:

1. H is finite-dimensional or
2. the coradical of H is cocommutative (e.g., H is cocommutative or pointed).

Then the Hopf annihilator of the adjoint representation is given by $\mathscr{H}(\operatorname{ann}{}^{\mathrm{ad}}H) = H\zeta(H)^{+}$.

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- $\zeta(H)^+ \subseteq \mathscr{H}(\mathrm{ann}\,{}^{\mathrm{ad}}H)$ since

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- Now $\zeta(H) \subseteq {}^{co\overline{H}}H$ since
$$\begin{aligned}\overline{z}_{(1)} \otimes z_{(2)} &= (\overline{z_{(1)} - \epsilon(z_{(1)})1} + \epsilon(z_{(1)})\overline{1}) \otimes z_{(2)} \\ &= \epsilon(z_{(1)})\overline{1} \otimes z_{(2)} = \overline{1} \otimes z\end{aligned}$$

The proof: Coinvariants II

- ${}^{co\overline{H}}H \subseteq \mathscr{Z}(H)$ since

$$\begin{aligned} ch &= c_{(1)}h\epsilon(c_{(2)}) = c_{(1)}hS(c_{(2)})c_{(3)} \\ &= {}^{c_{(1)}}hc_{(2)} = {}^{\overline{c}_{(1)}}hc_{(2)} = {}^{\overline{1}}hc = hc \end{aligned}$$

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- ${}^{co\overline{H}}H$ is a right subcomodule of H, thus:

$$\Delta({}^{co\overline{H}}H) \subseteq {}^{co\overline{H}}H \otimes H \subseteq \mathscr{Z}(H) \otimes H$$

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Theorem. (Chirvasitu, Kasprzak. preprint)

$$\zeta(H) = \{h \in H | \Delta(h) \in \mathscr{Z}(H) \otimes H\}$$

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- Giving ${}^{co\overline{H}}H \subseteq \zeta(H)$ and so ${}^{co\overline{H}}H = \zeta(H)$

The proof: faithful (co)flatness

Recall the assumption of Theorem 1 that:

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- H is a faithfully flat $\zeta(H)$-module
- H is a faithfully coflat $H/H\zeta(H)^+$-comodule

The proof: an equivalence

Theorem. (Takeuchi 79)

We have the following inverse maps:

$$\left\{ A \;\middle|\; \begin{array}{l} \text{a left H-comodule algebra} \\ \text{H faithfully flat over A} \end{array} \right\} \underset{HA^+}{\overset{{}^{coH/I}H}{\leftrightarrows}} \left\{ I \;\middle|\; \begin{array}{l} \text{I left H-module coideal} \\ \text{H faithfully coflat over H/I} \end{array} \right\}$$

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The result follows from the diagram below:

$$\begin{tikzcd} & H\zeta(H)^{+} \arrow[dl] \arrow[d, phantom, "\|"] \\ \zeta(H) \arrow[ur] \arrow[r, leftrightarrow] & \mathscr{H}(\mathrm{ann}^{\mathrm{ad}} H) \end{tikzcd}$$

$\square$

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Theorem. (Rieffel 67)

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${}^{\mathrm{ad}}H$ has the Chevalley property iff $H/H\zeta(H)^{+}$ is semisimple.

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Corollary 2. (J.)

${}^{\text{ad}}H$ has the Chevalley property impies H is unimodular.

Review: Drinfeld double

Now ${}^{\mathrm{ad}}H$ can be viewed as a right H-comodule with structure map Δ. With this ${}^{\mathrm{ad}}H$ becomes a Yetter-Drinfeld module, thus it is natural to consider the Drinfeld double.

Review: Drinfeld double

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Definition.

The *Drinfeld double* of H is the Hopf algebra $D(H)$. The coalgebra structure of $D(H)$ is given by:

$$D(H) \overset{coalg}{\cong} H^{*cop} \otimes H$$

The element $f \otimes h$ is denoted $f \bowtie h$. The multiplication is given by:

$$(f \bowtie h)(g \bowtie k) = f(h_{(1)} \rightharpoonup g \leftharpoonup S^{-1}(h_{(3)})) \bowtie h_{(2)}k$$

Conjugacy class definition

The Drinfeld double acts on H via the action below:

$$(f \bowtie h).k = ({}^{h}k) \leftharpoonup S^{-1}(f) \qquad (f \in H^{*} h, k \in H)$$

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Definition. (Cohen, Westreich 2010)

If H is a completely reducible $D(H)$-module then we say a *conjugacy class* is a simple $D(H)$-submodule of H.

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Example: group algebras

The action of $D(\mathbb{K}G)$ on $\mathbb{K}G$ is completely reducible.
The conjugacy classes, as defined above, are the modules arising from $D(\mathbb{K}G)$ acting as above on the $\mathbb{K}$-span of the classical conjugacy classes.

Results on conjugacy classes

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Motivation

Hopf annihilator

Conjugacy classes

Proposition 1. (J.)

For H a finite-dimensional Hopf algebra:

1. If H is a completely reducible $D(H)$-module then H is cosemisimple.
2. If H is cosemisimple and ${}^{\mathrm{ad}}H$ is a completely reducible then H is a completely reducible $D(H)$-module.

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Theorem 3. (J.)

Let H be a cosemisimple, involutory Hopf algebra with $\mathbb{K} = \overline{\mathbb{K}}$ then ${}^{\mathrm{ad}}H$ completely reducible implies char $\mathbb{K}$ does not divide the dimension of any of the conjugacy classes.

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