Stochastic Komatu-Loewner evolutions and SLEs

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Abstract

A stochastic Komatu-Loewner evolution $\mathrm{SKLE}_{\alpha,b}$ has been introduced in [CF] on a standard slit domain determined by certain continuous homogeneous functions α and b. We show that after a suitably reparametrization, $\mathrm{SKLE}_{\alpha,b}$ has the same distribution as the chordal Loewner evolution on $\mathbb{H}$ driven by a continuous semimartingale. When α is constant, we show that the distribution of $\mathrm{SKLE}_{\alpha,b}$, after a suitably reparametrization and up to some random hitting time, is equivalent to that of SLE_{α^2}. Moreover, a reparametrized $\mathrm{SKLE}_{\sqrt{6},-b_{\mathrm{BMD}}}$ has the same distribution as SLE_6 for BMD domain constant b_{BMD}.

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1 Introduction

A subset A of the upper half-plane $\mathbb{H}$ is called an $\mathbb{H}$-*hull* if A is bounded closed in $\mathbb{H}$ and $\mathbb{H} \setminus A$ is simply connected. Given an $\mathbb{H}$-hull A, there exists a unique conformal map f (one-to-one analytic function) from $\mathbb{H} \setminus A$ onto $\mathbb{H}$ satisfying a hydrodynamic normalization at infinity

$$f(z) = z + \frac{a}{z} + o(1/|z|),\ z \to \infty.$$

Such a map will be called a *canonical Riemann map from* $\mathbb{H} \setminus A$ and the constant a is called the *half-plane capacity* of A relative to f.

We consider a simple ODE called a chordal *Loewner differential equation*

$$\frac{dz(t)}{dt} = -2\pi\Psi^{\mathbb{H}}(z(t), \xi(t)), \quad z(0) = z \in \mathbb{H}, \tag{1.1}$$

where

$$\Psi^{\mathbb{H}}(z,\xi) = -\frac{1}{\pi}\frac{1}{z-\xi},\ z \in \mathbb{H},\ \xi \in \partial\mathbb{H},$$

that is the so-called *complex Poisson kernel* for the absorbing Brownian motion on $\mathbb{H}$ because

$$\Im\Psi^{\mathbb{H}}(z,\xi) = \frac{1}{\pi}\frac{y}{(x-\xi)^2+y^2} \quad \text{for } z = x+iy \in \mathbb{H} \text{ and } \xi \in \partial\mathbb{H},$$

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is the classical Poisson kernel in upper half space $\mathbb{H}$.

Given a continuous function $\xi(t) \in \partial\mathbb{H},\ 0 \le t < \infty$, the Cauchy problem of the ODE (1.1) admits a unique solution $z(t),\ 0 \le t < t_z$, with the maximal interval of definition $[0, t_z)$. Define

$$K_t = \{z \in \mathbb{H} : t_z \le t\}, \quad t \ge 0. \tag{1.2}$$

Then K_t is an $\mathbb{H}$-hull and $z(t)$ is the canonical Riemann map from $\mathbb{H} \setminus K_t$. The family $\{K_t : t \ge 0\}$ of growing hulls is called the chordal *Loewner evolution driven by* $\xi(t),\ 0 \le t < \infty$.

Let $B(t)$ be a one-dimensional Brownian motion with a fixed initial point $B(0) = \xi \in \partial\mathbb{H}$ and κ be a positive constant. The random Loewner evolution driven by the sample path of $B(\kappa t),\ 0 \le t < \infty$, is called the *stochastic Loewner evolution* (starting at ξ) and is denoted by SLE_κ. It was introduced by Oded Schramm [S] in his consideration of critical two-dimensional lattice models in statistical physics and their scaling limits. It is now also called the *Schramm-Loewner evolution*. Remarkable features as the *locality property* of SLE_6 and the *restriction property* of $\mathrm{SLE}_{8/3}$ were then revealed ([LSW1, LSW2]). SLE_κ was shown in [RS] to be generated by a continuous curve in the sense that, there exists a continuous path $\gamma : [0,\infty) \mapsto \overline{\mathbb{H}}$ such that $\mathbb{H} \setminus K_t$ is identical with the unbounded connected component of $\mathbb{H} \setminus \gamma[0,t]$ a.s. for each $t > 0$, and furthermore γ was shown to be a simple curve when $\kappa \le 4$, self-intersecting when $4 < \kappa \le 8$ and space-filling when $\kappa > 8$.

Several attempts have been made to extend both of the Loewner equation and the associated SLE from simply connected planar domains to multiply connected ones. Recently, motivated by [BF1, BF2, L2], Chen-Fukushima-Rohde [CFR] and Chen-Fukushima [CF] studied Komatu-Loewner equation and stochastic Komatu-Loewner evolution, respectively, in standard slit domains of finite multiplicity. Stochastic Komatu-Loewner evolution, denoted by $\mathrm{SKLE}_{\alpha,b}$, is a family of conformal maps that are determined by two functions α and b on the slit space S to be described below. They generate an increasing family of random growing $\mathbb{H}$-hulls.

The main purpose of this paper is to study the geometry of $\mathrm{SKLE}_{\alpha,b}$-hulls. We show that, after a suitably reparametrization, $\mathrm{SKLE}_{\alpha,b}$-hulls have the same disturbution as that of the Loewner evolution on $\mathbb{H}$ driven by a continuous semi-martingale. In particular, we show that when function α is a constant, after a reparametrization and under an equivalent martingale measure, $\mathrm{SKLE}_{\alpha,b}$ has the same distribution as the chordal SLE_{α^2} in $\mathbb{H}$ up to a stopping time. Hence when α is a positive constant, we conclude that $\mathrm{SKLE}_{\alpha,b}$-hulls are generated by continuous paths which are simple if $\alpha \le 2$, self-intersecting if $2 < \alpha \le 2\sqrt{2}$ and space-filling when $\alpha > 2\sqrt{2}$.

Fix $N \ge 1$. A *standard slit domain* (of N slits) is a domain of the type $D = \mathbb{H} \setminus \bigcup_{j=1}^N C_j$ for mutually disjoint line segments $C_j \subset \mathbb{H}$ parallel to $\partial\mathbb{H}$. The collection of all labelled standard slit domains (of N slits) is denoted by $\mathcal{D}$. For $D \in \mathcal{D}$, let $z_k = x_k + iy_k,\ z_k^r = x_k^r + iy_k$ be the left and right endpoints of the kth slit C_k of D. It is characterized by $\mathbf{y} := (y_1, \dots, y_N)$, $\mathbf{x} := (x_1, \dots, x_N)$ and $\mathbf{x}^r := (x_1^r, \dots, x_N^r)$ with the property that $\mathbf{y} > 0$, $\mathbf{x} < \mathbf{x}^r$, and either $x_j^r < x_k$ or $x_k^r < x_j$ whenever $y_j = y_k$ for $j \ne k$. Here for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, $\mathbf{y} > 0$ means each coordinate is strictly larger than 0; and $\mathbf{x} < \mathbf{y}$ means $\mathbf{y} - \mathbf{x} > 0$. With this characterization, the space $\mathcal{D}$ can be identified with the following subset of $\mathbb{R}^{3N}$

$$\mathcal{S} = \left\{(\mathbf{y}, \mathbf{x}, \mathbf{x}') \in \mathbb{R}^{3N} : \mathbf{y} > 0,\ \mathbf{x} < \mathbf{x}',\ \text{either } x_j' < x_k \text{ or } x_k' < x_j \text{ whenever } y_j = y_k,\ j \ne k\right\}.$$

For $\mathbf{s} \in \mathcal{S}$, denote by $D(\mathbf{s})$ the corresponding element in $\mathcal{D}$. For $\xi \in \mathbb{R}$, we denote by $\widehat{\xi} \in \mathbb{R}^{3N}$ the $3N$-vector whose first N-components are equal to 0 and the rest are equal to ξ.

For $\mathbf{s} \in \mathcal{S}$, we denote by $\Psi_{\mathbf{s}}(z, \xi)$, $z \in D(\mathbf{s})$, $\xi \in \partial\mathbb{H}$, the *BMD-complex Poisson kernel* for $D = D(\mathbf{s})$, namely, the unique analytic function in $z \in D$ vanishing at ∞ whose imaginary part is the Poisson kernel for the *Brownian motion with darning* (BMD) for D (see [CFR]).

A function f on $\mathcal{S}$ is called *homogeneous* with degree 0 (resp. -1) if $f(c\mathbf{s}) = f(\mathbf{s})$ (resp. $f(c\mathbf{s}) = c^{-1}f(\mathbf{s})$) for every positive constant $c > 0$. It is said to satisfy the *local Lipschitz condition* if the following property holds:

(L) For any $\mathbf{s} \in \mathcal{S}$ and any finite open interval $J \subset \mathbb{R}$, there exist a neighborhood $U(\mathbf{s})$ of $\mathbf{s}$ in $\mathcal{S}$ and a constant $L > 0$ such that

$$|f(\mathbf{s}^{(1)} - \widehat{\xi}) - f(\mathbf{s}^{(2)} - \widehat{\xi})| \leq L\,|\mathbf{s}^{(1)} - \mathbf{s}^{(2)}| \quad \text{for } \mathbf{s}^{(1)}, \mathbf{s}^{(2)} \in U(\mathbf{s}) \text{ and } \xi \in J. \tag{1.3}$$

We consider the strong solution $(\xi(t), \mathbf{s}(t)) \in \partial\mathbb{H} \times \mathcal{S}$ of the following stochastic differential equation (SDE) for a fixed initial point $(\xi, \mathbf{s}) \in \partial\mathbb{H} \times \mathcal{S}$

$$\begin{cases} \xi(t) = \xi + \int_0^t \alpha(\mathbf{s}(s) - \widehat{\xi}(s))dB_s + \int_0^t b(\mathbf{s}(s) - \widehat{\xi}(s))ds \\ \mathbf{s}_j(t) = \mathbf{s}_j + \int_0^t b_j(\mathbf{s}(s) - \widehat{\xi}(s))ds, \quad t \geq 0, \quad 1 \leq j \leq 3N, \end{cases} \tag{1.4}$$

where $\{B_s; s \geq 0\}$ is a one-dimensional Brownian motion with $B_0 = 0$, $\alpha(\mathbf{s})$ (resp. $b(\mathbf{s})$) is a homogneous function on $\mathcal{S}$ of degree 0 (resp. -1) satisfying condition **(L)**, and

$$b_j(\mathbf{s}) := \begin{cases} -2\pi\Im\Psi_{\mathbf{s}}(z_j, 0), & 1 \leq j \leq N, \\ -2\pi\Re\Psi_{\mathbf{s}}(z_j, 0), & N+1 \leq j \leq 2N, \\ -2\pi\Re\Psi_{\mathbf{s}}(z_j', 0), & 2N+1 \leq j \leq 3N. \end{cases} \tag{1.5}$$

It is known (see [CF]) that $b_j(\mathbf{s})$ is a homogeneous function on $\mathcal{S}$ of degree -1 satisfying condition **(L)**.

Putting the solution $(\xi(t), \mathbf{s}(t))$ of (1.4) into the *Komatu-Loewner equation* introduced in [CFR], we consider the equation

$$\frac{d}{dt}g_t(z) = -2\pi\Psi_{\mathbf{s}(t)}(g_t(z), \xi(t)) \quad \text{with } g_0(z) = z \in D. \tag{1.6}$$

The above equation has a unique maximal solution $g_t(z)$, $t \in [0, t_z)$, passing through $G = \bigcup_{t\in[0,\zeta)}\{t\} \times D_t$, where $D_t = D(\mathbf{s}(t))$ and $D = D_0$. Define

$$F_t = \{z \in D : t_z \leq t\}, \quad t \geq 0. \tag{1.7}$$

For $D \in \mathcal{D}$ and an $\mathbb{H}$-hull $A \subset D$, the conformal map f from $D \setminus A$ onto another set in $\mathcal{D}$ satisfying the hydrodynamic normalization at infinity will be called the *canonical map from* $D \setminus A$. The set F_t defined by (1.7) is an $\mathbb{H}$-hull and g_t is the canonical map from $D \setminus F_t$. This family of growing hulls $\{F_t\}$ is denoted by $\mathrm{SKLE}_{\alpha,b}$ and will be called a *stochastic Komatu-Loewner evolution.* SLE_κ can be viewed as a special case of $\mathrm{SKLE}_{\alpha,b}$ where no slit is present, α is constant with $\alpha^2 = \kappa$ and $b = 0$.

For $\mathrm{SKLE}_{\alpha,b}$-hull F_t defined by (1.7), we can consider the canonical Riemann map $g_t^0(z)$ from $\mathbb{H} \setminus F_t$, the half-plane capacity $a(t)$ of F_t relative to g_t^0 and a reparametrization $\{\check{F}_t\}$ of $\{F_t\}$ defined by $\check{F}_t = F_{a^{-1}(2t)}$, $0 \leq t < a(\infty)/2$. With $\mathrm{SKLE}_{\alpha,b}$ reparametrized in this way, it is shown in

Theorem 4.1 of this paper that it has the same distribution as the Schramm-Loewner evolution in $\mathbb{H}$ driven by a continuous semimartingale $\check{U}(t)$. We then prove that, when α is a constant, $\mathrm{SKLE}_{\alpha,b}$ up to some random hitting time and modulo a time change, has the same distribution as SLE_{α^2}, under a suitable Girsanov transformation; see Theorem 4.3. Moreover, we show in Theorem 4.2 that $\mathrm{SKLE}_{\sqrt{6},-b_{\rm BMD}}$, after a reparametrization, has the same distribution as SLE_6, where $b_{\rm BMD}$ is the BMD-domain constant defined by (2.14) that describes the discrepancy of a standard slit domain from $\mathbb{H}$ relative to BMD.

In order to establish Theorem 4.1 with rigor, we need to show that

(C) $g_t^0(z)$ is jointly continuous in $(t,z)\in[0,a]\times(\overline{\mathbb{H}}\setminus F_a)$ for each $a>0$.

A proof of this property will be carried out in Section 3 by combining the probabilistic representation of $\Im g_t^0(z)$ and $\Im g_t(z)$ obtained in [CFR] in terms of the absorbing Brownian motion $Z^{\mathbb{H}}$ on $\mathbb{H}$ and BMD for D with the continuity of $g_t(z)$ in t that is the solution of the ODE (1.6). A key ingredient of the proof is a hitting time analysis for $Z^{\mathbb{H}}$.

It is established in [CF, Theorem 6.11] that $\mathrm{SKLE}_{\sqrt{6},-b_{\rm BMD}}$ enjoys a locality property. In relation to this and the present Theorem 4.2, we will present in Section 5 a first rigorous proof of the locality of the chordal SLE_6 in the sense of [LSW3], and point out the missing pieces or gaps in other locality proofs in literature.

In the final Section 6, we recall and examine Komatu-Loewner equations and stochastic Komatu-Loewner evolutions for other canonical multiply connected domains than the standard slit one.

2 Riemann maps $\{g_t^0\}$ and a process $U(t)$ associated with SKLE

Let $\alpha>0$ and b be homogeneous functions on $\mathcal{S}$ of degree 0 and -1, respectively, that are local Lipschitz continuous. Let $(\xi(t),\mathbf{s}(t)),\ t<\zeta$, be the strong solution of the associated SDE (1.4) and $\{F_t\}$ be $\mathrm{SKLE}_{\alpha,b}$, namely, the family of growing hulls (1.7) on $D=D(\mathbf{s}(0))=\mathbb{H}\setminus K,\ K=\cup_{j=1}^N C_j$, driven by $(\xi(t),\mathbf{s}(t))$.

Denote by g_t the canonical map from $D\setminus F_t$ onto $D_t=D(\mathbf{s}(t))$, Φ the identity map from D into $\mathbb{H}$, and g_t^0 the canonical Riemann map from $\mathbb{H}\setminus F_t$ onto $\mathbb{H}$. According to [CF, Theorem 5.8], $\{F_t\}$ is right continuous with limit $\xi(t)$ in the sense that

$$\bigcap_{\varepsilon>0}\overline{g_t(F_{t+\varepsilon}\setminus F_t)}=\xi(t). \tag{2.1}$$

Define

$$\Phi_t(z)=g_t^0\circ\Phi\circ g_t^{-1}(z)\qquad\text{for } z\in D_t=D(\mathbf{s}(t)). \tag{2.2}$$

Lemma 2.1 *Φ_t admits an analytic extension to $D_t\cup\Pi D_t\cup\partial\mathbb{H}$ by the Schwarz reflection. Here* $\Pi z=\overline{z},\ z\in\mathbb{H}$.

Proof. Take an arbitrary smooth Jordan arc Γ in $\mathbb{H}$ with two end points $z_1,z_2\in\partial\mathbb{H}$ such that the open region V enclosed by Γ and the line segment connecting z_1,z_2 contains the set F_t with $\overline{V}\cap K=\emptyset$. Clearly, $V_t:=g_t(V)$ is the open region enclosed by $g_t(\Gamma)$ and the line segment ℓ_t connecting $g_t(z_i)$, $i=1,2$. In view of (2.1), $\xi(t)$ is located in the interior of the line segment ℓ_t. Furthermore, Φ_t is a Riemann map from the Jordan domain V_t onto the Jordan domain $g_t^0(V)$, which is enclosed by $g_t^0(\Gamma)$ and the line segment ℓ_t^0 connecting $g_t^0(z_i)$, $i = 1, 2$, and Φ_t maps ℓ_t onto ℓ_t^0 homeomorphically. Thus Φ_t admits a Schwarz reflection. $\square$

Define
$$U(t) = \Phi_t(\xi(t)). \tag{2.3}$$
We then have
$$\bigcap_{\varepsilon>0} \overline{g_t^0(F_{t+\varepsilon} \setminus F_t)} = U(t), \tag{2.4}$$
because, by (2.1) and (2.2),
$$\bigcap_{\varepsilon>0} \overline{g_t^0(F_{t+\varepsilon} \setminus F_t)} = \bigcap_{\varepsilon>0} \overline{g_t^0 \circ \Phi(F_{t+\varepsilon} \setminus F_t)} = \bigcap_{\varepsilon>0} \overline{\Phi_t \circ g_t(F_{t+\varepsilon} \setminus F_t)} = \Phi_t(\xi(t)) = U(t).$$

For $D \in \mathcal{D}$ and for an $\mathbb{H}$-hull $A \subset D$, we denote by $\mathrm{Cap}^{\mathbb{H}}(A)$ (resp. $\mathrm{Cap}^D(A)$) the half-plane capacity of A relative to the canonical Riemann map $g_A^{\mathbb{H}}$ from $\mathbb{H} \setminus A$ (resp. the canonical map g_A^D from $D \setminus A$).
$$\mathrm{Cap}^{\mathbb{H}}(A) = \lim_{z\to\infty} z(g_A^{\mathbb{H}}(z) - z), \quad \mathrm{Cap}^D(A) = \lim_{z\to\infty} z(g_A^D(z) - z).$$
Set $a(t) := \mathrm{Cap}^{\mathbb{H}}(F_t)$ and $b(t) := \mathrm{Cap}^D(F_t)$.

Lemma 2.2 *The right derivative of $a(t)$*
$$\frac{d^+a(t)}{dt} := \lim_{\partial\downarrow 0} \frac{a(t+\partial) - a(t)}{\partial} = 2\Phi_t'(\xi(t))^2. \tag{2.5}$$

Proof. For a set $A \subset \mathbb{H}$, we put $\mathrm{rad}(A) = \sup_{z\in A} |z|$. For a fixed $t > 0$, let $K_\varepsilon = g_t(F_{t+\varepsilon} \setminus F_t)$, $\varepsilon > 0$. By [CF, Theorem 5.8 (iii)], $\mathrm{rad}(K_\varepsilon - \xi(t)) \to 0$ as $\varepsilon \to 0$. Hence by the capacity comparison theorem [CF, Theorem 7.1], we have
$$\mathrm{Cap}^{\mathbb{H}}(K_\varepsilon) - \mathrm{Cap}^{D_t}(K_\varepsilon) = o(\varepsilon), \quad \varepsilon \to 0. \tag{2.6}$$
On the other hand, by [L1, (3.8)],
$$a(t+\varepsilon) - a(t) = \mathrm{Cap}^{\mathbb{H}}(g_t^0(F_{t+\varepsilon} \setminus F_t)). \tag{2.7}$$
Since $g_t^0(F_{t+\varepsilon} \setminus F_t) = g_t^0 \circ \Phi(F_{t+\varepsilon} \setminus F_t) = \Phi_t \circ g_t(F_{t+\varepsilon} \setminus F_t) = \Phi_t(K_\varepsilon)$, we obtain from [L1, (4.15)], (2.6) and (2.7)
$$a(t+\varepsilon) - a(t) = \mathrm{Cap}^{\mathbb{H}}(\Phi_t(K_\varepsilon)) = \Phi_t'(\xi(t))^2 \mathrm{Cap}^{\mathbb{H}}(K_\varepsilon) + o(\varepsilon) = \Phi_t'(\xi(t))^2 \mathrm{Cap}^{D_t}(K_\varepsilon) + o(\varepsilon),$$
which can be seen in an analogous manner to (2.7) to be equal to
$$\Phi_t'(\xi(t))^2(\mathrm{Cap}^D(F_{t+\varepsilon}) - \mathrm{Cap}^D(F_t)) + o(\varepsilon) = \Phi_t'(\xi(t))^2(b(t+\varepsilon) - b(t)) + o(\varepsilon).$$
As $b(t) = 2t$ by [CF, Theorem 5.12], we arrive at (2.5). $\square$

Proposition 2.3 *It holds that*

$$\frac{d^+ g_t^0(z)}{dt} = \frac{2\Phi_t'(\xi(t))^2}{g_t^0(z) - U(t)}, \quad z \in \mathbb{H} \setminus F_t. \tag{2.8}$$

in the right derivative sense.

Proof. Denote by Q the family of all $\mathbb{H}$-hulls. According to [L1, p69, Propositon 3.46],

$$g_{A-x}^{\mathbb{H}}(z) = g_A^{\mathbb{H}}(z+x) - x, \quad \mathrm{Cap}^{\mathbb{H}}(A-x) = \mathrm{Cap}^{\mathbb{H}}(A) \quad A \in Q,\ x \in \mathbb{R}, \tag{2.9}$$

and, there exists a constant $c > 0$ such that, for any $A \in Q$ and any z with $|z| \geq 2\mathrm{rad}(A)$,

$$\left| z - g_A^{\mathbb{H}}(z) + \frac{\mathrm{Cap}^{\mathbb{H}}(A)}{z} \right| \leq c \frac{\mathrm{rad}(A)\mathrm{Cap}^{\mathbb{H}}(A)}{|z|^2}. \tag{2.10}$$

For $z \in \mathbb{H} \setminus F_s$, we get from (2.7), (2.9) and (2.10)

$$\begin{aligned}
& g_{s+\varepsilon}^0(z) - g_s^0(z) \\
= \; & g_{g_s^0(F_{s+\varepsilon} \setminus F_s)}^{\mathbb{H}}(g_s^0(z)) - g_s^0(z) \\
= \; & g_{g_s^0(F_{s+\varepsilon} \setminus F_s) - U(s)}^{\mathbb{H}}(g_s^0(z) - U(s)) - (g_s^0(z) - U(s)) \\
= \; & \frac{a(s+\varepsilon) - a(s)}{g_s^0(z) - U(s)} + \mathrm{rad}(g_s^0(F_{s+\varepsilon} \setminus F_s) - U(s))(a(s+\varepsilon) - a(s)) O(1/(g_s^0(z) - U(s))^2)).
\end{aligned}$$

The formula (2.8) now follows from (2.4) and (2.5). □

To show that the right derivative in Proposition 2.3 can be strengthened to true derivative, we need the following proposition, whose proof is postponed to next section.

Proposition 2.4 *The Riemann maps* $\{g_t^0\}$ *enjoys the property* **(C)** *stated in Section 1.*

In the rest of this section, we shall take the validity of this proposition for granted. The following lemma can then be shown exactly in the same way as the proof of [CF, Proposition 6.7 (i)].

Lemma 2.5 $\Phi_t(z),\ \Phi_t'(z),\ \Phi_t''(z)$ *are jointly continuous in* $(t, z) \in [0, \zeta) \times (D_t \cup \partial\mathbb{H})$.

By the property **(C)** and the above lemma, the right hand side of (2.8) becomes continuous in t and so [L1, Lemma 4.3] applies in getting the following thoerem.

Theorem 2.6 $g_t^0(z)$ *is continuously differentiable in* t *and* (2.8) *becomes a genuine ODE:*

$$\frac{dg_t^0(z)}{dt} = \frac{2\Phi_t'(\xi(t))^2}{g_t^0(z) - U(t)}, \quad z \in \mathbb{H} \setminus F_t. \tag{2.11}$$

Remark 2.7 Strengthening from right time derivative in Proposition 2.3 to the genuine time derivative in Theorem 2.6 is very important since (2.8) does not uniquely characterize the conformal maps $\{g_t^0(z)\}$. This is because while the solution to (2.11) is unique, equation (2.8) may have numerous solutions. To see this, consider the case that $K = \emptyset$, that is, upper half space $\mathbb{H}$ with no slits. In this case, $\Phi_t(z) = z$ and (2.11) is the chordal Loewner equation with driving function $U(t)$. So for each $z \in \mathbb{H}$,

$$\frac{dg_t^0(z)}{dt} = \frac{2}{g_t^0(z) - U(t)}, \quad z(t) = z, \tag{2.12}$$

has a unique continuous solution $g_t^0(z)$ up to time t_z when g_t^0 and $U(t)$ collide. However, equation

$$\frac{d^+ z(t)}{dt} = \frac{2}{z(t) - U(t)}, \quad z(t) = z, \tag{2.13}$$

has infinitely many solutions. For instance, take any $\varepsilon \in (0, \zeta_z)$ and define $z(t) = g_t^0(z)$ for $t \in (0, \varepsilon]$. Let $z(\varepsilon)$ be any value in $\mathbb{H}$. Let $\widetilde{g}_t^0(z(\varepsilon))$, $0 \le t < t_{z(\varepsilon)}$ be the unique solution of

$$\frac{d\widetilde{g}_t^0(z(\varepsilon))}{dt} = \frac{2}{\widetilde{g}_t^0(z(\varepsilon)) - U(t+\varepsilon)}, \quad \widetilde{g}_0^0(z(\varepsilon)) = z(\varepsilon).$$

Define $z(t) = \widetilde{g}_{t-\varepsilon}^0(z(\varepsilon))$ for $t \in [\varepsilon, \varepsilon + t_{z(\varepsilon)})$. Then $\{z(t); 0 \le t < \varepsilon + t_{z(\varepsilon)}\}$ is a solution to equation (2.13). Indeed, we see by [L1, Lemma 4.3] that the solution $z(t)$ of (2.13) coincides with the solution $g_t^0(z)$ of (2.12) if and only if $z(t)$ is (left) continuous. □

For $\mathbf{s} \in \mathcal{S}$, let $b_{\rm BMD}(\mathbf{s})$ be the BMD-domain constant for the slit domain $D(\mathbf{s})$ introduced in [CF, §6.1]:

$$b_{\rm BMD}(\mathbf{s}) = 2\pi \lim_{z\to 0} \left(\Psi_{\mathbf{s}}(z, 0) + \frac{1}{\pi z} \right). \tag{2.14}$$

Theorem 2.8 *The process $U(t)$ on $\partial\mathbb{H}$ admits a semi-martingale decomposition*

$$\begin{aligned} dU(t) &= \Phi_t'(\xi(t))\alpha(\mathbf{s}(t) - \widehat{\xi}(t))dB_t + \Phi_t'(\xi(t)) \left(b_{\rm BMD}(\mathbf{s}(t) - \widehat{\xi}(t)) + b(\mathbf{s}(t) - \widehat{\xi}(t)) \right) dt \\ &\quad + \Phi_t''(\xi(t)) \left(-3 + \frac{1}{2}\alpha(\mathbf{s}(t) - \widehat{\xi}(t))^2 \right) dt. \end{aligned} \tag{2.15}$$

Proof. For a differentiable function $f_t(z) := f(t, z)$ defined on on open subset of $\mathbb{R}_+ \times \mathbb{C}$, we will use $\dot{f}$ and f' to denote its partial derivative in t and in $z \in \mathbb{C}$, respectively. Let $f_t(z) = g_t^{-1}(z)$. Then

$$\dot{f}_t(z) = 2\pi f_t'(z)\Psi_{\mathbf{s}(t)}(z, \xi(t)), \quad z \in D_t,$$

and $\Phi_t = g_t^0 \circ \Phi \circ f_t$ by (2.2). Thus by (1.6) and Theorem 2.6, for $z \in D_t$,

$$\begin{aligned} \dot{\Phi}_t(z) &= \dot{g}_t^0(f_t(z)) + (g_t^0)'(f_t(z))\dot{f}_t(z) \\ &= \frac{2\Phi_t'(\xi(t))^2}{g_t^0(f_t(z)) - U(t)} + (g_t^0)'(f_t(z)) \cdot 2\pi f_t'(z)\Psi_{\mathbf{s}(t)}(z, \xi(t)) \\ &= \frac{2\Phi_t'(\xi(t))^2}{\Phi_t(z) - \Phi_t(\xi(t))} + 2\pi\Phi_t'(z)\Psi_{\mathbf{s}(t)}(z, \xi(t)). \end{aligned} \tag{2.16}$$

In view of Lemma 2.5, by an argument similar to that in the paragraphs below (6.32) of [CF], we can deduce from (2.16) that $\Phi_t(z)$ is differentiable in t for every $z \in \partial\mathbb{H}$, and $\dot{\Phi}_t(z)$ is jointly continuous in $(t, z) \in (0, \infty) \times \partial\mathbb{H}$. Since $\xi(t)$ is the solution of the SDE (1.4), the above joint continuity together with Lemma 2.5 allows us to apply a generalized Itô formula to $U_t = \Phi_t(\xi(t))$; see Remark 2.9 below. We thus get

$$
\begin{aligned}
dU(t) &= \dot{\Phi}_t(\xi(t))dt + \Phi_t'(\xi(t))\left(\alpha(\mathbf{s}(t)-\widehat{\xi}(t))dB_t + b(\mathbf{s}(t)-\widehat{\xi}(t))dt\right) \\
&\quad + \frac{1}{2}\Phi_t''(\xi(t))\alpha(\mathbf{s}(t)-\widehat{\xi}(t))^2 dt
\end{aligned}
$$

An argument similar to that in the paragraphs below (6.32) of [CF] also yields the identity

$$
\dot{\Phi}_t(\xi(t)) = \lim_{z\to\xi(t),\, z\in D_t} \dot{\Phi}_t(z).
$$

Rewriting the right hand side of (2.16) as

$$
\left(\frac{2\Phi_t'(\xi(t))^2}{\Phi_t(z)-\Phi_t(\xi(t))} - \frac{2\Phi_t'(\xi(t))}{z-\xi(t)}\right) + 2\pi\Phi_t'(\xi(t))\left(\Psi_{\mathbf{s}(t)}(z,\xi(t)) + \frac{1}{\pi}\frac{1}{z-\xi(t)}\right),
$$

we obtain from (2.16) and [CF, Lemma 6.1]

$$
\dot{\Phi}_t(\xi(t)) = -3\Phi_t''(\xi(t)) + \Phi_t'(\xi(t))\, b_{\mathrm{BMD}}(\mathbf{s}(t)-\widehat{\xi}(t)).
$$

Therefore

$$
\begin{aligned}
dU(t) &= \left(-3\Phi_t''(\xi(t)) + \Phi_t'(\xi(t)) b_{\mathrm{BMD}}(\xi(t)-\widehat{\xi}(t))\right) dt \\
&\quad + \Phi_t'(\xi(t))\left(\alpha(\mathbf{s}(t)-\widehat{\xi}(t))dB_t + b(\mathbf{s}(t)-\widehat{\xi}(t))dt\right) + \frac{1}{2}\Phi_t''(\xi(t))\alpha(\mathbf{s}(t)-\widehat{\xi}(t))^2 dt,
\end{aligned}
$$

which is (2.15). □

Remark 2.9 (A generalized Itô formula) Exercise (IV.3.12) in the book [RY] formulates a generalized Itô formula for $g(X_t,\omega,t)$, the composition of an adapted random function $g(x,\omega,t)$, $x\in\mathbb{R}$, $t\geq 0$, and a continuous semimartingale X:

$$
dg(X_t,t) = g_x(X_t,t)dX_t + g_t(X_t,t)dt + \frac{1}{2}g_{xx}(X_t,t)d\langle X\rangle_t. \tag{2.17}
$$

We like to point out that in addition to the conditions i), ii), iii) and iv) stated in [RY, Exercise IV.3.12], the following condition

v) $g_x(x,\omega,t)$, $g_{xx}(x,\omega,t)$ and $g_t(x,\omega,t)$ are locally bounded in (x,t)

should also be required for the validity of (2.17) (a private communication by Masanori Hino). Of course, if these partial derivatives are jointly continuous in (x,t), then condition v) is satisfied. This type of generalized Itô formula has been frequently utilized in the literatures on SLE by referring to [RY, (IV.3.12)] but without verifying condition v) which is by no means trivial. This is part of the reasons why we spent considerable efforts in [CF] to establish the joint continuity of certain functions such as those summarized in Lemma 2.5 and of the function $\dot{\Phi}_t(z)$, $z\in\partial\mathbb{H}$, derived from the identity (2.16). □

3 Proof of property (C)

In this section, we present a proof of Proposition 2.4, using the probabilistic representation of $\Im g_t(z)$ in [CFR, Theorem 7.2] as well as that of $\Im g_t^0(z)$ obtained from [CFR, Theorem 7.2] by taking $D = \mathbb{H}$.

Recall that $g_t(z),\ t \in [0, t_z)$, is the unique solution of (1.6) with the maximal interval $[0, t_z)$ of existence, and $F_t = \{z \in D : t_z \le t\}$. We know that $g_t(z)$ is continuous in t, and g_t is the canonical map from $D \setminus F_t$. Let $G_t = \{z \in D : t_z < t\}$. Then

$$\bigcap_{s>t} F_s = F_t, \qquad \bigcup_{s<t} F_s = G_t. \tag{3.1}$$

Let g_t^0 be the canonical Riemann map from $\mathbb{H} \setminus F_t$. By virtue of Theorem 7.2 of [CFR] with $D = \mathbb{H}$ (see also [L1, (3.5)]), $\Im g_t^0(z)$ admits the expression

$$\Im g_t^0(z) = \Im z - \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{F_t}} : \sigma_{F_t} < \infty\right],\ z \in \mathbb{H} \setminus F_t, \tag{3.2}$$

where $Z^{\mathbb{H}} = (Z_t^{\mathbb{H}}, \zeta^{\mathbb{H}}, \mathbb{P}_z^{\mathbb{H}})$ is the absorbing Brownian motion (ABM) on $\mathbb{H}$, and $\sigma_{F_t} := \inf\{s > 0 : Z_s^{\mathbb{H}} \in F_t\}$.

Lemma 3.1 *Fix $a > 0$. $\Im g_t^0(z)$ is continuous in $t \in [0, a]$ for each $z \in \mathbb{H} \setminus F_a$ if and only if*

$$\mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{G_t}}; \sigma_{G_t} < \infty\right] = \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{F_t}}; \sigma_{F_t} < \infty\right], \tag{3.3}$$

for $t \in (0, a]$ and $z \in \mathbb{H} \setminus F_a$.

Proof. Since $\sigma_{F_s} \downarrow \sigma_{G_t}$ as $s \uparrow t$ by (3.1) (cf. [BG, Chapter 1, (10.4)]), we see from (3.2) that (3.3) is equivalent to the left continuoity of $\Im g_t^0(z)$ in t. On the other hand, $\Im g_t^0(z)$ is right continuous in t because $g_t^0(z)$ is right differentiable in t by Proposition 2.3. $\square$

Let $K = \bigcup_{j=1}^N C_j$ and $v_t^*(z) = \Im g_t(z)$. Denote by $Z^{\mathbb{H},*} = (Z_t^{\mathbb{H},*}, \mathbb{P}_z^{\mathbb{H},*})$ the BMD on $D^* = D \cup K^*$ with $K^* = \{c_1^*, \dots, c_N^*\}$ obtained from the ABM $Z^{\mathbb{H}}$ by shorting each slit C_i as a single point c_i^*. According to [CFR, Theorem 7.2], $v_t^*(z)$ can be expressed in terms of the ABM $Z^{\mathbb{H}}$ and BMD $Z^{\mathbb{H},*}$ as follows:

$$v_t^*(z) = v_t(z) + \sum_{j=1}^N \mathbb{P}_z^{\mathbb{H}}\left(\sigma_K < \sigma_{F_t},\ Z^{\mathbb{H}}_{\sigma_K} \in C_j\right) v_t^*(c_j^*), \quad z \in D \setminus F_t, \tag{3.4}$$

where

$$v_t(z) = \Im z - \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{F_t \cup K}}; \sigma_{F_t \cup K} < \infty\right], \tag{3.5}$$

$$v_t^*(c_i^*) = \sum_{j=1}^N \frac{M_{ij}(t)}{1 - R_i^*(t)} \int_{\eta_j} v_t(z)\nu_j(dz), \quad 1 \le i \le N. \tag{3.6}$$

Here $\eta_1, \cdots, \eta_N$ are mutually disjoint smooth Jordan curves surrounding $C_1, \cdots, C_N$, respectively,

$$\nu_i(dz) = \mathbb{P}^{\mathbb{H},*}_{c_i^*}\left(Z^{\mathbb{H},*}_{\sigma_{\eta_i}} \in dz\right), \quad 1 \le i \le N, \tag{3.7}$$

$$R_i^*(t) = \int_{\eta_i} \mathbb{P}_z^{\mathbb{H}}\left(\sigma_K < \sigma_{F_t},\ Z^{\mathbb{H}}_{\sigma_K} \in C_i\right) \nu_i(dz), \quad 1 \le i \le N, \tag{3.8}$$

and $M_{ij}(t)$ is the (i,j)-entry of the matrix $M(t) = \sum_{n=0}^{\infty}(Q^*(t))^n$ for a matrix $Q^*(t)$ with entries

$$q^*_{ij}(t) = \begin{cases} \mathbb{P}^{\mathbb{H},*}_{c^*_i}\left(\sigma_{K^*} < \sigma_{F_t},\ Z^{\mathbb{H},*}_{\sigma_{K^*}} = c^*_j\right)/(1 - R^*_i(t)) & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases} \quad 1 \leq i, j \leq N. \tag{3.9}$$

Lemma 3.2 *For every $1 \leq j \leq N$,*

$$\sup_{0 \leq t \leq a} v^*_t(c^*_j) < \infty \quad \text{for each } a > 0, \tag{3.10}$$

and

$$v^*_t(c^*_j) > 0 \quad \textit{for every } t > 0 \textit{ and } \ 1 \leq j \leq N. \tag{3.11}$$

Proof. For $0 \leq t \leq a$ and $1 \leq i \leq N$, let

$$\lambda_i(t) = \sum_{j=1}^{N} q^*_{ij}(t) \quad \text{and} \quad \gamma_i(t) = \mathbb{P}^{\mathbb{H},*}_{c^*_i}\left(\sigma_{K^*} < \sigma_{F_t},\ Z^{\mathbb{H},*}_{\sigma_{K^*}} \neq c^*_i\right), \quad 1 \leq i \leq N.$$

Note that $\lambda_i(t) = \gamma_i(t)/(1 - R^*_i(t))$ and

$$1 - R^*_i(t) = \gamma_i(t) + \int_{\eta_i} \mathbb{P}^{\mathbb{H}}_z\left(\sigma_{F_t} < \sigma_K\right)\nu_i(dz) + \int_{\eta_i} \mathbb{P}^{\mathbb{H}}_z\left(\sigma_{F_t \cup K} = \infty\right)\nu_i(dz). \tag{3.12}$$

Therefore

$$\begin{aligned} 1 - \lambda_i(t) &= \frac{1 - R^*_i(t) - \gamma_i(t)}{1 - R^*_i(t)} \geq \int_{\eta_i} \mathbb{P}^{\mathbb{H}}_z(\sigma_{F_t \cup K} = \infty)\nu_i(dz) \\ &\geq \inf_{1 \leq j \leq N} \int_{\eta_j} \mathbb{P}^{\mathbb{H}}_z(\sigma_{F_t \cup K} = \infty)\nu_j(dz) =: \delta_0 > 0. \end{aligned}$$

Hence $\lambda_i(t) \leq 1 - \delta_0$. Consequently, $(Q^*(t))^n\mathbf{1} \leq (1 - \delta_0)^n\mathbf{1}$ and so $M\mathbf{1} \leq \delta_0^{-1}\mathbf{1}$. Therefore we have by (3.6) and (3.12) that $v^*_t(c^*_i) \leq \sum_{j=1}^{N} \delta_0^{-2} m_j$ for all $t \in [0, a]$, where m_j is the maximum of the y-coordinate of points in η_j.

On the other hand, (3.6) implies $v^*_t(c^*_i) \geq \int_{\eta_i} v_t(z)\nu_i(dz)$. In view of (3.5), $v_t(z)$ is a non-negative harmonic function on $\mathbb{H} \setminus (F_t \cup K)$ that is strictly positive when $\Im z$ is large. Hence $v_t(z) > 0$ for any $z \in \mathbb{H} \setminus (F_t \cup K)$ and $t > 0$, yielding (3.11). □

Proposition 3.3 *The identity* (3.3) *holds, and so $\Im g^0_t(z)$ is continuous in $t \in [0, a]$ for every $z \in \mathbb{H} \setminus F_a$ and $a > 0$.*

Proof. Note that $v^*_t(z) = \Im g_t(z)$ is continuous in t since so is $g_t(z)$. By (3.4)-(3.5), for $z \in D \setminus F_t$,

$$v^*_t(z) = \Im z - \mathbb{E}^{\mathbb{H}}_z\left[\Im Z^{\mathbb{H}}_{\sigma_{F_t \cup K}}; \sigma_{F_t \cup K} < \infty\right] + \sum_{j=1}^{N} \mathbb{P}^{\mathbb{H}}_z\left(\sigma_K < \sigma_{F_t}, Z^{\mathbb{H}}_{\sigma_K} \in C_j\right) v^*_t(c^*_j). \tag{3.13}$$

For each fixed $t \in (0, a]$ and any sequence t_n increasing to t, by (3.10), there is a subsequence t_{n_k} such that $\lim_{k \to \infty} v^*_{t_{n_k}}(c^*_j) = a_j \in [0, \infty)$. Since $F_{t_n} \uparrow G_t$, we have

$$v^*_t(z) = \lim_{k \to \infty} v^*_{t_{n_k}}(z) = \Im z - \mathbb{E}^{\mathbb{H}}_z\left[\Im Z^{\mathbb{H}}_{\sigma_{G_t \cup K}}; \sigma_{G_t \cup K} < \infty\right] + \sum_{j=1}^{N} \mathbb{P}^{\mathbb{H}}_z\left(\sigma_K < \sigma_{G_t}, Z^{\mathbb{H}}_{\sigma_K} \in C_j\right) a_j. \tag{3.14}$$

Taking $z \to C_j$ in (3.13) and (3.14) yields $a_j = v_t^*(c_j^*)$ for each $1 \le j \le N$. Thus we have from (3.13) and (3.14) that

$$\begin{aligned}
&\mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{G_t\cup K}}; \sigma_{G_t\cup K} < \infty\right] - \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{F_t\cup K}}; \sigma_{F_t\cup K} < \infty\right] \\
= &\sum_{j=1}^{N}\left(\mathbb{P}_z^{\mathbb{H}}\left(\sigma_K < \sigma_{G_t}, Z^{\mathbb{H}}_{\sigma_K} \in C_j\right) - \mathbb{P}_z^{\mathbb{H}}\left(\sigma_K < \sigma_{F_t}, Z^{\mathbb{H}}_{\sigma_K} \in C_j\right)\right) v_t^*(c_j^*). \qquad (3.15)
\end{aligned}$$

Each term on the right hand side of (3.15) is non-negative since $G_t = F_{t-} \subset F_t$. On the other hand, $\Im z$ is a positive harmonic in $\mathbb{H}$ and so $\Im Z^{\mathbb{H}}_t$ is a non-negative supermartingale. By the optional sampling theorem, we have for every $z \in \mathbb{H}$ and any stopping time T, we have

$$\Im z \ge \mathbb{E}_z^{\mathbb{H}}[\Im Z^{\mathbb{H}}_T; T < \infty]. \qquad (3.16)$$

Since $\sigma_{G_t\cup K} \ge \sigma_{F_t\cup K}$, we have

$$\Im z \ge \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{F_t\cup K}}; \sigma_{F_t\cup K} < \infty\right] \ge \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{G_t\cup K}}; \sigma_{G_t\cup K} < \infty\right] \ge 0,$$

where in the second inequality we used the strong Markov property of $Z^{\mathbb{H}}$ at stopping time $\sigma_{F_t\cup K}$ and (3.16). Thus both sides of (3.15) have to be identically zero. As $v_t^*(c_j^*) > 0$ for each $1 \le j \le N$ by (3.11), we must have for $z \in D \setminus F_t$,

$$\mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{F_t\cup K}}; \sigma_{F_t\cup K} < \infty\right] = \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{G_t\cup K}}; \sigma_{G_t\cup K} < \infty\right], \qquad (3.17)$$

and

$$\mathbb{P}_z^{\mathbb{H}}\left(\sigma_K < \sigma_{G_t}, Z^{\mathbb{H}}_{\sigma_K} \in C_j\right) = \mathbb{P}_z^{\mathbb{H}}\left(\sigma_K < \sigma_{F_t}, Z^{\mathbb{H}}_{\sigma_K} \in C_j\right) \quad \text{for every } 1 \le j \le N. \qquad (3.18)$$

It follows from the above two displays that for $z \in \mathbb{H} \setminus (K \cup F_t)$,

$$\mathbb{P}_z^{\mathbb{H}}(\sigma_K < \sigma_{F_t}) = \mathbb{P}_z^{\mathbb{H}}(\sigma_K < \sigma_{G_t}) \quad \text{and} \quad \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{F_t}}; \sigma_{F_t} < \sigma_K\right] = \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{G_t}}; \sigma_{G_t} < \sigma_K\right]. \quad (3.19)$$

Take a bounded smooth domain $V \subset \mathbb{H}$ such that $K \subset V$ and $V \cap F_t = \emptyset$. Let $\Gamma = \partial V$. Define $\sigma_1 = \sigma_K$, $\tau_1 = \inf\{t \ge \sigma_1 : Z^{\mathbb{H}}_t \in \Gamma\}$, and for $n \ge 1$,

$$\sigma_{n+1} = \inf\{t > \tau_n : Z^{\mathbb{H}}_t \in K\}, \quad \tau_{n+1} = \inf\{t > \sigma_{n+1} : Z^{\mathbb{H}}_t \in \Gamma\}.$$

We claim that the following holds for every $n \ge 1$ and $z \in \mathbb{H} \setminus (K \cup F_t)$,

$$\mathbb{P}_z^{\mathbb{H}}(\sigma_n < \sigma_{F_t}) = \mathbb{P}_z^{\mathbb{H}}(\sigma_n < \sigma_{G_t}) \quad \text{and} \quad \mathbb{P}_z^{\mathbb{H}}(\tau_n < \sigma_{F_t}) = \mathbb{P}_z^{\mathbb{H}}(\tau_n < \sigma_{G_t}). \qquad (3.20)$$

We prove this by induction. Clearly the first identity in (3.20) holds for $n = 1$ by (3.19), while by the continuity of the sample paths of $Z^{\mathbb{H}}$,

$$\mathbb{P}_z^{\mathbb{H}}(\tau_1 < \sigma_{F_t}) = \mathbb{P}_z^{\mathbb{H}}(\sigma_K < \sigma_{F_t}) = \mathbb{P}_z^{\mathbb{H}}(\sigma_K < \sigma_{G_t}) = \mathbb{P}_z^{\mathbb{H}}(\tau_1 < \sigma_{G_t}).$$

So (3.20) holds for $n = 1$. Assume that (3.20) holds for $n \ge 1$. Then by the strong Markov property of $Z^{\mathbb{H}}$ and (3.19),

$$\begin{aligned}
&\mathbb{P}_z^{\mathbb{H}}(\sigma_{n+1} < \sigma_{F_t}) = \mathbb{P}_z^{\mathbb{H}}\left(\tau_n + \sigma_K \circ \theta_{\tau_n} < \sigma_{F_t},\ \tau_n < \sigma_{F_t}\right) \\
= \ &\mathbb{E}_z^{\mathbb{H}}\left[\mathbb{P}^{\mathbb{H}}_{Z_{\tau_n}}(\sigma_K < \sigma_{F_t}); \tau_n < \sigma_{F_t}\right] = \mathbb{E}_z^{\mathbb{H}}\left[\mathbb{P}^{\mathbb{H}}_{Z_{\tau_n}}(\sigma_K < \sigma_{G_t}); \tau_n < \sigma_{G_t}\right] = \mathbb{P}_z^{\mathbb{H}}(\tau_{n+1} < \sigma_{G_t}),
\end{aligned}$$

and by the continuity of $Z^{\mathbb{H}}$,

$$\begin{aligned} & \mathbb{P}_z^{\mathbb{H}}(\tau_{n+1} < \sigma_{F_t}) = \mathbb{P}_z^{\mathbb{H}}\left(\sigma_{n+1} + \tau_\Gamma \circ \theta_{\sigma_{n+1}} < \sigma_{F_t}\right) = \mathbb{P}_z^{\mathbb{H}}\left(\sigma_{n+1} < \sigma_{F_t}\right) \\ = \ & \mathbb{P}_z^{\mathbb{H}}\left(\sigma_{n+1} < \sigma_{G_t}\right) = \mathbb{P}_z^{\mathbb{H}}(\tau_{n+1} < \sigma_{G_t}). \end{aligned}$$

Hence (3.20) holds for $n+1$ and so for all $n \geq 1$ by induction.

Now, by the strong Markov property of $Z^{\mathbb{H}}$, (3.20) and (3.19), we have for $z \in \mathbb{H} \setminus (K \cup F_t)$

$$\begin{aligned} \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{F_t}}; \sigma_{F_t} < \infty\right] &= \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{F_t}}; \sigma_{F_t} < \sigma_K\right] + \sum_{n=1}^{\infty} \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{F_t}}; \sigma_n < \sigma_{F_t} < \sigma_{n+1}\right] \\ &= \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{F_t}}; \sigma_{F_t} < \sigma_K\right] + \sum_{n=1}^{\infty} \mathbb{E}_z^{\mathbb{H}}\left[\mathbb{E}^{\mathbb{H}}_{Z_{\tau_n}}[\Im Z^{\mathbb{H}}_{\sigma_{F_t}}; \sigma_{F_t} < \sigma_K]; \tau_n < \sigma_{F_t}\right] \\ &= \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{G_t}}; \sigma_{G_t} < \sigma_K\right] + \sum_{n=1}^{\infty} \mathbb{E}_z^{\mathbb{H}}\left[\mathbb{E}^{\mathbb{H}}_{Z_{\tau_n}}[\Im Z^{\mathbb{H}}_{\sigma_{G_t}}; \sigma_{G_t} < \sigma_K]; \tau_n < \sigma_{G_t}\right] \\ &= \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{G_t}}; \sigma_{G_t} < \sigma_K\right] + \sum_{n=1}^{\infty} \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{G_t}}; \sigma_n < \sigma_{G_t} < \sigma_{n+1}\right] \\ &= \mathbb{E}_z^{\mathbb{H}}\left[\Im Z^{\mathbb{H}}_{\sigma_{G_t}}; \sigma_{G_t} < \infty\right]. \end{aligned}$$

This establishes (3.3). The rest of the claim follows from Lemma 3.1. □

For $0 \leq s < t \leq a$, define $g^0_{t,s} = g^0_s \circ (g^0_t)^{-1}$, which is a conformal map from $\mathbb{H}$ onto $\mathbb{H} \setminus g^0_s(F_t \setminus F_s)$. Its inverse $(g^0_{t,s})^{-1}$ is the canonical Riemann map from $\mathbb{H} \setminus g^0_s(F_t \setminus F_s)$. Let $\ell_{t,s}$ be the set of all limitting points of $(g^0_{t,s})^{-1} \circ g^0_s(z) = g^0_t(z)$ as z approaches to $F_t \setminus F_s$. Then $\ell_{t,s}$ is a compact subset of $\partial\mathbb{H}$ and $(g^0_{t,s})^{-1}$ sends $\partial\mathbb{H} \cup \overline{g^0_s(F_t \setminus F_s)}$ into $\partial\mathbb{H}$ homeomorphically.

Let $\Lambda = \{x + iy : a < x < b,\ 0 < y < c\}$ be a finite rectangle such that $\ell_{t,s} \subset \{x + i0+ : a < x < b\}$. Then $\Im g^0_{t,s}(z) \leq \Im (g^0_t)^{-1}(z)$ by (3.2) that is uniformly bounded in $z \in \Lambda$ so that it admits finite limit

$$\Im g^0_{t,s}(x + i0+) = \lim_{y \downarrow 0} \Im g^0_{t,s}(x + iy) \quad \text{for a.e. } x \in (a, b). \tag{3.21}$$

The following lemma can be established in a similar way as that of [CF, Lemma 6.3]. We omit its proof here.

Lemma 3.4 *For $0 \leq s < t \leq a$, it holds that*

$$a(t) - a(s) = \frac{1}{\pi} \int_{\ell_{t,s}} \Im g^0_{t,s}(x + i0+)\ dx, \tag{3.22}$$

$$g^0_t(z) - g^0_s(z) = -\frac{1}{\pi} \int_{\ell_{t,s}} \frac{1}{g^0_t(z) - x} \Im g^0_{t,s}(x + i0+)\ dx, \quad z \in \mathbb{H} \setminus F_t. \tag{3.23}$$

Proof of Proposition 2.4. We know from Proposition 3.3 that $\Im g^0_t(z)$ is continuous in $t \in [0, a]$ for each $z \in \mathbb{H} \setminus F_a$. As $\Im g^0_t(z)$ is harmonic in $z \in \mathbb{H} \setminus F_a$, it is jointly continuous in $(t, z) \in [0, a] \times (\mathbb{H} \setminus F_a)$. By Lemma 3.4, we have

$$|g^0_s(z)| \leq |g^0_t(z)| + \sup_{x \in \ell_{t,s}} \frac{a(t)}{|g^0_t(z) - x|}, \quad s \in [0, t].$$

Therefore we can show as in the proof of [CFR, Theorem 7.4] that $g_t^0(z)$ is locally equi-continuous and locally uniformly bounded. The joint continuity of $g_t^0(z)$ then follows as in the proof of [CF, Lemma 6.5]. $\square$

4 Basic relations between $\mathrm{SKLE}_{\alpha,b}$ and SLE

In view of [CF, (7.20)] applied to the case $D = \mathbb{H}$ (see also [L1, (3.7)]), the half-plane capacity $a(t)$ of the hull F_t relative to g_t^0 admits the expression

$$a(t) = \frac{2R}{\pi} \int_0^\pi \mathbb{E}^{\mathbb{H}}_{Re^{i\theta}} \left[\Im Z^{\mathbb{H}}_{\sigma_{F_t}} : \sigma_{F_t} < \infty \right] d\theta,$$

in terms of the ABM $Z^{\mathbb{H}}$ on $\mathbb{H}$ for $R > 0$ with $F_t \subset \{z \in \mathbb{H} : |z| < R\}$. Since the SKLE $\{F_t\}$ is strictly increasing in t by virtue of [CF, Theorem 5.8], we can see as in the proof of [CF, Lemma 5.15] that $a(t)$ is strictly increasing in t.

By Lemma 2.2 and Lemma 2.5,

$$a(t) = 2 \int_0^t |\Phi_s'(\xi(s))|^2 ds. \tag{4.1}$$

We reparametrize the SKLE hulls $\{F_t\}$ by the inverse function a^{-1} of a and define

$$\check{F}_t = F_{a^{-1}(2t)}, \qquad 0 \le t < \tau_0 := a(\infty)/2. \tag{4.2}$$

Accordingly, the associated Riemann maps $\{g_t^0\}$ and the process $U(t)$ are time changed into

$$\check{g}_t^0 = g^0_{a^{-1}(2t)}, \quad \check{U}(t) = U(a^{-1}(2t)), \quad 0 \le t < \tau_0. \tag{4.3}$$

It then follows from (2.11) that $z(t) = \check{g}_t^0(z)$ is a solution of the Loewner equation

$$\frac{d}{dt} z(t) \;=\; \frac{2}{z(t) - \check{U}(t)}, \quad z(0) = z \in \mathbb{H}. \tag{4.4}$$

Theorem 4.1 $\{\check{F}_t; t \in [0, \tau_0)\}$ *has the same law as the Loewner evolution driven by the path of the continuous process* $\check{U}(t)$ *up to the random time* τ_0*; namely, for the unique solution* $z(t)$, $0 \le t < t_z$, *of* (4.4),

$$\{\check{F}_t; t \in [0, \tau_0)\} \textit{ has the same distribution as } \{\{z \in \mathbb{H} : t_z \le t\}; t \in [0, \tau_0)\}. \tag{4.5}$$

Let $M_t = \int_0^t \Phi_s'(\xi(s)) dB_s$. By (4.1), $\langle M \rangle_t = \int_0^t \Phi_s'(\xi(s))^2 ds = a(t)/2$ so that $\check{B}_t = M_{a^{-1}(2t)}$ is a Browninan motion. The formula (2.15) can be rewritten as

$$\begin{aligned} \check{U}(t) \;&=\; \xi + \int_0^t \widetilde{\Phi}_s'(\widetilde{\xi}(s))^{-1} \left(b(\widetilde{\mathbf{s}}(s) - \widehat{\widetilde{\xi}}(s)) + b_{\mathrm{BMD}}(\widetilde{\mathbf{s}}(s) - \widehat{\widetilde{\xi}}(s)) \right) ds \\ &\quad + \frac{1}{2} \int_0^t \widetilde{\Phi}_s''(\widetilde{\xi}(s)) \cdot \widetilde{\Phi}_s'(\widetilde{\xi}(s))^{-2} \left(\alpha(\widetilde{\mathbf{s}}(s) - \widehat{\widetilde{\xi}}(s))^2 - 6 \right) ds \\ &\quad + \int_0^t \alpha(\widetilde{\mathbf{s}}(s) - \widehat{\widetilde{\xi}}(s)) d\check{B}_s, \end{aligned} \tag{4.6}$$

where $\widetilde{\Phi}'_s(z) := \Phi'_{a^{-1}(2s)}(z)$, $\widetilde{\Phi}''_s(z) := \Phi''_{a^{-1}(2s)}(z)$, $\widetilde{\xi}(t) := \xi(a^{-1}(2t))$ and $\widetilde{\mathbf{s}}_j(t) = \mathbf{s}_j(a^{-1}(2t))$ for $1 \le j \le 3N$. Note that since $\Phi_t(z)$ is univalent in z on the region $D_t \cup \Pi D_t \cup \partial\mathbb{H}$, $\Phi'_t(z)$ never vanishes there. (4.6) particularly means that $\check{U}(t)$ is a continuous semimartingale.

From Theorem 4.1 and the identity (4.6), we can obtain immediately the following two theorems.

Theorem 4.2 $\mathrm{SKLE}_{\sqrt{6},-b_{\mathrm{BMD}}}$ *being reparametrized as* (4.2) *has the same distribution as* SLE_6 *over the time interval* $[0, \tau_0)$.

Theorem 4.3 *For a positive constant* α, *there exists a sequence of hitting times* $\{\sigma_n\}$ *increasing to* τ_0 *such that* $\mathrm{SKLE}_{\alpha,b}$ *being reparametrized as* (4.2) *has the same distribution as* SLE_{α^2} *over each time interval* $[0, \sigma_n]$ *under a suitable Girsanov transform.*

When α is a positive constant, it follows from Theorem 4.3 and [RS] that $\mathrm{SKLE}_{\alpha,b}$ is generated by a continuous curve γ and that γ is simple when $\alpha \le 2$, self-intersecting when $2 < \alpha \le 2\sqrt{2}$ and space-filling when $\alpha > 2\sqrt{2}$.

5 Locality property of SLE_6 in canonical domains

It has been demonstrated in [CF, Theorem 6.11] that $\mathrm{SKLE}_{\alpha,-b_{\mathrm{BMD}}}$ enjoys the locality property for a positive constant α if and only if $\alpha = \sqrt{6}$. The proof is being carried out independently of the locality of SLE_6. The next subsection will concern the question:

(Q) Is there any alternative proof of the locality of $\mathrm{SKLE}_{\sqrt{6},-b_{\mathrm{BMD}}}$ based on Theorem 4.2 ?

5.1 Locality of chordal SLE_6 and $\mathrm{SKLE}_{\sqrt{6},-b_{\mathrm{BMD}}}$

Let Φ be a locally real conformal transformation from an $\mathbb{H}$-neighborhood $\mathcal{N}$ of a subset of $\partial\mathbb{H}$ into $\mathbb{H}$ in the sense of [L1, §4.6]. Theorem 6.13 of [L1] claimed a *locality* of SLE_6 *relative to* Φ in the following sense: the SLE_6-hulls $\{K_t\}$ have the same law as $\{\Phi(K_t)\}$ until the exit time from $\Phi(\mathcal{N} \cup \partial\mathbb{H})$ up to a time change. The proof was based on a generalized Loewner equaiton

$$\frac{dg^*_t(z)}{dt} = \frac{2\Phi'_t(\xi(t))^2}{g^*_t(z) - U^*(t)}, \quad g^*_0(z) = z, \quad U^*(t) = \Phi_t(\xi(t)), \tag{5.1}$$

for the canonical Riemann map $g^*_t(z)$ from $\mathbb{H} \setminus \Phi(K_t)$. Here $\xi(t) = \xi + \sqrt{6}B_t,\ \xi \in \partial\mathbb{H}$, and

$$\Phi_t := g^*_t \circ \Phi \circ g_t^{-1}, \tag{5.2}$$

where $g_t(z)$ is the solution of the Loewner equation (1.1).

But the equation (5.1) was rigorously proved in [L1] only in the right derivative sense just as the proof of Proposition 2.3 of this paper. In order to make it a genuine ODE, we need to verify the joint continuity of $g^*_t(z)$ in (t, z), which can be shown when Φ is the canonical Riemann map φ_A from $\mathbb{H} \setminus A$ for any $\mathbb{H}$-hull $A \subset \mathbb{H}$ by using the probabilistic representation of $\Im\Phi_t(z)$. Indeed, in this case, we have

$$\Phi_t(z) = \varphi_{g_t(A)}(z), \quad z \in \mathbb{H} \setminus g_t(A), \tag{5.3}$$

for the canonical Riemann map $\varphi_{g_t(A)}$ from $\mathbb{H}\setminus g_t(A)$ and so $\Im\Phi_t(z)$ admits a probabilistic expression

$$\Im\Phi_t(z) = \Im z - \mathbb{E}^{\mathbb{H}}_z \left[\Im Z^{\mathbb{H}}_{\sigma_{g_t(A)}}; \sigma_{g_t(A)} < \infty\right] \tag{5.4}$$

in terms of the ABM $(Z_t^{\mathbb{H}}, \mathbb{P}_z^{\mathbb{H}})$ on $\mathbb{H}$ in view of [L1, (3.5)]. Define

$$q_t(z) = \Im g_t(z) - \mathbb{E}_z^{\mathbb{H}}\left[\Im g_t(Z_{\sigma_A}^{\mathbb{H}}); \sigma_A < \infty\right], \quad z \in \mathbb{H} \setminus (F_t \cup A). \tag{5.5}$$

Due to the invariance of the ABM under the conformal map g_t, we have $\Im\Phi_t(g_t(z)) = q_t(z)$. Since $g_t^* = \Phi_t \circ g_t \circ \varphi_A^{-1}$ by (5.2), we obtain for each $T < T_A := \inf\{t : \overline{K}_t \cap \overline{A} \neq \emptyset\}$

$$\Im g_t^*(z) = q_t(\varphi_A^{-1}(z)), \quad t \in [0, T], \quad z \in \mathbb{H} \setminus \varphi_A(K_T). \tag{5.6}$$

As $g_t(z)$ is the solution of the Loewner equation (1.1), $\Im g_t(z)$ is jointly continuous and bounded by $\Im z$. Hence $q_t(z)$ is continuous in t for each $z \in \mathbb{H} \setminus (K_t \cup A)$ by (5.5) and so is $\Im g_t^*(z)$ for each $z \in \mathbb{H} \setminus \varphi_A(K_T)$. This continuity implies the joint continuity of $g_t^*(z)$ just as in the last part of Section 3 and so (5.1) becomes a genuine ODE. Using a generalized Itô formula as the proof of Theorem 2.8, we can then obtain $U^*(t) = \varphi_A(\xi) + \sqrt{6}\int_0^t \Phi_s'(\xi(s))dB_s$, $t < T$. Define $a(t) = 2\int_0^t \Phi_s'(\xi(s))^2 ds$. We have thus given a first rigorous proof of the locality of SLE_6 relative to φ_A.

Proposition 5.1 (Locality of chordal SLE_6 in the sense of [LSW3]). *For any $\mathbb{H}$-hull A with $\xi \notin \overline{A}$,, let φ_A be the canonical Riemann map from $\mathbb{H} \setminus A$ and $\{K_t; t \geq 0\}$ an SLE_6 starting from ξ. Then the reparametrized family of image hulls $\{\varphi_A(K_{a^{-1}(2t)}); 0 \leq t < a(T_A)/2\}$ has the same law as an SLE_6 $\{\widetilde{K}_t; 0 \leq t < \widetilde{T}\}$ starting from $\varphi_A(\xi)$, where $\widetilde{T} := \inf\{t > 0 : \widetilde{K}_t \cap \varphi_A(\partial A) \neq \emptyset\}$.*

Notice that, in view of [RS], SLE_6-hulls $\{K_t\}$ are generated by continuous self-interesecting curves, thus so are the image hulls $\{\varphi_A(K_t)\}$. Accordingly, the classical argument for a Jordan arc yielding the left continuity in t of $g_t^*(z)$ (see [CFR, §6]) cannot be applied and no proof of the continuity of g_t^* in t seems to be available other than the probabilistic method we employed above.

Now, for any standard slit domain D and any $\mathbb{H}$-hull $A \subset D$, consider the canonical conformal map Φ_A from $D \setminus A$. Note that Φ_A is a specific locally real conformal map from the $\mathbb{H}$-neighborhood $D \setminus A$ of $\partial\mathbb{H} \setminus \overline{A}$. If we could verify the locality of SLE_6 relative to Φ_A, then the locality of $\mathrm{SKLE}_{\sqrt{6}, -b_{\mathrm{BMD}}}$ could be readily deduced from Theorem 4.2. But Φ_t defined by (5.2) for $\Phi = \Phi_A$ does not satisfy (5.3) unless $D = \mathbb{H}$ so that the above probabilistic method does not work for proving the locality of SLE_6 relative to Φ_A.

So the answer to question **(Q)** remains negative at present. However it may be still possible to show the locality of SLE_6 relative to Φ_A, for example, from the point of view that SLE_6 is the scaling limit of the critical percolation exploration process on triangular lattices, although we feel that its rigorous proof would get lengthy.

In this connection, we emphasize that the locality of SLE_6 in the sense of Proposition 5.1 is enough to derive the *splitting property* and the *restriction property* of SLE_6 that were formulated in [LSW1, Corollaries 2.3 and 2.4] and utilized in [LSW1, LSW2] in order to identify the values of various Brownian intersection exponents. See [W, p 139] for a proof of this deduction where $\frac{\Phi_\varepsilon(z)}{\Phi_\varepsilon'(0)}$ should be replaced by $\frac{\Phi_\varepsilon(z)-\phi_\varepsilon(0)}{\Phi_\varepsilon'(0)}$ however. Analogously we may study BMD intersection exponents in relation to [CF, Theorem 6.11].

5.2 Locality of radial SLE_6 relative to modified canonical maps

So far, only chordal SLEs and chordal SKLEs have been considered.

Consider a linear transformation $\psi(z) = i\dfrac{1+z}{1-z}$ from the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto $\mathbb{H}$, that sends 1 to ∞. Its inverse ψ^{-1} sends $\partial\mathbb{H}$ onto $\partial\mathbb{D} \setminus \{1\}$. Let $\{K_t\}$ and $\{\widehat{K}_t\}$ be the radial SLE_κ on $\mathbb{D}$ starting at $\xi \in \partial\mathbb{D} \setminus \{1\}$ and the chordal SLE_κ on $\mathbb{H}$ starting at $\psi(\xi) \in \partial\mathbb{H}$, respectively. A Basic relation of their distributions was investigated in [LSW2] by using the map ψ.

More specifically, define $\widetilde{K}_t = \psi^{-1}(\widehat{K}_t)$. $\{\widetilde{K}_t\}$ is then a family of random growing hulls on $\mathbb{D}$ starting at $\xi \in \partial\mathbb{D} \setminus \{1\}$. In §4.1 of [LSW2], the following statement was established by a right application of a generalized Itô formula mentioned in Remark 2.9 ($e_t = g_t(1)$ in its proof is a random variable). The hitting time of a point $a \in \overline{\mathbb{D}}$ of the closure of a growing hull in $\mathbb{D}$ is denoted by σ_a.

Proposition 5.2 (Theorem 4.1 of [LSW2]) *The radial* SLE_6 $\{K_t\}$ *restricted on* $[0, \sigma_1)$ *has under a reparametrization the same law as the* ψ^{-1}*-image* $\{\widetilde{K}_t\}$ *of the chordal* SLE_6 *restricted on* $[0, \sigma_0)$.

For a hull A on $\mathbb{D}$ with $0 \notin A$, the unique Riemann map Φ_A from $\mathbb{D} \setminus A$ onto $\mathbb{D}$ satisfying $\Phi_A(0) = 0,\ \Phi_A'(0) > 0$, is called the *canonical map* from $\mathbb{D} \setminus A$. We also define a *modified canonical map* $\widetilde{\Phi}_A$ from $\mathbb{D} \setminus A$ (onto $\mathbb{D}$) by

$$\widetilde{\Phi}_A = \psi^{-1} \circ \varphi_{\psi(A)} \circ \psi, \quad \text{when } 1 \notin \overline{A},$$

where $\varphi_{\psi(A)}$ is the canonical Riemann map from $\mathbb{H} \setminus \psi(A)$ (onto $\mathbb{H}$). A modified canonical map $\widetilde{\Phi}_A$ is different from the canonical map Φ_A in general.

The radial SLE_κ $\{K_t\}$ starting at $\xi \in \partial\mathbb{D}$ is said to enjoy the *locality property* if, for any hull $A \subset \mathbb{D}$ with $0 \notin A$ and $\xi \notin \overline{A}$, $\{\Phi_A(K_t)\}$ has under a reparametrization the same law as $\{K_t\}$ starting at $\Phi_A(\xi)$ until the hitting time $\tau_A = \inf\{t : \overline{K}_t \cap \overline{A} \neq \emptyset\}$ for the canonical map Φ_A from $\mathbb{D} \setminus A$. It readily follows from Proposition 5.1 and Proposition 5.2 that the radial SLE_6 $\{K_t\}$ starting at $\xi \in \partial\mathbb{D}$ enjoys the locality but relative to the modified canonical map $\widetilde{\Phi}_A$:

Corollary 5.3 *For any hull* $A \subset \mathbb{D}$ *with* $0 \notin A$ *and* $\xi, 1 \notin \overline{A}$. $\{\widetilde{\Phi}_A(K_t)\}$ *has under a reparametrization the same law as the radial* SLE_6 $\{K_t\}$ *until a hitting time not greater than* τ_A *for the modified canonical map* $\widetilde{\Phi}_A$ *from* $\mathbb{D} \setminus A$.

In order to show the locality of the radial SLE_6 (relative to canonical maps), one may need to make analogous considerations to the proof of Proposition 5.1 first by deriving a generalized Loewner equation in the right derivative sense and then using the absorbing Brownian motion on $\mathbb{D}$. We leave its proof to interested readers.

6 K-L equations and SKLEs for other canonical domains

In this section, we recall and examine Komatu-Loewner equations and stochastic Komatu-Loewner evolutions studied in literature for other canonical multiply connected planar domains (cf. [C, G]).

6.1 Annulus

The annulus $\mathbb{A}_q = \{z \in \mathbb{C} : q < |z| < 1\}$ for $q \in (0, 1)$ occupies a special place among multiply connected planar domains. The first extension of the Loewner equation from simply connected domains to annuli goes back to Y. Komatu [K1]. Fix an annulus $\mathbb{A}_Q$ for $0 < Q < 1$, and a Jordan arc $\gamma = \{\gamma(t) : 0 \le t \le t_\gamma\}$ with $\gamma(0) \in \partial\mathbb{D}$ and $\gamma(0, t_\gamma] \subset \mathbb{A}_Q$. There exists a strictly increasing function $\alpha : [0, t_\gamma] \mapsto [Q, Q_\gamma]$ with $\alpha(t_\gamma) = Q_\gamma < 1$ and the following property: if $\alpha(t) = q$, then there is a unique conformal map g_q from $\mathbb{A}_Q \setminus \gamma[0, t]$ onto $\mathbb{A}_q$ such that $g_q(Q) = q$. A differential equation for g_q in the left derivative in q was derived in [K1] in terms of the Weierstrass as well as Jacobi elliptic functions. But the continuity of α and right differentiability of g_q in q were not rigorously established although an annulus variant of the Carathéodory convergence theorem was indicated in [K1] to cover these points.

Recently [FK] utilizes this variant of the Carathéodory theorem to show that α is indeed continuous and that $g_q(z)$, $Q \le q \le Q_\gamma$, satisfies a genuine ODE

$$\frac{\partial \log g_q(z)}{\partial \log q} = \mathcal{K}_q(g_q(z), \lambda(q)) - i\Im\mathcal{K}_q(q, \lambda(q)), \qquad g_Q(z) = z, \tag{6.1}$$

where $\mathcal{K}_q(z, \zeta)$, $z \in \mathbb{A}_q$, $\zeta \in \partial\mathbb{D}$, is Villat's kernel defined by $\mathcal{K}_q(z, \zeta) = \mathcal{K}_q(z/\zeta)$. Here

$$\mathcal{K}_q(z) := \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1 + q^{2n}z}{1 - q^{2n}z},$$

and $\lambda(q) := g_q(\gamma(t))$ (where $t > 0$ is such that $\alpha(t) = q$) is a continuous function of q taking values on the outer boundary $\partial\mathbb{D}$. Since α is continuous, the curve γ can be parametrized as $\{\gamma(q) : Q \le q \le Q_\gamma\}$ so that $g_q(z)$ is a conformal map from $\mathbb{A}_Q \setminus \gamma[0, q]$ onto $\mathbb{A}_q$ with the normalization $g_q(Q) = q$. We may further let $P = -\log Q$, $P_\gamma = -\log Q_\gamma$, $S_p(z, \zeta) = \mathcal{K}_{e^{-p}}(z, \zeta)$ and change the parameter q into s by $q = e^{s-P}$ for $0 \le s \le s_\gamma = P - P_\gamma$. Then (6.1) becomes, for $z \in \mathbb{A}_Q \setminus \gamma[0, s]$ and $s \in [0, s_\gamma]$,

$$\frac{\partial \log g_s(z)}{\partial s} = S_{P-s}(g_s(z), \lambda(s)) - i\Im S_{P-s}(e^{s-P}, \lambda(s)), \quad g_0(z) = z, \tag{6.2}$$

where $\lambda(s) := g_s(\gamma(s))$. Note that g_s is the conformal mapping from $\mathbb{A}_Q \setminus \gamma[0, s]$ onto $\mathbb{A}_{Qe^s}$ with $g_s(Q) = Qe^s$. By using the stated variant of the Carathéodory convergence theorem, it is also shown in [FK] that, the equation (6.1) in the right derivative sense is still valid if we take, in place of the Jordan curve γ, a family $\{F_q\}$ of growing hulls in $\mathbb{A}_Q$ that is right continuous with limit $\lambda(q)$ in a sense similar to (5.24) of [CF].

In [Z1], D. Zhan defined the annulus SLE$_\kappa$ to be the growing hulls $\{K_s\}$ in $\mathbb{A}_Q$ driven by $\lambda(s) = e^{iB(\kappa s)}$ for the one-dimensional Brownian motion $B(s)$ based on the equation

$$\frac{\partial \log g_s(z)}{\partial s} = S_{P-s}(g_s(z), \lambda(s)), \qquad g_0(z) = z. \tag{6.3}$$

As was noted in the proof of [Z1, Proposition 2.1], for any pair $(g_q(z), \lambda(s))$ satisfying equation (6.2), its rotation $e^{i\theta(s)}(g_q(z), \lambda(s))$ satisfies (6.3) where $\theta(s) = \int_0^s \Im S_{P-r}(e^{r-P}, \lambda(r))dr$, and so the growing hulls based on (6.2) driven by $\lambda(s)$ are the same as those based on (6.3) driven by $e^{i\theta(s)}\lambda(s)$.

For each $\kappa > 0$, it was shown in [Z1] that the distribution of the annulus SLE$_\kappa$ defined by (6.3) is related to that of the radial SLE$_\kappa$ stopped upon hitting a compact set containing the origin. When $\kappa = 6$, they were further identified up to a time change. In this connection, we point out a gap in the proof of the differentiability of the function $f_t(w)$ in t for each $w \in \mathbf{C}_0$ in [Z1, page 350]. See Remark 2.9. The proof of [L1, Prop. 4.40, Th. 6.13] involves a similar gap.

For a given continuous function $\lambda(q)$, $Q \le q < 1$, taking value in $\partial\mathbb{D}$, the ODE (6.1) admits a unique solution $g_q(z)$ that can be verified to satisfy the normalization condition $g_q(Q) = q$, due to the fact that $\Re\mathcal{K}_q(q, e^{i\theta}) = 1$ for every $q \in (0,1)$ and $\theta \in [0, 2\pi)$. It may be worthwhile to consider an SKLE on the annulus based directly on the equation (6.1) or on its modified version driven by a general diffusion process on $\partial\mathbb{D}$ along the lines of [CF] and this paper.

D. Zhan further extended the notion of annulus SLE_κ, $\kappa \le 4$, in a certain way to specify the end points of the curves and investigated its properties such as reversibility and restriction property (see [Z2] and references therein). Recently, G. Lawler [L2] defined SLE_κ for $\kappa \le 4$ in more general multiply connected domains using the Brownian loop measure and compared it with Zhan's one in the annulus case. As is noted in Remark 6.12 of [CF], we can hardly expect a straightforward generalization of the restriction property of $\mathrm{SLE}_{8/3}$ to the chordal $\mathrm{SKLE}_{\sqrt{8/3},-b_{\mathrm{BMD}}}$ due to an effect of the second order BMD-domain constant c_{BMD}. It would be interesting to find connections of the conditional laws induced by $\mathrm{SKLE}_{\sqrt{\kappa},b}$ with Lawler's measures.

6.2 Circularly slit annulus

Parallel to the BMD complex Poisson kernel, the notion of the BMD Schwarz kernel $\mathbf{S}(z,\zeta)$ is introduced in [FK] for a general multiply connected planar domain D as an analytic function in $z \in D$ whose real part is the BMD-Poisson kernel. In particular, it is shown in [FK] that the Villat's kernel multiplied by $1/(2\pi)$ is a BMD Schwarz kernel for the annulus.

A domain D of the form $D = \mathbb{A}_q \setminus \bigcup_{j=1}^{N-1} C_j$ is called a *circularly slit annulus* if C_j are mutually disjoint concentric circular slits contained in $\mathbb{A}_q$. We denote by $\mathcal{D}$ the collection of all circularly slit annuli. We fix $D = \mathbb{A}_Q \setminus \bigcup_{j=1}^{N-1} C_j \in \mathcal{D}$ and consider a Jordan arc $\gamma : [0, t_\gamma] \mapsto D$ with $\gamma(0) = \partial\mathbb{D}$. We can then find a strictly increasing function $\alpha:\ [0, t_\gamma] \mapsto [Q, Q_\gamma]$, $(\alpha(t_\gamma) = Q_\gamma < 1)$ such that, for $q = \alpha(t)$, there exists a unique conformal map $g_q:\ D \setminus \gamma[0,t] \ \mapsto\ D_q = \mathbb{A}_q \setminus \bigcup_{j=1}^{N-1} C_j(q) \in \mathcal{D}$, with $g_q(Q) = q$.

The first extension of the Loewner equation to a circularly slit annulus goes back to Y. Komatu [K2] and the resulting Komatu-Loewner equation for g_q is rewritten by [BF2] and then by [FK] as

$$\frac{\partial^- \log g_q(z)}{\partial \log q} = 2\pi \widehat{\mathbf{S}}_q(g_q(z), \lambda(q)),\ q \in \alpha(0, t_\gamma] \subset (Q, Q_\gamma],\ g_Q(z) = z, \tag{6.4}$$

where the left hand side denotes the left derivative and $\widehat{\mathbf{S}}_q(z,\zeta) = \mathbf{S}_q(z,\zeta) - i\Im\mathbf{S}_q(q,\zeta)$ is the normalized BMD Schwarz kernel for $D_q \in \mathcal{D}$. When $N = 1$, $2\pi\widehat{\mathbf{S}}_q$ is just the normalized Villat's kernel with the stated explicit expression and (6.4) is reduced to (6.1). When $N > 1$, the problem of the continuity of α and right differentiability of g_q remains open. Recently C. Boehm and W. Lauf [BL] establish a Komatu-Loewner equation for a circularly slit disk as a genuine ODE by using an extended version of the Carathéodory convergence theorem. A method similar to [BL] or to [CFR] might work to make (6.4) a genuine ODE and we may then conceive an SKLE for it in analogue to [CF].

6.3 Circularly slit disk

A domain D of the form $D = \mathbb{D} \setminus \bigcup_{j=1}^{N-1} C_j$ is called a *circularly slit disk* if C_j are mutually disjoint concentric circular slits contained in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For a circularly slit disk D, Bauer and Friedrich have obtained a radial Komatu-Loewner equation [BF1, (44)] with a kernel explicitly expressed in terms of the Green function and harmonic measures, which could be identical with a BMD Schwarz kernel for the image domain D_t. The differntiability problem for $g_t(z)$ in this equation seems to have been settled by the aforementioned approach of [BL].

Moreover, an $\mathrm{SKLE}_{\sqrt{\kappa},b}$ on D is formulated in [BF1] for any constant $\kappa > 0$ in a quite analogous manner to [CF] and it is claimed that $\mathrm{SKLE}_{\sqrt{6},b}$ enjoys the locality property relative to canonical maps for a specific choice (Ansatz) of the drift coefficient b of the driving process on $\partial\mathbb{D}$. Our natural guess is that $b = -b_{\mathrm{BMD}}$. However the establishment of a generalized Komatu-Loewner equation [BF1, (63)] for image hulls by a canonical map as a genuine ODE requires the continuity of g_t^* (which corresponds to $\widetilde{g}_t$ in [CF]). But this has been left unconfirmed even in the radial SLE case with no circular slit.

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