On The Endomorphisms of Some Noncommutative Algebras

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Automorphisms of Polynomial Algebras

Let $k$ be a field. The $k$-algebra automorphisms of $k[x_1, \cdots, x_n]$ have been extensively researched in the literature.

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- \textbf{(H.W.E. Jung 1942 & W. Van der Kulk 1953)} The structure of $\text{Aut}_k k[x, y]$ is also well understood.
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- (H.W.E. Jung 1942 & W. Van der Kulk 1953) The structure of $\text{Aut}_k k[x, y]$ is also well understood.
- (M. Nagata 1972) It was conjectured by Nagata that $k[x, y, z]$ has a wild automorphism

$$(x, y, z) \mapsto (x - 2(xz + y^2)y - (xz + y^2)^2z, y + (xz + y^2)z, z).$$

This conjecture was proved by U.U. Umirbaev and I.P. Shestakov in 2004.
The Jacobian Conjecture

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- The Jacobian Conjecture remains open (except for the case $n = 1$ which is trivial).
Weyl Algebras and The Dixmier Conjecture

Let $A_1(\mathbb{C})$ be the $\mathbb{C}$– algebra generated by $x, y$ subject to the relation $xy - yx = 1$.

- (J. Dixmier 1968 & L. Makar-Limanov 1983) The group $\text{Aut}_\mathbb{C}A_1(\mathbb{C})$ was classified by Dixmier. Makar-Limanov determined $\text{Aut}_k(A_1(A))$ for $k$ of positive characteristic.
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Three Questions Considered

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There has much interest in the following questions:

▶ Determine the automorphism group for a noncommutative algebra.

▶ Identify noncommutative algebras whose endomorphisms are all automorphisms.

▶ Establish a criterion for an algebra endomorphism to be an automorphism.
The automorphism groups have been classified for the following algebras.

- (J. Alev & M. Chamarie 1992) The tensor products of quantum plane algebras \( \mathbb{k}_q[x, y] \), the 2 \( \times \) 2 quantum matrix algebra \( R_q[M_2] \), the quantum space \( \mathbb{k}_q[x_1, \cdots, x_n] \), and the quantized enveloping algebra \( U_q(sl_2) \).
Classifying the Automorphism Group

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▶ (J. Alev & F. Dumas 1993) The quantum Heisenberg algebra \( U_q^+(sl_3) \), the quantum Weyl algebra \( A_1^q(\mathbb{k}) \) of rank 1, the Weyl-Hayashi algebra, and other related algebras.
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Andruskiewitsch-Dumas Conjecture and Launois-Lenagan Conjecture

- In 2003, N. Andruskiewitsch and F. Dumas conjectured that

$$\text{Aut}_{\mathbb{k}}(U_q^+(\mathfrak{g})) \cong (\mathbb{k}^\times)^n \rtimes \text{Aut}(\Gamma).$$

This was verified for $U_q^+(\mathfrak{so}_5)$ by S. Launois in 2004 and for $U_q^+(\mathfrak{sl}_4)$ by S. Launois and S. Lopes in 2006.

- In 2007, S. Launois and T.H. Lenagan conjectured that

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Two Major Breakthroughs

- (M. Yakimov 2013) M. Yakimov resolved both conjectures in full generality in 2013.

- (M. Yakimov & K. Goodearl 2013) There is a far-reaching result by M. Yakimov and K. Goodearl on the unipotent automorphisms for a variety of quantum nilpotent algebras.

- (S. Ceken, J.H. Palmieri, Y.-H. Wang & J.J. Zhang 2015) In the root of unity case, the automorphism group is determined for many quantum algebras using the idea of discriminants. There have been many works following this direction.
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Let $A = \mathbb{k}_Q[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$ with $n \geq 2$ be the quantum Laurent polynomial algebra defined by $Q = (q_{ij})_{1 \leq i,j \leq m}$.

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- One may refer the property with “all algebra endomorphisms being bijective” to as the \textbf{Dixmier Property or Condition}. 
Let $R = \mathbb{C}[t, t^{-1}]$ and $A_n^{(t)}$ be the $R$–algebra generated by $x_1, y_1, \ldots, x_n, y_n$ subject to the relations: $x_iy_i - ty_ix_i = 1$, $x_iy_j = y_jx_i$, $x_ix_j = x_jx_i$, and $y_iy_j = y_jy_i$. Set $A_n^q = A_n^{(t)}/(t - q)$ for $q \neq 1 \in \mathbb{C}^\times$ and $z = \prod_{i=1}^{n}(x_iy_i - y_ix_i)$.

- (E. Backelin 2011) Each $R$–algebra endomorphism of $(A_n^{(t)})_z$ is an automorphism.
Endomorphisms of Quantized Weyl Algebras

Let $R = \mathbb{C}[t, t^{-1}]$ and $A_n^{(t)}$ be the $R-$algebra generated by $x_1, y_1, \ldots, x_n, y_n$ subject to the relations: $x_i y_i - ty_i x_i = 1$, $x_i y_j = y_j x_i$, $x_i x_j = x_j x_i$, and $y_i y_j = y_j y_i$. Set $A_q^n = A_n^{(t)}/(t - q)$ for $q \neq 1 \in \mathbb{C}^\times$ and $z = \prod_{i=1}^n (x_i y_i - y_i x_i)$.

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- **(E. Backelin 2011)** The algebra $A_q^n$ has injective but not surjective endomorphisms; and $A_q^n$ has endomorphisms which are neither injective nor surjective.
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- **(E. Backelin 2011)** When $q$ is a root of unity, the localization $(A_n^q)_z$ is an Azumaya algebra over its center.
More Algebras with the Dixmier Property

For $0 \neq a(h) \in \mathbb{K}[h^{\pm 1}]$, let $A(a(h), q)$ be the $\mathbb{k}$-algebra generated by $x, y, h^{\pm 1}$ subject to the relations

$$xh = qhx, \quad yh = q^{-1}hy, \quad xy = a(qh), \quad yx = a(h).$$

(S. Launois & A. Kitchin 2014) Assume that $a$ is not a monomial. Each $\mathbb{k}$-algebra monomorphism of $A(a(h), q)$ is an automorphism. If $A(a(h), q)$ is simple, then each $\mathbb{k}$-algebra endomorphism of $A(a(h), q)$ is an algebra automorphism.
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▶ (S. Launois & A. Kitchin 2015) When $q$ is not a root of unity, each $\mathbb{k}$-algebra endomorphism of $A(n, d, q) = \bigotimes_{i=1}^{n} A(h^d - 1, q)$ for $d \in \mathbb{N}$. 
More Algebras with the Dixmier Property

- **(Tang 2015)** The localized down-up algebra: $A_{r,s}(\mathbb{k})_S$ with $r, s$ being independent in $\mathbb{K}^\times$. Here $A_{r,s}(\mathbb{k})$ is a $\mathbb{k}$-algebra generated by $u, d$ subject to: $u^2d - (r + s)udu + rsdu^2 = 0$, $d^2u - (r + s)dud + rsud^2 = 0$. We denote by $S$ the set $\{(ud - rdu)(ud - sdu)^i | i \geq 0\}$. 
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- **(Tang 2017)** The algebra: $\bigotimes_{i=1}^{n} A\left(\frac{h_i-1}{q_i-1}, q_i\right)$ with $q_1, \cdots, q_n$ being independent in $k^\times$. 
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- (Tang 2017) The algebra: $\mathbb{k}_p[s^\pm 1, t^\pm 1] \bigotimes A(\frac{h - 1}{q - 1}, q)$ with $p, q \in \mathbb{k}^\times$ being non-root of unity.
Some Criteria for Bijectivity

There have been some criteria in the literature for singling algebra automorphisms out of algebra endomorphisms:

- (V.A. Artamonov & R. Wisbauer 2001) Assume that $n \geq 3$ and $q_{ij}$ are independent in $\mathbb{k}^\times$. Let $\varphi$ be a $\mathbb{k}$-algebra endomorphism of $\mathbb{k}Q[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n]$. If there exist $i, j, k$ such that $\varphi(x_i), \varphi(x_j), \varphi(x_k) \neq 0$, then $\varphi$ is an algebra automorphism.
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- **(N. Lauritzen & J.F. Thomsen 2017)** Assume that $\text{char} \mathbb{k} = 0$. Each birational $\mathbb{k}$-algebra endomorphism of $A_n(\mathbb{k})$ is an automorphism.
Benkart’s Multi-paramter Weyl Algebras

(G. Benkart 2013) For any \( r, s \in \mathbb{k}^\times \), let \( A_{r,s}(\mathbb{k}) \) be the \( \mathbb{k} \)-algebra generated by \( \rho^{\pm 1}, \sigma^{\pm 1} \) and \( x, y \) subject to the following relations: 
\[
\rho\sigma = \sigma\rho, \quad \rho x = rx\rho, \quad \rho y = r^{-1}\rho y, \quad \sigma x = sx\sigma, \quad \sigma y = s^{-1}y\sigma, \quad y\chi = \frac{r^2\rho^2 - s^2\sigma^2}{r^2 - s^2}, \quad xy = \frac{\rho^2 - \sigma^2}{r^2 - s^2}.
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Set \( A_{r,s}(n) = \bigotimes_{i=1}^n A_{r_i,s_i}(\mathbb{k}) \).
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For any $r, s \in k^\times$ and $b(\rho, \sigma) \in k[\sigma^{\pm 1}, \rho^{\pm 1}]$, let $B(b, r, s)$ be the $k$-algebra generated by $\rho^{\pm 1}, \sigma^{\pm 1}$ and $x, y$ subject to the following relations $\rho \sigma = \sigma \rho$ and

$$\rho x = rx \rho, \quad \rho y = r^{-1}y \rho;$$

$$\sigma x = s x \sigma, \quad \sigma y = s^{-1}y \sigma;$$

$$xy = b(r \rho, s \sigma), \quad yx = b(\rho, \sigma).$$
Endomorphisms for Benkart’s Algebras

Benkart’s algebras have the Dixmier Property.

- If \( b(\rho, \sigma) \) is a monomial and \( r^i s^j = 1 \) implies that \( i = j = 0 \), then each endomorphism of \( B(b, r, s) \) is an automorphism.
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- If $b(\rho, \sigma) \in \mathbb{k}[\rho^{\pm1}, \sigma^{\pm1}]$ is not a monomial with no monomials $m_1, m_2 \in \mathbb{k}[\rho^{\pm1}, \sigma^{\pm1}]$ such that $b(m_1, m_2) = b(rm_1, sm_2) = 0$ and $r^i s^j = 1$ implies that $i = j = 0$, then each endomorphism of $B(b, r, s)$ is an automorphism.
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- An Example: One can set \(b(\rho, \sigma) = \frac{\rho^d - \sigma^d}{r - s}\) where \(d \in \mathbb{Z}\) and \(r^i s^j = 1\) implies that \(i = j = 0\).
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- **A Counter-Example:** Set \( b(\rho, \sigma) = (\rho - \sigma)(s\rho - t\sigma) \). Then \( B(b, r, s) \) has non-injective endomorphisms.
A Tensor Product Phenomenon

The following algebras have the Dixmier Property.

- The algebra: $\bigotimes_{i=1}^{n} k[h^{\pm 1}](a_i(h), q_i)$ where $q_i$ is a non-root of unity and $a_i$ is a non-monomial with no $c_i \in k^\times$ such that $a_i(c_i) = a_i(q_i c_i) = 0$ for $i = 1, \ldots, n$. 
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- The algebra: $[\bigotimes_{i=1}^{n} A(a_i(h), q_i)] \otimes k_Q[t_1^{\pm 1}, \cdots, t_m^{\pm 1}]$ where $q_i$ is not a root of unity and $a_i(h)$ is not a monomial with no $c_i \in k^\times$ such that $a_i(c_i) = a_i(q_i c_i) = 0$ for $i = 1, \cdots, n$ and $k_Q[t_1^{\pm 1}, \cdots, t_m^{\pm 1}]$ is a simple algebra.
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- The algebra: $\bigotimes_{i=1}^{n} \mathbb{k}[h^{\pm 1}](a_i(h), q_i)$ where $q_i$ is a not a root of unity and $a_i$ is a non-monomial with no $c_i \in \mathbb{k}^\times$ such that $a_i(c_i) = a_i(q_i c_i) = 0$ for $i = 1, \ldots, n$.

- The algebra: $\left(\bigotimes_{i=1}^{n} A(a_i(h), q_i)\right) \otimes \mathbb{k}_Q[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ where $q_i$ is not a root of unity and $a_i(h)$ is not a monomial with no $c_i \in \mathbb{k}^\times$ such that $a_i(c_i) = a_i(q_i c_i) = 0$ for $i = 1, \ldots, n$ and $\mathbb{k}_Q[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ is a simple algebra.

- The algebra: $\bigotimes_{i=1}^{n} B(b_i(\rho, \sigma), r_i, s_i)$ where $r_1, s_1, \ldots, r_n, s_n$ are independent in $\mathbb{k}^\times$ and $b_i$ is not a monomial with no monomials $m_1^i, m_2^i \in \mathbb{k}[\rho^{\pm 1}, \sigma^{\pm 1}]$ such that $b_i(m_1^i, m_2^i) = 0$ and $b_i(r_i m_1^i, s_i m_2^i) = 0$ for $i = 1, \ldots, n$. 
What Is the Underlying Idea?

- The algebra has a lot of units.
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- There is a strong corelation between the units and generators.
- There is $q$-commuting relation between the generators and units.
What Might Be Next?

▶ Understand the nature of these examples in an intrinsic way.
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- Understand the nature of these examples in an intrinsic way.
- Produce new families of examples in a systematic way.
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- Produce new families of examples in a systematic way.
- Establish a general result on the tensor product behavior.
- Research on the root of unity case.
THANK YOU!