Reflections at infinity of time changed RBMs on a domain with Liouville branches

Zhen-Qing Chen and Masatoshi Fukushima

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Abstract

Let Z be the transient reflecting Brownian motion on the closure of an unbounded domain $D \subset \mathbb{R}^d$ with N number of Liouville branches. We consider a diffusion X on $\overline{D}$ having finite lifetime obtained from Z by a time change. We show that X admits only a finite number of possible symmetric conservative diffusion extensions Y beyond its lifetime characterized by possible partitions of the collection of N ends and we identify the family of the extended Dirichlet spaces of all Y (which are independent of time change used) as subspaces of the space $\mathrm{BL}(D)$ spanned by the extended Sobolev space $H^1_e(D)$ and the approaching probabilities of Z to the ends of Liouville branches.

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1 Introduction

The boundary problem of a Markov process X concerns all possible Markovian prolongations Y of X beyond its life time ζ whenever ζ is finite. For a conservative but transient Markov process, we can still consider its extension, after a time change to speed up the original process. Let $Z = (Z_t, \mathbf{Q}_z)$ be a conservative right process on a locally compact separable metric space E and ∂ be the point at infinity of E. Suppose Z is transient relative to an excessive measure m: for the 0-order resolvent R of Z, $Rf(z) < \infty$, m-a.e. for some strictly positive function (or equivalently, for any non-negative function) $f \in L^1(E; m)$. Then

$$\mathbf{Q}_z\left(\lim_{t\to\infty} Z_t \;=\; \partial\right) = 1 \quad \text{for q.e. } x \in E,$$

if Rf is lower semicontinuous for any non-negative Borel function f ([FTa]). The last condition is not needed when X is m-symmetric ([CF2]). Here, 'q.e.' means 'except for an m-polar set.

Take any strictly positive bounded function $f \in L^1(E; m)$. Then $A_t = \int_0^t f(Z_s)ds,\ t \geq 0$ is a strictly increasing PCAF of Z with $\mathbf{E}^{\mathbf{Q}}_z[A_\infty] = Rf(x) < \infty$ for q.e. $x \in E$.

The time changed process $X = (X_t, \zeta, \mathbf{P}_x)$ of Z by means of A is defined by

$$X_t = Z_{\tau_t},\ t \geq 0, \quad \tau = A^{-1}, \quad \zeta = A_\infty, \quad \mathbf{P}_x = \mathbf{Q}_x,\ x \in E.$$

Since $\mathbf{P}_x(\zeta < \infty,\ \lim_{t\to\zeta} X_t = \partial) = \mathbf{P}_x(\zeta < \infty) = 1$ for q.e. $x \in E$, the boundary problem for X at ∂ makes perfect sense. We denote X also by X^f to indicate its depedence on the function f. For different choices of f, X^f have a common geometric structure related each other only by time changes. Thus a study of the boundary problem for $X = X^f$ is a good way to have a closer look at the geometric behaviors of a conservative transient process Z around ∂. A strong Markov process $\widehat{X}$ on a topological space $\widehat{E}$ is said to be an extension of X on E if (i) E can be embedded homomorphically as a dense open subset of $\widehat{E}$, (ii) the part process of $\widehat{X}$ killed upon leaving E has the same distribution as X, and (iii) $\widehat{X}$ has no sojourn on $\widehat{E} \setminus E$; that is, $\widehat{X}$ spends zero Lebesgue amount of time on $\widehat{E} \setminus E$.

In this paper, we consider as Z the transient reflecting Brownian motion on the closure of an unbounded domain $D \subset \mathbb{R}^d$ with N number of Liouville branches. Our main aim is to prove in Section 5 that a time changed process X^f of Z admits essentially only a finite number of possible symmetric conservative diffusion extensions Y beyond its lifetime. They are characterized by the partition of the collection of N ends. Moreover, all the corresponding extended Dirichlet spaces $(\mathcal{E}^Y, \mathcal{F}_e^Y)$ are identified in terms of the extended Dirichlet space of Z and the approaching probabilities of Z to the ends of Liouville branches in an extremely simple manner. These extended Dirichlet spaces are independent of the choice of f. The L^2-generator of each extension Y is also characterized in Section 6 by means of zero flux conditions at the ends of branches. Each extension Y may be called a *many point reflection at infinity of* X^f generalizing the notion of the one point reflection in [CF3] in the present specific context. The characterization of possible extensions also uses quasi-homeomorphism and equivalence between Dirichlet forms. See the Appendix, Section 8, of this paper for details.

In fact, our results are valid for a time changed process X^μ of Z by means of a more general finite smooth measure μ on $\overline{D}$ than $f(x)dx$. This is demonstrated in Section 7.

Although we formulate our results for the reflecting Brownian motion on an unbounded domain in $\mathbb{R}^d$ with several Liouville branches, all of them except for Theorem 6.1 remain valid without any essential change for the reflecting diffusion process associated with the uniformly elliptic second order self-adjoint partial differential operator with measurable coefficients that was constructed in [C] and [FTo]. Since we need strong Feller property of the reflecting diffusion process, we assume the underlying unbounded domain is Lipschitz in the sense of [FTo]; see Remark 5.3. Thus we are effectively investigating common path behaviours at infinity holding for such a general family of diffusion processes.

Acknowledgement This paper is a direct outgrowth of our paper [CF1] and Chapter 7 of our book [CF2]. In relation to them, we had very valuable discussions with Krztsztof Burdzy on boundaries of transient reflecting Brownian motions. We would like to express our sincere thanks to him.

2 Preliminaries

For a domain $D \subset \mathbb{R}^d$, let us consider the spaces

$$\mathrm{BL}(D) = \{u \in L^2_{\mathrm{loc}}(D) : \; |\nabla u| \in L^2(D)\}, \quad H^1(D) = \mathrm{BL}(D) \cap L^2(D), \tag{2.1}$$

The space $\mathrm{BL}(D)$ called the *Beppo Levi space* was introduced by J. Deny and J. L. Lions [DL] as the space of Schwartz distributions whose first order derivatives are in $L^2(D)$, which can be identified with the function space described above. The quotient space $\dot{\mathrm{BL}}(D)$ of $\mathrm{BL}(D)$ by the space of all constant functions on D is a real Hilbert space with inner product

$$\mathbf{D}(u, v) = \int_D \nabla u(x) \cdot \nabla v(x) dx.$$

See §1.1 of V.G, Maz'ja [M] for proofs of the above stated facts, where the space $\mathrm{BL}(D)$ is denoted by $L^1_2(D)$ and studied in a more general context of the spaces $L^\ell_p(D), \; \ell \geq 1, \; p \geq 1$.

Define

$$(\mathcal{E}, \mathcal{F}) = (\frac{1}{2}\mathbf{D}, H^1(D)), \tag{2.2}$$

which is a Dirichlet form on $L^2(D)$. The collection of those domains $D \subset \mathbb{R}^d$ for which (2.2) is regular on $L^2(\overline{D})$ will be denoted by $\mathcal{D}$. It is known that $D \in \mathcal{D}$ if D is either a domain of continuous boundary or an extendable domain relative to $H^1(D)$ (cf. [CF1, p 866]). For $D \in \mathcal{D}$, the diffusion process Z on $\overline{D}$ associated with (2.2) is by definition the *reflecting Brownian motion* (RBM in abbreviation) which is known to be conservative. Furthermore, the space $\mathrm{BL}(D)$ is nothing but the *reflected Dirichlet space* of the form (2.2) ([CF2, §6.5]). The Dirichlet form (2.2) is either recurrent or transient and the latter case occurs only when $d \geq 3$ and D is unbounded. For $D_1, \; D_2 \in \mathcal{D}$ with $D_1 \subset D_2$, (2.2) is transient for D_2 whenever so it is for the smaller domain D_1. If (2.2) is recurrent, then we have the identity

$$\mathrm{BL}(D) = H^1_e(D)$$

where $H^1_e(D)$ denotes the *extended Dirichlet space* of the form (2.2) or of the RBM Z ([CF2]) that may be called the *extended Sobolev space of order* 1.

Suppose $D \in \mathcal{D}$ and (2.2) is transient. Then $H^1_e(D)$ is a Hilbert space with inner product $\frac{1}{2}\mathbf{D}$ possessing the space $C^\infty_c(\overline{D})$ as its core. $H^1_e(D)$ can be regarded as a proper closed subspace of the quotient space $\dot{\mathrm{BL}}(D)$. Define

$$\mathbb{H}^*(D) = \{u \in \mathrm{BL}(D) : \; \mathbf{D}(u, v) = 0 \text{ for every } v \in H^1_e(D)\}. \tag{2.3}$$

Any function $u \in \mathrm{BL}(D)$ admits a unique decomposition

$$u = u_0 + h, \quad u_0 \in H^1_e(D), \quad h \in \mathbb{H}^*(D). \tag{2.4}$$

Any function $h \in \mathbb{H}^*(D)$ is of finite Dirichlet integral and harmonic on D. Furthermore, the quasi-continuous version of h is harmonic on $\overline{D}$ with respect to the RBM Z.

In what follows, we restrict our attention to the case where the form (2.2) is transient and so we assume that $d \geq 3$ and $D \in \mathcal{D}$ is unbounded.

Definition 2.1 A domain $D \in \mathcal{D}$ is called a *Liouville domain* if the form (2.2) is transient and $\dim \mathbb{H}^*(D) = 1$.

A domain $D \in \mathcal{D}$ is a Liouville domain if and only if the form (2.2) is transient and any function $u \in \mathrm{BL}(D)$ admits a unique decomposition

$$u = u_0 + c, \qquad \text{where } u_0 \in H_e^1(D) \text{ and } c \in \mathbb{R}. \tag{2.5}$$

We shall denote by $c(u)$ the constant c in (2.5) uniquely associated with $u \in \mathrm{BL}(D)$ for a Liouville domain D.

A trivial but important example of a Liouville domain is $\mathbb{R}^d$ with $d \geq 3$, see M. Brelot [B]. Another important example of a Liouville domain is provided by an unbounded uniform domain that has been shown by P. Jones [1] (see also [HK]) to be an extendable domain relative to the space $\mathrm{BL}(D)$.

A domain $D \subset \mathbb{R}^d$ is called a *uniform domain* if there exists $C > 0$ such that for every $x, y \in D$, there is a rectifiable curve γ in D connecting x and y with $\mathrm{length}(\gamma) \leq C|x - y|$, and moreover

$$\min\{|x - z|,\ |z - y|\} \leq C\mathrm{dist}(z, D^c) \quad \text{for every } z \in \gamma.$$

It was proved in Theorem 3.5 of [CF1] that any unbounded uniform domain is a Liouville domain in the sense of Definition 2.1. An unbounded uniform domain is such a domain that is broaden toward the infinity. The truncated infnite cone $C_{A,a} = \{(r, \omega) : r > a,\ \omega \in A\} \subset \mathbb{R}^d$ for any connected open set $A \subset S^{d-1}$ with Lipschitz boundary is an unbounded uniform domain. To the contrary, (2.2) is recurrent for the cylinder $D = \{(x, x') \in \mathbb{R}^d : x \in \mathbb{R},\ |x'| < 1\}$. See R. G. Pinsky [P] for transience criteria for other types of domains. On the other hand, it has been shown in [CF2, Proposition 7.8.5] that (2.2) is transient but $\dim(\mathbb{H}^*(D)) = 2$ for a special domain

$$D = B_1(\emptyset) \,\cup\, \Big\{(x, x') \in \mathbb{R}^d : x \in \mathbb{R},\ |x| > |x'|\Big\}, \quad d \geq 3. \tag{2.6}$$

with two symmetric cone branches. Here $B_r(\emptyset)$, $r > 0$, denotes an open ball with radius r centered at the origin. This domain is not uniform because of a presence of a bottleneck. We shall consider much more general domains than this. But before proceeding to the main setting of the present paper, we state a simple property of Liouville domains:

Proposition 2.2 *For $D_1, D_2 \in \mathcal{D}$ with $D_1 \subset D_2$, Suppose D_1 is a Liouville domain and $D_2 \setminus D_1$ is bounded. Then D_2 is a Liouville domain. Furthermore, for any $u \in \mathrm{BL}(D_2)$, it holds that $c(u) = c(u\big|_{D_1})$.*

Proof. The proof is similar to that of [CF1, Proposition 3.6]. Note that (2.2) is transient for D_2. We show that any $u \in \mathrm{BL}(D_2)$ admits a decomposition (2.5) with $u_0 \in H_e^1(D_2)$ and $c = c(u\big|_{D_1})$. Due to the normal contraction property of $\mathrm{BL}(D_2)$ and the transience of $(\frac{1}{2}\mathbf{D}, H^1(D))$, we may assume that u is bounded on D_2. By noting that $u\big|_{D_1} \in \mathrm{BL}(D_1)$ and D_1 is a Liouville domain, we let $c = c(u\big|_{D_1})$ and $u_0(x) = u(x) - c,\ x \in D_2$. Then $u_0\big|_{D_1} \in H_e^1(D_1)$. To prove that $u_0 \in H_e^1(D_2)$, choose an open ball $B_r(\emptyset) \supset \overline{D_2 \setminus D_1}$ and a function $w \in C_c^\infty(\mathbb{R}^d)$ with $w(x) = 1,\ x \in B_r(\emptyset)$. Clearly $wu_0 \in H_e^1(D_2)$.

It remains to show $(1-w)u_0 \in H^1_e(D_2)$. Take $g_n \in H^1(D_1)$ converging to u_0 a.e. on D_1 and in the Dirichlet norm on D_1. By truncation, we may assume that g_n is uniformly bounded on D_1. Then

$$\begin{aligned}&\int_{D_2} |\nabla[(1-w(x))g_n(x)]|^2 dx \\ \leq\ & 2\sup_{x\in\mathbb{R}^d}(1-w(x))^2\int_{D_1}|\nabla g_n(x)|^2dx + 2\sup_{x\in D_1}|g_n(x)|^2\int_{\mathbb{R}^d}|\nabla w(x)|^2dx,\end{aligned}$$

which is uniformly bounded in n, yielding by the Banach-Saks theorem that $(1-w)u_0 \in H^1_e(D_2)$. □

We shall work under the regularity condition

(A.1) D is of a Lipschitz boundary ∂D,

which means the following: there are constants $M > 0$, $\delta > 0$ and a locally finite covering $\{U_j\}_{j\in J}$ of ∂D such that, for each $j \in J$, $D \cap U_j$ is a upper part of a graph of a Lipschitz continuous function under an appropriate coodinate system with the Lipschitz constant bounded by M and $\partial D \subset \bigcup_{j\in J}\{x \in U_j : \mathrm{dist}(x, \partial U_j) > \delta\}$. According to [FTo], there exists then a conservative diffusion process $Z = (Z_t, \mathbf{Q}_x)$ on $\overline{D}$ associated with the regular Dirichlet form (2.2) on $L^2(\overline{D})$ whose resolvent $\{G^Z_\alpha; \alpha > 0\}$ has the strong Feller property in the sense that

$$G^Z_\alpha(bL^1(D)) \subset bC(\overline{D}). \tag{2.7}$$

Z is a precise version of the RBM on $\overline{D}$. In particular, the transition probability of Z is absolutely continuous with respect to the Lebesgue measure.

Under the condition **(A.1)** and the transience assumption on (2.2), the RBM $Z = (Z_t, \mathbf{Q}_x)$ on $\overline{D}$ enjoys the properties that

$$\mathbf{Q}_x\left(\lim_{t\to\infty} Z_t = \partial\right) = 1 \quad \text{for every } x \in \overline{D}, \tag{2.8}$$

where ∂ denotes the point at infinity of $\mathbb{R}^d$, and

$$\mathbf{Q}_x\left(\lim_{t\to\infty} u(Z_t) = 0\right) = 1 \quad \text{for every } x \in \overline{D}, \tag{2.9}$$

for any $u \in H^1_e(D)$, u being taken to be quasi-continuous. See [CF2, §7.8, (4^o)].

In the rest of this paper, we fix a domain D of $\mathbb{R}^d$, $d \geq 3$, satisfying **(A.1)** and

(A.2) $\qquad D \setminus \overline{B_r(\emptyset)} = \bigcup_{j=1}^N C_j$

for some $r > 0$ and an integer N, where $C_1, \cdots, C_N$ are Liouville domains with Lipschitz boundaries such that $\overline{C}_1, \cdots, \overline{C}_N$ are mutually disjoint. D may be called a *Lipschitz domain with N number of Liouville branches*.

Let ∂_j be the point at infinity of the unbounded closed set $\overline{C}_j$ for each $1 \leq j \leq N$. Denote the N-points set $\{\partial_1, \cdots, \partial_N\}$ by F and put $\overline{D}^* = \overline{D} \cup F$. $\overline{D}^*$ can be made to be a compact Hausdorff space if we employ as a local base of neighborhoods of each point $\partial_j \in F$ the neighborhoods of ∂_j in $\overline{C}_j \cup \{\partial_j\}$. $\overline{D}^*$ may be called the *N-points compactification of* $\overline{D}$.

Obviously the Dirichlet form (2.2) is transient for D. We shall verify in Section 4 that $\dim(\mathbb{H}^*(D)) = N$. Here we note the following implication of Proposition 2.2; if a domain D is of the type **(A.2)** for different $0 < r_1 < r_2$, and if D is a domain with N number of Liouville branches relative to r_2, then so it is relative to r_1.

3 Approaching probabilities of RBM Z and limits of BL-functions along Z_t

For each $1 \le j \le N$, define the approaching probability of the RBM $Z = (Z_t, \mathbf{Q}_x)$ to ∂_j by

$$\varphi_j(x) = \mathbf{Q}_x\left(\lim_{t\to\infty} Z_t = \partial_j\right), \quad x \in \overline{D}. \tag{3.1}$$

Proposition 3.1 *It holds that*

$$\sum_{j=1}^N \varphi_j(x) = 1 \quad \textit{for every} \quad x \in \overline{D}, \tag{3.2}$$

and, for each $1 \le j \le N$,

$$\varphi_j(x) > 0 \quad \textit{for every} \quad x \in \overline{D}. \tag{3.3}$$

Proof. (3.2) is a consequence of (2.8). As φ_j is a non-negative harmonic function on the domain D, it is either identically zero on D or strictly positive on D. Since $\varphi_j(x) = Q_t\varphi_j(x),\ x \in \overline{D}$, where Q_t is the transition semigroup of the RBM Z, which has a strictly positive transition density kernel, the above dichotomy extends from D to $\overline{D}$.

Suppose $\varphi_j(x) \equiv 0$ on $\overline{D}$. Then by (2.8)

$$\mathbf{Q}_x\left(\sigma_{\partial B_r(\emptyset)} < \infty\right) = 1, \quad \text{for any } x \in \overline{C}_j \setminus B_{r+1}(\emptyset). \tag{3.4}$$

Let $Z^j = (Z^j_t, \mathbf{Q}^j_x),\ x \in \overline{C}_j$, be the RBM on $\overline{C}_j$, which is transient as C_j is a Liouville domain. Since Z and Z^j share the common part process on $\overline{C}_j \setminus \partial B_r(\emptyset)$, (3.4) remains valid if $\mathbf{Q}_x$ is replaced by $\mathbf{Q}^j_x$. By the Markov property of Z^j and the consevativeness of Z^j, we have

$$\mathbf{Q}^j_x\left(\sigma_{\partial B_r(\emptyset)} \circ \theta_\ell < \infty \text{ for every integer } \ell\right) = 1,$$

for any $x \in \overline{C}_j \setminus B_{r+1}(\emptyset)$. This however contradicts to the transience property (2.8) of Z^j. □

Proposition 3.2 *For any $u \in \mathrm{BL}(D)$, let $c_j(u) = c(u|_{C_j})$ for $1 \le j \le N$. Then*

$$\mathbf{Q}_x\left(Z_{\infty-} = \partial_j,\ \lim_{t\to\infty} u(Z_t) = c_j(u)\right) = \mathbf{Q}_x\left(Z_{\infty-} = \partial_j\right),\ x \in \overline{D}, \quad 1 \le j \le N. \tag{3.5}$$

If $c_j(u) = 0$ for every $1 \le j \le N$, then $u \in H^1_e(D)$.

Proof. We prove (3.5) for $j = 1$. Let $r > 0$ be the radius in **(A.2)** and $Z^1 = (Z^1_t, \mathbf{Q}^1_x)$ be the RBM on $\overline{C_1}$. The hitting times of $B_r(\emptyset)$ and $B_R(\emptyset)$ for $R > r$ will be denoted by σ_r and σ_R, respectively. Observe that Z and Z^1 share in common the part process on $\overline{C}_1 \setminus \partial B_r(\emptyset)$. Since C_1 is a Liouville domain, we see from (2.5) and (2.9) that

$$\mathbf{Q}^1_x \left(\lim_{t\to\infty} u(Z^1_t) = c_1(u) \right) = 1 \quad \text{for every } x \in \overline{C}_1.$$

For $R > r$, we consider the event

$$\Gamma_R = \{ Z_{\sigma_R} \in \overline{C}_1,\ \sigma_r \circ \theta_{\sigma_R} = \infty \}.$$

Then $\Gamma_R \cap \{Z_{\infty-} = \partial\}$ increses as R increases and $\{Z_{\infty-} = \partial_1\} = \bigcup_{R>r} [\Gamma_R \cap \{Z_{\infty-} = \partial\}]$. In view of (2.8), we have for $x \in \overline{D}$,

$$\begin{aligned}
&\mathbf{Q}_x(Z_{\infty-} = \partial_1) = \lim_{R\to\infty} \mathbf{Q}_x(\Gamma_R \cap \{Z_{\infty-} = \partial\}) = \lim_{R\to\infty} \mathbf{Q}_x(\Gamma_R) \\
&= \lim_{R\to\infty} \mathbf{E}^{\mathbf{Q}_x} \left[\mathbf{Q}_{Z_{\sigma_R}}(\sigma_r = \infty); Z_{\sigma_R} \in \overline{C}_1 \right] \\
&= \lim_{R\to\infty} \mathbf{E}^{\mathbf{Q}_x} \left[\mathbf{Q}^1_{Z_{\sigma_R}}(\sigma_r = \infty); Z_{\sigma_R} \in \overline{C}_1 \right] \\
&= \lim_{R\to\infty} \mathbf{E}^{\mathbf{Q}_x} \left[\mathbf{Q}^1_{Z_{\sigma_R}}(\sigma_r = \infty,\ \lim_{t\to\infty} u(Z^1_t) = c_1(u)); Z_{\sigma_R} \in \overline{C}_1 \right].
\end{aligned}$$

In exactly the same way, we can see that $\mathbf{Q}_x(Z_{\infty-} = \partial_1,\ \lim_{t\to\infty} u(Z_t) = c_1(u))$ equals the last expression in the above display, proving (3.5) for $j = 1$.

Suppose $u \in \mathrm{BL}(D)$ satisfies $c_j(u) = 0$ for every $1 \le j \le N$. Then $u\big|_{C_j} \in H^1_e(C_j)$ for every $1 \le j \le N$ and we can conclude as the proof of Proposition 2.2 that $u \in H^1_e(D)$. $\square$

We remark that, in view of Proposition 2.2 the constants $c_j(u)$, $1 \le j \le N$, in the above proposition are independent of the choice of the radius r in **(A.2)**.

4 Reflecting extension X^* of a time changed RBM X and dimension of $\mathbb{H}^*(D)$

Fix a strictly positive bounded integrable function f on $\overline{D}$ and define

$$A_t = \int_0^t f(Z_s)ds, \quad t \ge 0. \tag{4.1}$$

A_t is a positive continuous additive functional (PCAF) of the RBM $Z = (Z_t, \mathbf{Q}_x)$ on $\overline{D}$ in the strict sense with full support. Notice that

$$\mathbf{Q}_x(A_\infty < \infty) = 1 \quad \text{for every } x \in \overline{D}, \tag{4.2}$$

because $\mathbf{E}^{Q_x}[A_\infty] = G^Z_{0+} f(x) < \infty$ for a.e. $x \in \overline{D}$ due to the transience of Z ([CF2, Proposition 2.1.3]) and hence

$$\mathbf{Q}_x(A_\infty = \infty) = \mathbf{Q}_x(A_\infty \circ \theta_t = \infty) = \mathbf{E}^{\mathbf{Q}_x}[\mathbf{Q}_{Z_t}(A_\infty = \infty)] = 0 \quad \text{for every } x \in \overline{D}, \tag{4.3}$$

on account of the stated absolute continuity of the transition function of Z.

Let $X = (X_t, \zeta, \mathbf{P}_x)$ be the time changed process of Z by means of A:

$$X_t = Z_{\tau_t}, \quad \tau = A^{-1}, \quad \zeta = A_\infty, \quad \mathbf{P}_x = \mathbf{Q}_x \text{ for } x \in \overline{D}.$$

The Markov process $X = X^f$ is a diffusion process on $\overline{D}$ symmetric with respect to the measure $m(dx) = f(x)dx$ and the Dirichlet form $(\mathcal{E}^X, \mathcal{F}^X)$ of X on $L^2(\overline{D}; m)$ is given by

$$\mathcal{E}^X = \frac{1}{2}\mathbf{D}, \qquad \mathcal{F}^X = H^1_e(D) \cap L^2(\overline{D}; m). \tag{4.4}$$

Since the extended Dirichlet space and the reflected Dirichlet space are invariant under a time change by a fully supported PCAF ([CF2, Cor.5.2.12, Prop.6.4.6]), these spaces for $\mathcal{E}^X$ are still given by $H^1_e(D)$ and $\mathrm{BL}(D)$, respectively. But the life time ζ of X is finite $\mathbf{P}_x$-a.s. for every $x \in \overline{D}$ in view of (4.2) so that we may consider the problem of extending X after ζ, particularly, from $\overline{D}$ to its N-points compactification $\overline{D}^* = \overline{D} \cup F$ with $F = \{\partial_1, \cdots, \partial_N\}$.

We can rewrite the approaching probability φ_j of Z to ∂_j defined by (3.1) as

$$\varphi_j(x) = \mathbf{P}_x\left(\zeta < \infty, \quad X_{\zeta-} = \partial_j\right), \quad x \in \overline{D}, \quad 1 \le j \le N, \tag{4.5}$$

in terms of the time changed process X. The measure $m(dx) = f(x)dx$ is extended from $\overline{D}$ to $\overline{D}^*$ by setting $m(F) = 0$. An m-symmetric conservative diffusion process X^* on $\overline{D}^*$ will be called a *symmetric conservative diffusion extension* of X if its part process on $\overline{D}$ being killed upon hitting F is equivalent in law with X. The resolvent of X is denoted by $\{G^X_\alpha, \ \alpha > 0\}$.

Proposition 4.1 *There exists a unique symmetric conservative diffusion extension X^* of X from $\overline{D}$ to $\overline{D}^* = \overline{D} \cup F$. The process X^* is recurrent. Let $(\mathcal{E}^*, \mathcal{F}^*)$ and $\mathcal{F}^*_e$ be the Dirichlet form of X^* on $L^2(\overline{D}^*, m)$ $(= L^2(D; m))$ and its extended Dirichlet space, respectively. Then*

$$\mathcal{F}^*_e = H^1_e(D) \ \oplus \ \left\{ \sum_{j=1}^N c_j \varphi_j \ : \ c_j \in \mathbb{R} \right\} \ \subset \ \mathrm{BL}(D), \tag{4.6}$$

$$\mathcal{E}^*(u, v) = \frac{1}{2}\, \mathbf{D}(u, v), \qquad u, v \in \mathcal{F}^*_e. \tag{4.7}$$

Proof. We apply a general existence theorem of a many-point extension formulated in [CF2, Theorem 7.7.4] to the m-symmetric diffusion X on $\overline{D}$ and the N-points compactification $\overline{D}^* = \overline{D} \cup F$ of $\overline{D}$. We verify conditions **(M.1), (M.2), (M.3)** for X required in this theorem. $\psi_j(x) := \mathbf{P}_x(\zeta < \infty, X_{\zeta-} = \partial_j)$ is positive for every $x \in \overline{D}, 1 \le j \le N$, by (3.3) and (4.5), and so **(M.1)** is satisfied. Since $m(\overline{D}) = \int_{\overline{D}} f dx < \infty$, the m-integrability **(M.2)** of the function $u^{(j)}_\alpha(x) = \mathbf{E}_x\left[e^{-\alpha\zeta}; X_{\zeta-} = \partial_j\right], \ x \in \overline{D}$, is trivially fulfilled, $1 \le j \le N$. For any $1 \le j \le N$ and any compact set $V \subset \overline{D}$, $\inf_{x \in V} G^X_\alpha \psi_j(x)$ is positive because $G^X_\alpha \psi_j = G^X_{0+} u^{(j)}_\alpha = G^Z_{0+}(u^{(j)}_\alpha f)$ is lower semi-continuous on account of (2.7) and $u^{(j)}_\alpha$ is positive on $\overline{D}$. Accordingly, condition **(M.3)** is also satisfied.

Therefore there exists an m-symmetric diffusion extension X^* of X from $\overline{D}$ to $\overline{D}^*$ admitting no killing on F. We can then use a general characterization theorem [CF2, Theorem

7.7.3] to conclude that such an extension X^* of X is unique in law and its extended Dirichlet space $(\mathcal{F}_e^*, \mathcal{E}^*)$ is given by (4.6) and (4.7) as $\psi_j = \varphi_j,\ 1 \le j \le N$. In particular, (3.2) implies $1 \in \mathcal{F}_e^*,\ \mathcal{E}^*(1,1) = 0$, so that X^* is recurrent and consequently conservative. This also means the unique existence of an m-symmetric conservative diffusion extension X^* of X to $\overline{D}^*$. □

Theorem 4.2 $\dim(\mathbb{H}^*(D)) = N$ *and*

$$\mathbb{H}^*(D) = \left\{ \sum_{j=1}^N c_j\ \varphi_j\ :\ c_j \in \mathbb{R} \right\}. \tag{4.8}$$

The m-symmetric conservative diffusion extension X^* of the time changed RBM X constructed in Proposition 4.1 *is a reflecting extension of X in the sense that the extended Dirichlet space $(\mathcal{F}_e^*, \mathcal{E}^*)$ of X^* equals* $(\mathrm{BL}(D), \frac{1}{2}\mathbf{D})$ *the reflected Dirichlet space of X.*

Proof. By Proposition 4.1, $\{\varphi_j; 1 \le j \le N\} \subset \mathbb{H}^*(D) \subset \mathrm{BL}(D)$. For $1 \le j,k \le N$, let $c_k^{(j)} = c_k(\varphi_j)$. We claim that

$$c_k^{(j)} = \delta_{jk}, \qquad 1 \le k \le N. \tag{4.9}$$

Let τ_n be the exit time of Z from the set $\overline{D} \cap B_n(\emptyset),\ n \ge 1$. Then $\{\varphi_j(Z_{\tau_n})\}_{n \ge 1}$ is a bounded $\mathbf{Q}_x$-martingale and possesses an a.s. limit Φ with $\varphi_j(x) = \mathbf{E}^{Q_x}[\Phi]$. By (3.5),

$$\Phi = \sum_{k=1}^N c_k^{(j)} \mathbf{1}_{\{Z_{\infty-} = \partial_k\}}. \tag{4.10}$$

For $k \ne j$, put $F_{k,n} = C_k \cap \{|x| = n\}$. Then by (3.5) again

$$\begin{aligned} c_k^{(j)} \varphi_k(x) &= \lim_{n\to\infty} \mathbf{E}^{\mathbf{Q}_x} \left[\varphi_j(Z_{\tau_n}) \mathbf{1}_{\{Z_{\infty-} = \partial_k\}} \right] \le \limsup_{n\to\infty} \mathbf{E}^{\mathbf{Q}_x} \left[\varphi_j(Z_{\tau_n}) \mathbf{1}_{\{Z_{\tau_n} \in C_k\}} \right] \\ &= \limsup_{n\to\infty} \mathbf{E}^{\mathbf{Q}_x} \left[\mathbf{Q}_x \left(Z_{\infty-} \circ \theta_{\tau_n} = \partial_j,\ Z_{\tau_n} \in C_k \middle| \mathcal{F}_{\tau_n} \right) \right] \\ &\le \lim_{n\to\infty} \mathbf{Q}_x \left(Z_{\infty-} = \partial_j,\ \sigma_{F_{k,n}} < \infty \right) = 0, \end{aligned}$$

yielding $c_k^{(j)} = 0,\ k \ne j$. Taking $\mathbf{Q}_x$-expectation in (4.10) proves the claim (4.9).

Next for any $u \in \mathrm{BL}(D)$, let $u_0 = u - \sum_{j=1}^N c_j(u)\varphi_j$. Then $u_0 \in \mathrm{BL}(D)$ with $c_j^{u_0} = 0$ for every $1 \le j \le N$. So by Proposition 3.2, $u_0 \in H_e^1(D)$. This establishes (4.8). The linear independence of $\{\varphi_j; 1 \le j \le N\}$ follows from (4.9), while (4.6) and (4.8) yield the last assertion of the theorem. □

Remark 4.3 This theorem for the special domain (2.6) was stated in [CF2, Proposition 7.8.5]. We take this opportunity to mention that the proof of the latter given in the book [CF2] contained a flaw (on the third line of page 386), that should be corrected in the above way. □

5 Partitions Π of F and all possible symmetric diffusion extensions Y of a time changed RBM X

We continue to consider the N-points compactification $\overline{D}^* = \overline{D} \cup F$ of $\overline{D}$ introduced at the end of Section 1. A map Π from the boundary set $F = \{\partial_1, \cdots, \partial_N\}$ onto a finite set $\widehat{F} = \{\widehat{\partial}_1, \cdots, \widehat{\partial}_\ell\}$ with $\ell \leq N$ is called a *partition* of F. We let $\overline{D}^{\Pi,*} = \overline{D} \cup \widehat{F}$. We extend the map Π from F to $\overline{D}^*$ by setting $\Pi x = x,\ x \in \overline{D}$, and introduce the quotient topology on $\overline{D}^{\Pi,*}$ by Π. In other words, we employ $\mathcal{U}_\Pi = \{U \subset \overline{D}^{\Pi,*} : \Pi^{-1}(U) \text{ is an open subset of } \overline{D}^*\}$ as the family of open subsets of $\overline{D}^{\Pi,*}$. Then $\overline{D}^{\Pi,*}$ is a compact Hausdorff space and may be called an *ℓ-points compactification of $\overline{D}$ obtained from $\overline{D}^*$ by identifying the points in the set $\Pi^{-1}\widehat{\partial}_i \subset F$ as a single point $\widehat{\partial}_i$ for each $1 \leq i \leq \ell$.*

Given a partition Π of F, the approaching probabilities $\widehat{\varphi}_i$ of the RBM $Z = (Z_t, \mathbf{Q}_x)$ to $\widehat{\partial}_i \in \widehat{F}$ are defined by

$$\widehat{\varphi}_i(x) = \sum_{j \in \Pi^{-1}\widehat{\partial}_i} \varphi_j(x), \quad x \in \overline{D}, \quad 1 \leq i \leq \ell. \tag{5.1}$$

As in the preceding section, we define the time changed process $X = (X_t, \zeta, \mathbf{P}_x)$ on $\overline{D}$ of Z by means of a strictly positive bounded integrable function f on $\overline{D}$. The measure $m(dx) = f(x)dx$ is extended from $\overline{D}$ to $\overline{D}^{\Pi,*}$ by setting $m(\widehat{F}) = 0$. Just as in Proposition 4.1, there exists then a unique m-symmetric conservative diffusion extension $X^{\Pi,*}$ of X from $\overline{D}$ to $\overline{D}^{\Pi,*}$ and the Dirichlet form $(\mathcal{E}^{\Pi,*}, \mathcal{F}^{\Pi,*})$ of $X^{\Pi,*}$ on $L^2(\overline{D}^{\Pi,*}; m)$ $(= L^2(D; m))$ admits the extended Dirichlet space $(\mathcal{F}_e^{\Pi,*}, \mathcal{E}^{\Pi,*})$ expressed as

$$\mathcal{F}_e^{\Pi,*} = H_e^1(D) \ \oplus \ \left\{ \sum_{i=1}^{\ell} c_i \widehat{\varphi}_i \ : \ c_i \in \mathbb{R} \right\} \ \subset \ \mathrm{BL}(D), \tag{5.2}$$

$$\mathcal{E}^{\Pi,*}(u, v) = \frac{1}{2}\, \mathbf{D}(u, v), \qquad u, v \in \mathcal{F}_e^{\Pi,*}. \tag{5.3}$$

$X^{\Pi,*}$ is recurrent. $\mathcal{E}^{\Pi,*}$ is a quasi-regular Dirichlet form on $L^2(\overline{D}^{\Pi,*}; m)$.

We now prove that the family $\{\overline{X}^{\Pi,*} : \Pi \text{ is a partition of } F\}$ exhausts all possible m-symmetric conservative diffusion extensions of the time changed RBM X on $\overline{D}$.

Let E be a Lusin space into which $\overline{D}$ is homeomorpically embedded as an open subset. The measure $m(dx) = f(x)dx$ on $\overline{D}$ is extended to E by setting $m(E \setminus \overline{D}) = 0$. Let $Y = (Y_t, \mathbf{P}_x^Y)$ be an m-symmetric conservative diffusion process on E whose part process on $\overline{D}$ is identical in law with X. We denote by $(\mathcal{E}^Y, \mathcal{F}^Y)$ and $\mathcal{F}_e^Y$ the Dirichlet form of Y on $L^2(E; m)$ and its extended Dirichlet space. We call Y an *m-symmetric conservative diffusion extension* of X. The following theorem extends [CF1, Theorem 3.4].

Theorem 5.1 *There exists a partition Π of F such that, as Dirichlet forms on $L^2(\overline{D}; m)$,*

$$(\mathcal{E}^Y, \mathcal{F}^Y) = (\mathcal{E}^{\Pi,*}, \mathcal{F}^{\Pi,*}). \tag{5.4}$$

Y under $\mathbf{P}_{g\cdot m}$ and $X^{\Pi,*}$ under $\mathbf{P}^{\Pi,*}_{g\cdot m}$ have the same finite dimensional distribution for any non-negative $g \in L^2(\overline{D}; m)$. Furthermore, a quasi-homeomorphic image of Y is identical with $X^{\Pi,*}$ in the sense of Theorem 8.2 *in Appendix.*

Proof. As has been noted in the preceding section, the extended Dirichlet space $(\mathcal{F}^X_e, \mathcal{E}^X)$ and the reflected Dirichlet space $((\mathcal{F}^X)^{\rm ref}, (\mathcal{E}^X)^{\rm ref})$ of the Dirichlet form (4.4) are given by

$$\mathcal{F}^X_e = H^1_e(D), \qquad \mathcal{E}^X = \frac{1}{2}\mathbf{D}, \tag{5.5}$$

$$(\mathcal{F}^X)^{\rm ref} = {\rm BL}(D) = H^1_e(D) \oplus \mathbb{H}^*(D), \qquad (\mathcal{E}^X)^{\rm ref} = \frac{1}{2}\mathbf{D}, \tag{5.6}$$

respectively.

$\mathcal{E}^Y$ is a quasi-regular Dirichlet form on $L^2(E; m)$ and Y is properly associated with it by virtue of Z.-M. Ma and M. Röckner [MR]. By Chen-Ma-Röckner [CMR], $\mathcal{E}^Y$ is therefore quasi homeomorphic with a regular Dirichlet form. In particular, via a quasi homeomorphism j in [CF2, Theorems 3.1.13]), we can assume that E is a locally compact separable metric space, $\mathcal{E}^Y$ is a regular Dirichlet form on $L^2(E; m)$, Y is an associated Hunt process on E, and $\widetilde{F} := E \setminus \overline{D}$ is quasi-closed. Since Y is a conservative extension of the non-conservative process X, $\widetilde{F}$ must be non $\mathcal{E}^Y$-polar. Y can be also shown to be irreducible as in the proof of [CF2, Lemma 7.2.7 (ii)]. Thus we are in the same setting as in §7.1 of [CF2] and Theorem 7.1.6 in it applies to Y and $\widetilde{F}$.

Every function in $\mathcal{F}^Y_e$ will be taken to be $\mathcal{E}^Y$-quasi continuous. As Y is a diffusion with no killing inside, the jumping measure J and the killing measure k in the Beurling-Deny decomposition of $\mathcal{E}^Y$ vanish so that we have by [CF2, Theorem 7.1.6]

$$H^1_e(D) \subset \mathcal{F}^Y_e \subset {\rm BL}(D), \quad \mathcal{H}^Y := \{\mathbf{H}u : u \in \mathcal{F}^Y_e\} \subset \mathbb{H}^*(D), \tag{5.7}$$

$$\mathcal{E}^Y(u,u) = \frac{1}{2}\mathbf{D}(u,u) + \frac{1}{2}\mu^c_{\langle \mathbf{H}u\rangle}(\widetilde{F}), \quad u \in \mathcal{F}^Y_e, \tag{5.8}$$

where $\mathbf{H}u(x) = \mathbf{E}^Y_x[u(Y_{\sigma_{\widetilde{F}}})],\ x \in E$.

Let us prove that

$$\mu^c_{\langle u\rangle}(\widetilde{F}) = 0 \qquad u \in \mathcal{H}^Y. \tag{5.9}$$

To this end, we consider a finite measure ν on E defined by

$$\nu(B) = \int_{\overline{D}} \mathbf{P}^Y_x\left(Y_{\sigma_{\widetilde{F}}} \in B,\ \sigma_{\widetilde{F}} < \infty\right) m(dx), \quad B \in \mathcal{B}(E).$$

ν vanishes off $\widetilde{F}$ and charges no $\mathcal{E}^Y$-polar set. In view of [CF2, Lemma 5.2.9 (i)], $\widetilde{F}$ is a quasi support of ν in the following sense: $\nu(E\setminus\widetilde{F}) = 0$ and $\widetilde{F} \subset \widehat{F}$ q.e. for any quasi closed set $\widehat{F}$ with $\nu(E \setminus \widehat{F}) = 0$.

Now, for $u \in \mathcal{H}^Y$, (4.8) and (5.7) imply that $u = \sum_{j=1}^N c_j\varphi_j$ for some constants c_j. Take $\widehat{F} = \{\xi \in E : u(\xi) \in \{c_1, \cdots, c_N\}\}$. Since u is quasi continuous, $\widehat{F}$ is a quasi closed set. As u is continuous along the sample path of Y (cf. [CF2, Theorem 3.1.7]), we have $\nu(E \setminus \widehat{F}) = \mathbf{P}_m(u(Y_{\sigma_{\widetilde{F}}}) \notin \{c_1, \cdots, c_N\}) = 0$ on account of Proposition 3.2 and (4.9).

Accordingly $\widetilde{F} \subset \widehat{F}$ q.e., namely, u takes only finite values $\{c_1, \cdots, c_N\}$ q.e. on $\widetilde{F}$. By the *energy image density property* of $\mu^c_{\langle u\rangle}$ due to N. Bouleau and F. Hirsch [BH] (cf. [CF2, Theorem 4.3.8]), we thus get (5.9).

Relation (5.7) and Proposition 3.2(ii) imply that every function $u \in \mathcal{H}^Y (\subset \mathrm{BL}(D))$ admits a limit $u(\partial_j)$ at each boundary point $\partial_j \in F$ along the path of Z. Define an equivalence relation $\sim$ on F by $\partial_j \sim \partial_k$ if and only if $u(\partial_j) = u(\partial_k)$ for every $u \in \mathcal{H}^Y$. Notice that, for every $1 \le j \le N$, there exists $u \in \mathcal{H}^Y$ with $u(\partial_j) \ne 0$. Otherwise, for the resolvent $\{G^Y_\alpha : \alpha > 0\}$ of Y, $G^Y_\alpha 1 \in \mathcal{F}^Y_e (\subset \mathrm{BL}(D))$ approaches to zero at some ∂_j along the path of Z, contradiction to the conservativeness of Y. Let Π be the corresponding partition of F: Π maps F onto $\{\widehat{\partial}_1, \cdots, \widehat{\partial}_\ell\}$ the set of all equivalence classes with respect to $\sim$. Then $\mathcal{H}^Y = \left\{ \sum_{i=1}^{\ell} c_i \widehat{\varphi}_i : \ c_i \in \mathbb{R} \right\}$ for $\widehat{\varphi}_i$ define by (5.1). Hence (5.2), (5.3), (5.7), (5.8) and (5.9) lead us to the desired identity (5.4).

Since the both Dirichlet forms share a common semigroup on $L^2(\overline{D}; m)$, we get the first conclusion of the theorem. Further the Dirichlet spaces

$$(E,\ m,\ \mathcal{E}^Y, \mathcal{F}^Y), \qquad (\overline{D}^{\Pi,*},\ m,\ \mathcal{E}^{\Pi,*}, \mathcal{F}^{\Pi,*})$$

are equivalent in the sense of Appendix (Section 8) by the identity map Φ from $\mathcal{F}^Y_b$ onto $\mathcal{F}^{\Pi,*}_b$ so that we get the second conclusion from Theorem 8.2. □

Remark 5.2 (i) For different choices of f, the family of all symmetric conservative extensions Y of X^f is invariant up to time changes because it shares a common family of extended Dirichlet spaces (5.2)-(5.3). The same can be said for more general time changed RBM X^μ, which will be formulated in Section 7.

(ii) We can replace the conservativness assumption on Y by a weaker one that Y is a proper extension of X with no killing on $E \setminus \overline{D}$. Then the above theorem remains valid if $X^{\Pi,*}$ is allowed to be replaced by its subprocess being killed upon hitting some (but not all) $\widehat{\partial}_i$. □

Remark 5.3 (Symmetric diffusion for a uniformly elliptic differential operator) Given measurable functions $a_{ij}(x),\ 1 \le i, j \le d$, on D such that

$$a_{ij}(x) = a_{ji}(x), \quad \Lambda^{-1}|\xi|^2 \le \sum_{1\le i,j\le d} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \quad x \in D,\ \xi \in \mathbb{R}^d, \tag{5.10}$$

for some constant $\Lambda \ge 1$, we consider a Dirichlet form

$$(\mathcal{E}, \mathcal{F}) = (\mathbf{a}, H^1(D)) \tag{5.11}$$

on $L^2(D)$ where

$$\mathbf{a}(u, v) = \int_D \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) dx, \quad u, v \in H^1(D).$$

If we replace the Dirichlet form (2.2) on $L^2(D)$ and the assoicated RBM Z on $\overline{D}$, respectively, by the Dirichlet form (5.11) on $L^2(D)$ and the associated reflecting diffusion process on $\overline{D}$ constructed in [FTo], all results from Section 3 to Section 5 still hold without any change as we shall see now.

By this replacement, the extended Dirichlet space and the reflected Dirichlet space are still $H^1_e(D)$ and $\mathrm{BL}(D)$, respectively, although the inner product $\frac{1}{2}\mathbf{D}$ is replaced by $\mathbf{a}$. The transience of (5.11) is equivalent to that of (2.2). The space $\mathbb{H}^*(D)$ is now defined by (2.3) with $\mathbf{a}$ in place of $\frac{1}{2}\mathbf{D}$. But, by noting that $\mathbf{a}(c,c)=0$ for any constant c and by taking the characterization of a Liouville domain stated below Definition 2.1 into account, we readily see that $D \in \mathcal{D}$ is a Liouville domain relative to (5.11) if and only if so it is relative to (2.2). □

Remark 5.4 (All possible symmetric conservative diffusion extensions of a one-dimensional minimal diffusion) Consider a minimal diffusion X on a one-dimensional open interval $I=(r_1,r_2)$ with no killing inside for which both boundaries r_1, r_2 are regular. Let E be a Lusin space into which I is homeomorphically embedded as an open subset. The speed measure m of X is extended to E by setting $m(E\setminus I)=0$. Let Y be an m-symmetric conservative diffusion extension of X from I to E. Then, by removing some m-polar open set for Y from $\widetilde{F}=E\setminus I$, a homeomorphic image of Y is identical with either the two point extension of X to $[r_1.r_2]$ or its one-point extension to the one-point compactification of I. This fact was implicitly indicated in [F2, §5] and [F3, §5] without proof. This can be shown in a similar manner to the proof of Theorem 5.1 by establishing the counterpart of the identity (5.9) and by noting that, for the one-point and two-point extensions of X, every non-empty subset of the state space has a positive 1-capacity uniformly bounded away from zero due to the bound [CF2, (2.2.31)] and so a quasi-homeomorphism is reduced to a homeomorphism.

To put it another way, Theorem 5.1 reveals that the time changed RBM X on an unbounded domain with N-Liouville branches has a very similar structure to the one-dimensional diffusion only by changing two boundary points to N boundary points. □

We note that the connected sum of non-parabolic manifolds being studied by Y. Kuz'menko and S. Molchanov [KM], A. Grigor'yan and L. Salloff-Coste [GS] bears a strong similarity to the present paper in the setting although the main concern in these papers was the heat kernel estimates.

6 Characterization of L^2-generator of extension Y by zero flux condition at infnity

For a strictly positive bounded integrable function f on D, we put $m(dx)=f(x)dx$ and denote by $(\cdot,\cdot)$ the inner product for $L^2(D;m)$. Let Y be any $m-$symmetric conservative diffusion extension of the time changed process $X=X^f=(X_t,\zeta,\mathbf{P}_x)$ of the RBM Z

on $\overline{D}$. Let $\Pi : F \mapsto \{\widehat{\partial}_1, \cdots, \widehat{\partial}_\ell\},\ \ell \le N$, be the corresponding partition of the boundary $F = \{\partial_1, \cdots, \partial_N\}$ appearing in Theorem 5.1. The Dirichlet form $(\mathcal{E}^Y, \mathcal{F}^Y)$ of Y on $L^2(D; m)$ is then described as

$$\begin{cases} \mathcal{F}^Y = \left\{ u = u_0 + \sum_{i=1}^{\ell} c_i \widehat{\varphi}_i :\ u_0 \in H^1_e(D) \cap L^2(D; m),\ c_i \in \mathbb{R} \right\}, \\ \mathcal{E}^Y(u, v) = \frac{1}{2} \mathbf{D}(u, v), \qquad u, v \in \mathcal{F}^Y, \end{cases}$$

where $\widehat{\varphi}_i,\ 1 \le i \le \ell$, are defined by (5.1).

Let $\mathcal{A}$ be the L^2-generator of Y, that is, $\mathcal{A}$ is a self-adjoint operator on $L^2(D; m)$ such that $u \in \mathcal{D}(\mathcal{A}),\ \mathcal{A}u = v \in L^2(D; m)$ if and only if $u \in \mathcal{F}^Y$ with $\mathcal{E}^Y(u, w) = -(v, w)$ for every $w \in \mathcal{F}^Y$. In view of Proposition 3.2, the condition (7.3.4) of [CF2] is fulfilled by Y. Therefore Theorem 7.7.3 (vii) of [CF2] is well applicable in getting the following characterization of $\mathcal{A}$:

$$u \in \mathcal{D}(\mathcal{A}) \quad \text{if and only if} \quad u \in \mathcal{D}(\mathcal{L}) \text{ and } \mathcal{N}(u)(\widehat{\partial}_i) = 0,\ 1 \le i \le \ell.$$

In this case, $\mathcal{A}u = \mathcal{L}u$.

Here $\mathcal{L}$ is a linear operator defined as follows: $u \in \mathcal{D}(\mathcal{L}),\ \mathcal{L}u = v \in L^2(D; m)$ if and only if $u \in \mathrm{BL}(D) \cap L^2(D; m)$ and $\frac{1}{2}\mathbf{D}(u, w) = -(v, w)$ for every $w \in H^1_e(D) \cap L^2(D; m)$, or equivalently, for every $w \in C^1_c(\overline{D})$. $\mathcal{N}(u)(\widehat{\partial}_i)$ is the *flux of u at $\widehat{\partial}_i$* defined by

$$\mathcal{N}(u)(\widehat{\partial}_i) = \frac{1}{2}\mathbf{D}(u, \widehat{\varphi}_i) + (\mathcal{L}u, \widehat{\varphi}_i), \quad 1 \le i \le \ell.$$

It can be readily verified that $u \in \mathcal{D}(\mathcal{L})$ if and only if $u \in \mathrm{BL}(D) \cap L^2(D; m)$, Δu in the Schwartz distribution sense is in $L^2(D)$ and

$$\mathbf{D}(u, w) \ + \ \int_D \Delta u(x) \cdot w(x) dx \ = \ 0 \quad \text{for every } w \in C^1_c(\overline{D}). \tag{6.1}$$

In this case, $\mathcal{L}u(x) = \frac{1}{2f(x)}\ \Delta u(x),\ x \in D$. The equation (6.1) can be interpreted as the requirement that the *generalized normal derivative* of u vanishes on ∂D. Thus we have

Theorem 6.1 *$u \in \mathcal{D}(\mathcal{A})$ if and only if $u \in \mathrm{BL}(D) \cap L^2(D; m)$, Δu in the Schwartz distribution sense belongs to $L^2(D)$, the equation* (6.1) *is satisfied and*

$$\Big(\mathcal{N}(u)(\widehat{\partial}_i) = \Big)\ \frac{1}{2}\mathbf{D}(u, \widehat{\varphi}_i) + \frac{1}{2}\int_D \Delta u(x) \widehat{\varphi}_i(x) dx = 0, \quad 1 \le i \le \ell. \tag{6.2}$$

In this case,

$$\mathcal{A}u(x) = \frac{1}{2f(x)}\ \Delta u(x), \quad \text{a.e. on } D. \tag{6.3}$$

Suppose $u \in \mathcal{D}(\mathcal{A})$ is smooth on $\overline{D}$. Then $\frac{\partial u}{\partial \mathbf{n}} = 0$ on ∂D due to the condition (6.1) so that the zero flux condition (6.2) at $\widehat{\partial}_j$ can be expressed as

$$\lim_{r \uparrow \infty} \int_{D \cap \partial B_r(\o)} u_r(x) \widehat{\varphi}_i(x) d\sigma_r(dx) = 0, \quad 1 \le i \le \ell, \tag{6.4}$$

where σ_r is the surface measure on $\partial B_r(\o)$.

The last part of Section 7.6 (4°) of [CF2] has treated a very special case of the above where $D = \mathbb{R}^d$, $d \geq 3$, and Y is the one-point reflection at the infinity of $\mathbb{R}^d$ of a time changed Brownian motion on $\mathbb{R}^d$.

In [F3], the L^2-generator of any symmetric diffusion extension Y of a one-dimensional minimal diffusion X is identified. In this case, the Dirichlet form of Y admits its reproducing kernel which enables us to identify also the C_b-generator of Y, recovering the general boundary condition due to W. Feller and K. Itô-H. P. McKean.

7 Extensions of more general time changed RBMs

All the results in Sections 4-6 except for (6.3) hold for more general time changed RBMs than X^f. Let $Z = (Z_t, \mathbf{Q}_x)$, f, $X = X^f = (X_t, \zeta, \mathbf{P}_x)$, $X^* = (X_t^*, \mathbf{P}_x^*)$ be as in Section 4.

We consider a finite smooth measure μ on $\overline{D}$ with full quasi-support $\overline{D}$ relative to the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of (2.2). Let A^μ be the PCAF of Z with Revuz measure μ and $X^\mu = (X_t^\mu, \zeta^\mu, \mathbf{P}_x^\mu)$ be the time changed process of Z by A^μ. The Markov process X^μ is μ-symmetric and its Dirichlet form $(\mathcal{E}^{X^\mu}, \mathcal{F}^{X^\mu})$ on $L^2(\overline{D}; \mu)$ is given by

$$\mathcal{E}^{X^\mu} = \frac{1}{2}\mathbf{D}, \quad \mathcal{F}^{X^\mu} = H_e^1(D) \cap L^2(\overline{D}; \mu). \tag{7.1}$$

Proposition 7.1 *It holds that*

$$\mathbf{Q}_x(A_\infty^\mu < \infty) = 1 \quad \text{for q.e. } x \in \overline{D}, \tag{7.2}$$

$$\mathbf{P}_x^\mu(\zeta^\mu < \infty,\ X_{\zeta^\mu -}^\mu = \partial_i) = \varphi_i(x) > 0, \quad \text{for q.e. } x \in \overline{D} \text{ and } 1 \leq i \leq N. \tag{7.3}$$

Proof. Fix a strictly positive bounded integrable function h_0. By the transience of Z and [CF2, Theorem A.2.13 (v)], $G_{0+}^Z h_0(x) < \infty$ for q.e. $x \in \overline{D}$. For integer $k \geq 1$, let

$$\Lambda_k := \left\{ x \in \overline{D} : G_{0+}^Z h_0(x) \leq 2^k \right\} \quad \text{and} \quad h(x) = \sum_{k=1}^\infty 2^{-2k} \mathbf{1}_{\Lambda_k}(x) h_0(x).$$

Then h is a strictly positive bounded integrable function on $\overline{D}$ with $G_{0+}^Z h(x) \leq \mathbf{1}$ q.e. on $\overline{D}$. From [CF2, (4.1.3)], we have

$$\int_{\overline{D}} \mathbf{E}^{\mathbf{Q}_x}\left[A_\infty^\mu\right] h(x)dx = \langle G_{0+}^Z h, \mu \rangle \leq \mu(\overline{D}) < \infty. \tag{7.4}$$

It follows that $\mathbf{E}^{\mathbf{Q}_x}\left[A_\infty^\mu\right] < \infty$ a.e $x \in \overline{D}$ and hence q.e. $x \in \overline{D}$ by [CF2, Theorem A.2.13 (v)], yielding (7.2). (7.3) follows from (7.2) and Proposition 3.1. □

Since $m(dx) = f(x)dx$ has its quasi-support $\overline{D}$ relative to $(\mathcal{E}, \mathcal{F})$, the Dirichlet form $(\mathcal{E}^X, \mathcal{F}^X)$ of (4.4) shares the common quasi-notion with $(\mathcal{E}, \mathcal{F})$ ([CF2, Theorem 5.2.11]). Hence the quasi-support of μ relative to $(\mathcal{E}^X, \mathcal{F}^X)$ is still $\overline{D}$.

The Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(\overline{D}^*, m)$ of X^* is quasi-regular. According to the quasi-homeomorphism method already used in Section 4, we may assume it to be regular. The measure μ on $\overline{D}$ is extended to $\overline{D}^*$ by setting $\mu(F)=0$. We claim that the quasi-support of μ relative to this Dirichlet form equals $\overline{D}^*$ by using a criteria [CF2, Theorem 3.3.5].

Assume that $u \in \mathcal{F}^*$ is $\mathcal{E}^*$-quasi-continuous and that $u=0$ μ-a.e. Then $u\big|_{\overline{D}}$ is $\mathcal{E}^X$-quasi-continuous ([CF2, Theorem 3.3.8]) so that $u=0$ q.e. on $\overline{D}$. According to the same reference, there exists a Borel m-polar set $C \subset \overline{D}$ relative to X^* such that $u(x)=0$ for every $x \in \overline{D}\setminus C$. Since u is continuous along the path of X^* ([CF2, Theorem 3.1.7]), we have for each $1 \le i \le N$

$$\mathbf{P}^*_m\left(u(\partial_i)=\lim_{t\uparrow\sigma_F} u(X^*_t),\ \sigma_C=\infty,\ \sigma_F<\infty,\ X^*_{\sigma_F}=\partial_i\right)=\mathbf{P}_m(\zeta<\infty, X_{\zeta-}=\partial_i)>0,$$

and so u vanishes on F and hence q.e. on $\overline{D}^*$, as was to be proved.

Theorem 7.2 *There exists a unique μ-symmetric conservative diffusion $\widetilde{X}^{*,\mu}$ on $\overline{D}^*$ which is a q.e. extension of X^μ in the sense that the part of the former on $\overline{D}$ coincides in law with the latter for q.e. starting points $x\in\overline{D}$. The extended Dirichlet space of $\widetilde{X}^{*,\mu}$ equals* $(\mathrm{BL}(D), \frac{1}{2}\mathbf{D})$ *the reflected Dirichlet space of X^μ.*

Proof. Let B^0_t and B_t be the PCAFs of X and X^*, respectively, with Revuz measure μ. According to [CF2, Proposition 4.1.10]

$$B^0_t = B_{t\wedge\sigma_F}. \tag{7.5}$$

Let $\widetilde{X}^\mu$ and $\widetilde{X}^{*,\mu}$ be the time changed processes of X and X^* by means of B^0_t and B_t, respectively. The Markov process $\widetilde{X}^\mu$ is then the part of $\widetilde{X}^{*,\mu}$ on $\overline{D}$ by (7.5). Since X^* is recurrent, so is $\widetilde{X}^{*,\mu}$ in view of [CF2, Theorem 5.2.5]. Therefore $\widetilde{X}^{*,\mu}$ is a μ-symmetric conservative diffusion extension of $\widetilde{X}^\mu$.

On the other hand, the Dirichlet form of $\widetilde{X}^\mu$ on $L^2(\overline{D};\mu)$ is identical with (7.1) the Dirichlet form of X^μ on $L^2(\overline{D};\mu)$, and consequently $\widetilde{X}^{*,\mu}$ is a q.e. extension of X^μ. The last statement follows from the invariance of extended and reflected Dirichlet spaces under time changes by fully supported PCAFs.

The uniqueness of such a μ-symmetric conservative Markovian extension of X^μ to $\overline{D}^*$ follows from [CF2, Theorem 7.7.3] . □

Similarly, all results in Section 4 and 5 with μ in place of $dm = f dx$ remain valid except for (6.3).

Remark 7.3 One can give an alternative proof of Theorem 7.2 without invoking the time change of X^* but still using the quasi-regularity of $(\mathcal{E}^*, \mathcal{F}^*)$. Indeed, the following proposition combined with (7.3) and [CF2, Theorem 7.7.3] readily yields Theorem 7.2.

Each function in $\mathcal{F}^*_e$ is taken to be $\mathcal{E}^*$-quasi continuous. Define

$$\widehat{\mathcal{F}} = \mathcal{F}^*_e \cap L^2(\overline{D};\mu) \quad \text{and} \quad \widehat{\mathcal{E}}(u,v)=\mathcal{E}^*(u,v)=\frac{1}{2}\mathbf{D}(u,v) \text{ for } u,v\in\widehat{\mathcal{F}}. \tag{7.6}$$

Proposition 7.4 (i) *$(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ is a quasi-regular Dirichlet form on $L^2(\overline{D}^*;\mu)$.*

(ii) *Its associated strong Markov process* $\widehat{X}$ *on* $\overline{D}^*$ *is a* μ*-symmetric conservative diffusion which is a q.e. extension of* X^μ.

(iii) *Each* ∂_j *is non-*$\widehat{\mathcal{E}}$*-polar.*

Proof. (i) As $\overline{D}$ is a quasi-support of μ, $u=0$ μ–a.e. for $u\in\widehat{\mathcal{F}}$ implies $u=0$ a.e. on $\overline{D}$ and $\mathbf{D}(u,u)=0$. This together with the transience of $(\mathcal{F}^*_e,\mathcal{E}^*)$ implies that $(\widehat{\mathcal{E}},\widehat{\mathcal{F}})$ is a well defined Dirichlet form on $L^2(\overline{D}^*;\mu)$.

Since $(\mathcal{E}^*,\mathcal{F}^*)$ is a quasi-regular Dirichlet form on $L^2(\overline{D}^*;m)$, by [CF2, Remark 1.3.9], there is an increasing sequence of compact subsets $\{F_k\}$ of $\overline{D}^*$ so that

(a) there is an increasing sequence of compact subsets $\{F_k\}$ of $\overline{D}^*$ so that $\cup_{k\ge1}\mathcal{F}^*_{F_k}$ is $\mathcal{E}^*_1$-dense in $\mathcal{F}^*$.

(b) there is an $\mathcal{E}^*_1$-dense of countable set $\Lambda_0:=\{f_j; j\ge1\}$ of bounded functions of $\mathcal{F}^*$ so that $\{f_j;j\ge1\}\subset C(\{F_k\})$ and they separate points of $\cup_{k\ge1}F_k$.

By the contraction of the Dirichlet form, we may and do assume without loss of generality that for every integer $n\ge1$ and $f\in\Lambda_0$, $((-n)\vee f)\wedge n\in\Lambda_0$. We claim that $\cup_{k\ge1}\mathcal{F}^*_{F_k,b}\subset\cup_{k\ge1}\widehat{\mathcal{F}}_{F_k,b}$ is $\widehat{\mathcal{E}}_1$-dense in $\widehat{\mathcal{F}}_b$. Let $u\in\widehat{\mathcal{F}}_b$. Since $\widehat{\mathcal{F}}_b=\mathcal{F}^*_b$, there are $u_k\in\mathcal{F}^*_{F_k}$ so that $u_k\to u$ in $\mathcal{E}^*_1$-norm. Using truncation if needed, we may and do assume $\|u_k\|_\infty\le\|u\|_\infty+1$. Taking a subsequence if needed, we may also assume that u_k converges to u $\mathcal{E}^*$-q.e. on $\overline{D}^*$. Since μ is a finite smooth measure, we conclude that u_k is $\widehat{\mathcal{E}}_1$-convergent to u. This proves the claim. As $\widehat{\mathcal{F}}_b$ is $\widehat{\mathcal{E}}_1$ dense in $\widehat{\mathcal{F}}$, it follows that $\{F_k\}$ is an $\widehat{\mathcal{E}}$-nest on $\overline{D}^*$.

A similar argument shows that $\Lambda_0\subset\widehat{\mathcal{F}}_b=\mathcal{F}^*_b$ is $\widehat{\mathcal{E}}_1$-dense in $\widehat{\mathcal{F}}_b$ and hence in $\widehat{\mathcal{F}}$. This proves the assertion (i).
(ii) Since $1\in\widehat{\mathcal{F}}$ and $\mathbf{D}(1,1)=0$, the associated μ-symmetric diffusion $\widehat{X}$ on $\overline{D}^*$ is recurrent and conservative. For $R>r$, take $\psi\in C^\infty_c(\overline{D})$ with $\psi=1$ on $B_{R+1}(\mathbf{0})$. Then, for any bounded $u\in\widehat{\mathcal{F}}$, $\psi u\in H^1_e(D)$ and so

$$\{v\in\widehat{\mathcal{F}}: v=0 \text{ q.e. on } \overline{D}^*\setminus B_R(\mathbf{0})\}=\{v\in H^1_e(D)\cap L^2(\overline{D};\mu): v=0 \text{ q.e. on } \overline{D}\setminus B_R(\mathbf{0})\},$$

namely, the part of $\widehat{\mathcal{E}}$ on $\overline{D}\cap B_R(\mathbf{0})$ coincides with the part of $\mathcal{E}^{X^\mu}$ on $\overline{D}\cap B_R(\mathbf{0})$. By letting $R\to\infty$, we see that the part of $\widehat{\mathcal{E}}$ on $\overline{D}$ coincides with $\mathcal{E}^{X^\mu}$, proving (ii).
(iii) The non-$\widehat{\mathcal{E}}$-polarity of ∂_j follows from (ii) and (7.3). $\square$

8 Appendix: equivalence and quasi-homeomorphism

In dealing with boundary problems for symmetric Markov processes, it is convenient to introduce an equivalence of Dirichlet spaces following [FOT, A.4] as will be stated below.

We say that a quadruplet $(E,m,\mathcal{E},\mathcal{F})$ is a *Dirichlet space* if E is a Hausdorff topological space with a countable base, m is a σ-finite positive Borel measure on E and $\mathcal{E}$ with domain $\mathcal{F}$ is a Dirichlet form on $L^2(E;m)$. The inner product in $L^2(E;m)$ is denoted by $(\cdot,\cdot)_E$. For a given Dirichlet space $(E,m,\mathcal{E},\mathcal{F})$, the notions of an $\mathcal{E}$*-nest*, an $\mathcal{E}$*-polar set*, an $\mathcal{E}$-quasi-*continuous numercal function* and '$\mathcal{E}$*-quasi-everywhere*' ('$\mathcal{E}$-q.e.' in abbreviation) are defined as in [CF2, Definition 1.2.12]. The *quasi-regularity* of the Dirichlet space is defined just as in [CF2, Definition 1.3.8]. We note that the space $\mathcal{F}_b = \mathcal{F} \cap L^\infty(E;m)$ is an algebra.

Remark 8.1 In Section 1.2 and the first half of Section 1.3 of [CF2], it is assumed that

$$\mathrm{supp}[m] = E. \tag{8.1}$$

We need not assume it. Generally, if we let $E' = \mathrm{supp}[m]$, then $E \setminus E'$ is $\mathcal{E}$-polar according to the definition of the $\mathcal{E}$-polarity. If $(E, m, \mathcal{E}, \mathcal{F})$ is quasi-regular, so is $(E', m\big|_{E'}, \mathcal{E}, \mathcal{F})$ accordingly. Therefore we may assume (8.1) if we like by replacing E with E'. □

Given two Dirichlet spaces

$$(E, m, \mathcal{E}, \mathcal{F}), \qquad (\tilde{E}, \tilde{m}, \tilde{\mathcal{E}}, \tilde{\mathcal{F}}), \tag{8.2}$$

we call them *equivalent* if there is an algebraic isomorphism Φ from $\mathcal{F}_b$ onto $\tilde{\mathcal{F}}_b$ preserving three kinds of metrics: for $u \in \mathcal{F}_b$

$$\|u\|_\infty = \|\Phi u\|_\infty, \ (u,u)_E = (\Phi u, \Phi u)_{\tilde{E}}, \ \mathcal{E}(u,u) = \tilde{\mathcal{E}}(\Phi u, \Phi u).$$

One of the two equivalent Dirichlet spaces is called a *representation* of the other.

The underlying spaces E, $\tilde{E}$ of two Dirichlet spaces (8.2) are said to be *quasi-homeomorphic* if there exist $\mathcal{E}$-nest $\{F_n\}$, $\tilde{\mathcal{E}}$-nest $\{\tilde{F}_n\}$ and a one to one mapping q from $E_0 = \cup_{n=1}^\infty F_n$ onto $\tilde{E}_0 = \cup_{n=1}^\infty \tilde{F}_n$ such that the restricition of q to each F_n is a homeomorphism onto $\tilde{F}_n$. $\{F_n\}$, $\{\tilde{F}_n\}$ are called the *nests attached to the quasi-homemorphism* q. Any quasi-homeomorphism is quasi-notion-preseving.

We say that the equivalnce Φ of two Dirichlet spaces (8.2) *is induced by a quasi-homeomorphism* q of the underlying spaces if

$$\Phi u(\tilde{x}) = u(q^{-1}(\tilde{x})), \quad u \in \mathcal{F}_b, \quad \tilde{m}\text{−a.e. } \tilde{x}.$$

Then $\tilde{m}$ is the image measure of m and $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is the image Dirichlet form of $(\mathcal{E}, \mathcal{F})$.

Theorem 8.2 *Assume that two Dirichlet spaces* (8.2) *are quasi-regular and that they are equivalent. Let* $X = (X_t, \mathbb{P}_x)$ *(resp.* $\tilde{X} = (\tilde{X}_t, \tilde{\mathbb{P}}_x)$*) be an* m*-symmetric right process on* E *(resp. an* $\tilde{m}$*-symmetric right process on* $\tilde{E}$*) properly associated with* $(\mathcal{E}, \mathcal{F})$ *on* $L^2(E;m)$ *(resp.* $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ *on* $L^2(\tilde{E};\tilde{m})$*). Then the equivalence is induced by a quasi-homeomorphism* q *with attached nests* $\{F_n\}$, $\{\tilde{F}_n\}$ *such that* $\tilde{X}$ *is the image of* X *by* q *in the following sense: there exist an* m*-inessential Borel subset* N *of* E *containing* $\cap_{n=1}^\infty F_n^c$ *and an* $\tilde{m}$*-inessential Borel subset* $\tilde{N}$ *of* $\tilde{E}$ *containing* $\cap_{n=1}^\infty \tilde{F}_n^c$ *so that* q *is one to one from* $E \setminus N$ *onto* $\tilde{E} \setminus \tilde{N}$ *and*

$$\tilde{X}_t = q(X_t), \qquad \tilde{\mathbb{P}}_{\tilde{x}} = \mathbb{P}_{q^{-1}\tilde{x}}, \qquad \tilde{x} \in \tilde{E} \setminus \tilde{N}. \tag{8.3}$$

Proof. Since both Dirichlet spaces in (8.2) are assumed to be quasi-regular, they are equivalent to some regular Dirichlet spaces and the equivalences are induced by some quasi-homeomorphisms q_1, q_2 in view of [CF2, Theorem 1.4.3]. Since two Dirichlet spaces in (8.2) are also assumed to be equivalent, so are the corresponding two regular Dirichlet spaces, the equivalence being induced by a quasi-homeomorphism q_3 on account of [FOT, Theorem A.4.2] combined with [CF2, Theorem 1.2.14]. Hence the equivalence of the quasi-regular Dirichlet spaces in (8.2) is induced by the quasi-homeomorphism $q = q_1 \circ q_3 \circ q_2^{-1}$ between E and $\tilde{E}$. Let $\{F_n\}, \{\tilde{F}_n\}$ be the nests attached to q.

According to [CF2, Theorem 3.1.13], we may assume without loss of generality that both X and $\tilde{X}$ are Borel right processes. Further the $\mathcal{E}$-polarity is equivalent to the m-polar for X. By virtue of [CF2, Theorem A.2.15], we can therefore find an m-inessential Borel set $N_1 \subset E$ containing $\cap_{n=1}^{\infty} F_n^c$. Consider the set $\tilde{N}_1 \subset \tilde{E}$ defined by $q(E \setminus N_1) = \tilde{E} \setminus \tilde{N}_1$. $\tilde{N}_1$ is an $\tilde{\mathcal{E}}$-polar Borel set and q is one to one from $E \setminus N_1$ onto $\tilde{E} \setminus \tilde{N}_1$.

Define the process $\widehat{X} = (\widehat{X}_t, \widehat{\mathbb{P}}_{\tilde{x}})_{\tilde{x} \in \tilde{E} \setminus \tilde{N}_1}$ by

$$\widehat{X}_t = q(X_t), \qquad \widehat{\mathbb{P}}_{\tilde{x}} = \mathbb{P}_{q^{-1}\tilde{x}}, \qquad \tilde{x} \in \tilde{E} \setminus \tilde{N}_1.$$

On account of [FFY, Lemma 3.1], we can then see that $\widehat{X}$ is an $\tilde{m}$-symmetric Markov process on $\tilde{E} \setminus \tilde{N}_1$ properly associated with the Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ on $L^2(\tilde{E}; \tilde{m})$. Since the $\tilde{m}$-symmetric Borel right process $\tilde{X}$ is also properly associated with the Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ on $L^2(\tilde{E}; \tilde{m})$, the same method as in the proof of [CF2, Theorem 3.1.12] combined with [CF2, Theorem A.2.15] leads us to finding an $\tilde{m}$-inessential Borel set $\tilde{N}$ containing $\tilde{N}_1$ for $\tilde{X}$ such that the Markov processes $\tilde{X}\big|_{\tilde{E} \setminus \tilde{N}}$ and $\widehat{X}\big|_{\tilde{E} \setminus \tilde{N}}$ are identical in law. It now suffices to define the set N by $E \setminus N = q^{-1}(\tilde{E} \setminus \tilde{N})$. □

Remark 8.3 Owing to the works of S. Albeverio, Z.-M. Ma, M. Röckner and P. J. Fitzsimmons, the quasi-regularity of a Dirichlet form has been known to be not only a sufficient condition but also a necessary one for the existence of a properly associated right process. It is further shown in [CMR] that a Dirichlet form is quasi-regular if and only if it is quasi-homeomorphic to a reglar Dirichlet form on a locally compact separable metric space. These facts are formulated by Theorem 1.5.3 and Theorem 1.4.3, respectively, of [CF2] under the assumption (8.1) which is not needed actually. But we may assume it without loss of generality as will be seen below.

Indeed, let E be a Lusin space, m be a σ-finite measure on E and X be an m-symmetric Borel right process on E. Then, for $E_0 = \text{supp}[m]$, $E \setminus E_0$ is an m-negligible open set so that it is m-polar for X by [CF2, Theorem A.2.13 (iii)]. Hence, by [CF2, Theorem A.2.15], there exists a Borel set $E_1 \subset E_0$ such that $E \setminus E_1$ is m-inessential for X. E_1 is the support of $m\big|_{E_1}$ because, for any $x \in E_1$ an any neighborhood $O(x)$ of x, $m(O(x) \cap E_1) = m(O(x)) - m(O(x) \cap (E \setminus E_1)) > 0$. Hence it suffices to replace E by E_1.

In Theorem 5.1, the extension process Y is assumed to live on a Lusin space E into which $\overline{D}$ is homeomorphically embedded as an open subset. In this particular case, the above set E_1 can be choosen to contain $\overline{D}$ on account of the proof of [CF2, Theorem A.2.15]. Therefore, in Theorem 5.1 (resp. Remark 5.4), we can assume more strongly that $\overline{D}$ (resp. I) is homeomorphically embedded into the state space E of Y as a dense open subset. □

References

[BH] N. Bouleau and F. Hirsch *Dirichlet Forms and Analysis on Wiener Space*, De Gruyter, 1991

[B] M. Brelot, Etude et extension du principe de Dirichlet, *Ann. Inst. Fourier* **5** (1953/54), 371-419.

[C] Z.-Q. Chen, On reflecting diffusion processes and Skorokhod decompositions. *Probab. Theory Relat. Fields,* **94** (1993), 281-315.

[CF1] Z.-Q. Chen and M. Fukushima, On unique extension of time changed reflecting Brownian motions, *Ann. Inst. Henri Poincaré Probab. Statist.* **45** (2009), 864-875.

[CF2] Z.-Q. Chen and M. Fukushima, *Symmetric Markov Processes, Time Change and Boundary Theory*, Princeton University Press, 2011.

[CF3] Z.-Q. Chen and M. Fukushima, One-point reflections, *Stochastic Process Appl.* **125** (2015), 1368-1393.

[CMR] Z.-Q. Chen, Z.-M. Ma and M. Röckner, Quasi-homeomorphisms of Dirichlet forms, *Nagoya Math. J* **136** (1994), 1-15.

[DL] J. Deny and J.L. Lions, Les espaces du type de Beppo Levi, *Ann. Inst. Fourier* **5** (1953/54), 305-370.

[FFY] X. Fang, M. Fukushima and J. Ying, On regular Dirichlet subspaces of $H^1(I)$ and associated linear diffusions, *Osaka J. Math.* **42** (2005), 1-15.

[F1] M. Fukushima, On boundary conditions for multi-dimensional Brownian motions with symmetric resolvent densities, *J. Math. Soc. Japan* **21** (1969), 58-93 .

[F2] M. Fukushima, From one dimensional diffusions to symmetric Markov processes, *Stochastic Process Appl.* **120** (2010), 590-604.

[F3] M. Fukushima, On general boundary conditions for one-dimensional diffusions with symmetry, *J. Math. Soc. Japan* **66** (2014), 289-316/

[FOT] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, De Grruyter, 1994, 2nd Edition. 2011.

[FTa] M. Fukushima and M. Takeda, *Markov Processes* (in Japanese), Baifukan, Tokyo, 2008, Chinese translation by P. He, ed. by J. Ying, Science Press, Beijing, 2011.

[FTo] M. Fukushima and M. Tomisaki, Construction and decomposition of reflecting diffusions on Lipschitz domains with Hölder cusps, *Probabb. Theory Relat. Fields* **106** (1996), 521-557.

[GS] A. Grigor'yan and L. Salloff-Coste, Heat kernels on manifolds with ends, *Ann.Inst. Fourier, grenoble* **59** (2009), 1917-1997.

[KM] Y. Kuz'menko and S. Molchanov, Counterexamples to Liouville-type theorems, *Moscow Univ. Math. Bull.* **34** (1979), 35-39.

[HK] D.A. Herron and P. Koskella, Uniform, Sobolev extension and quasiconformal circle domains, *J. Anal. Math.* **57**(1991), 172-202.

[1] P. W. Jones, Quasiconformal mappings and extendibility of functions in Sobolev spaces. *Acta Math.* **147** (1981), 71-88.

[M] V.G. Maz'ja, *Sobolev Spaces*. Springer, 1985.

[MR] Z.-M. Ma and M. Röckner, *Introduction to the Theory of (non-symmetric) Dirichlet Forms*, Springer, 1992.

[P] Ross G. Pinsky, Transience/recurence for normally reflected Brownian motion in unbounded domains, *Ann. Probab.* **37**(2009), 676-686.

Zhen-Qing Chen

Department of Mathematics, University of Washington, Seattle, WA 98195, USA
E-mail: `zqchen@uw.edu`

Masatoshi Fukushima:

Branch of Mathematical Science, Osaka University, Toyonaka, Osaka 560-0043, Japan.
Email: `fuku2@mx5.canvas.ne.jp`