Markov processes with darning and their approximations

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Abstract

In this paper, we study darning of general symmetric Markov processes by shorting some parts of the state space into singletons. A natural way to construct such processes is via Dirichlet forms restricted to the function spaces whose members take constant values on these collapsing parts. They include as a special case Brownian motion with darning, which has been studied in details in [1, 2, 5]. When the initial processes have discontinuous sample paths, the processes constructed in this paper are the genuine extensions of those studied in Chen and Fukushima [2]. We further show that, up to a time change, these Markov processes with darning can be approximated in the finite dimensional sense by introducing additional jumps with large intensity among these compact sets to be collapsed into singletons. For diffusion processes, it is also possible to get, up to a time change, diffusions with darning by increasing the conductance on these compact sets to infinity. To accomplish these, we give a version of the semigroup characterization of Mosco convergence to closed symmetric forms whose domain of definition may not be dense in the L^2-space. The latter is of independent interest and potentially useful to study convergence of Markov processes having different state spaces. Indeed, we show in Section 5 of this paper that Brownian motion in a plane with a very thin flag pole can be approximated by Brownian motion in the plane with a vertical cylinder whose horizontal motion on the cylinder is a circular Brownian motion moving at fast speed.

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1 Introduction

K. Ito [18] introduced the notion of Poisson point process of excursions around one point a in the state space of a standard Markov process X. He was motivated by giving a systematic construction of Markovian extensions of the absorbing diffusion process X^0 on the half line $(0, \infty)$ subject to Feller's general boundary conditions [19]. Ito had constructed the most general jump-in process from the exit boundary 0 by using Poisson point process of excursions. Recent study [14, 6, 2] reveals that Ito's program works equally well in the study of Markov processes transformed by collapsing certain compact subsets of the state space into singletons. These processes are called Markov processes with darning in [2]. (When the underlying process is a Markov chain on a discrete state space, such a procedure of collapsing subsets of state space is also called shorting in some literature.) However, in order to use excursion theory, it is assumed in [14, 6, 2] that the original Markov process enters these compact subsets in a continuous way. This condition is automatically satisfied for diffusion processes but not for general symmetric Markov processes that may have discontinuous trajectories.

The purpose of this paper is two-folds. First, we extend the notion and construction of Markov processes with darning to any symmetric Markov process, without assuming that the processes enter the compact subsets to be collapsed in a continuous way. In this generality, we can no longer use Poisson point process of excursions for the construction. We will use instead a Dirichlet form approach, which turns out to be quite effective. The second goal is to investigate approximation schemes for general Markov processes with darning by more concrete processes, which can be used for simulation. For this, we present a version of Mosco convergence of closed symmetric forms whose domain may not be dense in the underlying L^2-space. This is because due to the collapsing of the compact holes, the domain of the Dirichlet form for the Markov process with darning is not dense in the L^2-space on the original state space. We now describe the content of this paper in some details. For basic definitions and properties of symmetric Dirichlet forms, we refer the reader to [2, 13].

Let E be a locally compact separable metric space and m a Radon measure on E with full support. Suppose $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(E;m)$ in the sense that $C_c(E) \cap \mathcal{F}$ is dense both in $C_c(E)$ with respect to the uniform norm in $\mathcal{F}$ and with respect to the Hilbert norm $\sqrt{\mathcal{E}_1(u,u)} := \sqrt{\mathcal{E}(u,u) + (u,u)_{L^2(E;m)}}$. Here and in the sequel, we use $:=$ as a way of definition and $C_c(E)$ is the space of continuous functions on E with compact support. Every f in $\mathcal{F}$ admits an $\mathcal{E}$-quasi-continuous m-version, which is unique up to an $\mathcal{E}$-polar set. We always take such a quasi-continuous version for functions in $\mathcal{F}$. There is an m-symmetric Hunt process X on E associated with $(\mathcal{E}, \mathcal{F})$, which is unique up to an $\mathcal{E}$-polar set. It is known that for any regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E;m)$ it admits the following unique Beurling-Deny decomposition (see [2, 13]):

$$\mathcal{E}(u,u) = \mathcal{E}^c(u,u) + \frac{1}{2}\int_{E\times E} (u(x)-u(y))^2 J(dx,dy) + \int_E u(x)^2 \kappa(dx), \quad u \in \mathcal{F},$$

where $\mathcal{E}^c$ is a symmetric non-negative definite bilinear form on $\mathcal{F}$ that satisfies strong local property, where $J(dx,dy)$ is a σ-finite measure on $E \times E \setminus$ diagonal, and κ is a σ-finite smooth measure on E. The measures $J(dx,dy)$ and $\kappa(dx)$ are called the jumping measure and killing measure of the process X (or equivalently, of the Dirichlet form $(\mathcal{E}, \mathcal{F})$). Indeed, if we use $(N(x,dy), H_t)$ to denote a Lévy system of X, where $N(x,dy)$ is a kernel on $E_\partial := E \cup \{\partial\}$ and $t \mapsto H_t$ is a positive continuous additive functional (PCAF) of X, then

$$J(dx,dy) = N(x,dy)\mu_H(dx) \qquad \text{and} \qquad \kappa(dx) = N(x,\{\partial\})\mu_H(dx).$$

Here μ_H is the Revuz measure of the PCAF H and ∂ is the cemetery point for X added to E as a one-point compactification.

Let $F = \cup_{j=1}^N K_j$ be the union of N disjoint compact subsets K_j of positive $\mathcal{E}_1$-capacity. Set $D := E \setminus F$. In this paper, we will construct a new Markov process X^* from X by darning (or shorting) each hole K_j into a single point a_j^*. This new process has state space $E^* := D \cup \{a_1^*, \cdots, a_N^*\}$ and is m^*-symmetric, where $m^* := m$ on D and $m^*(E^* \setminus D) := 0$. Moreover, the jumping measure J^*

and the killing measure κ^* of X^* on E^* should have the properties inherited from J and κ without incurring additional jumps or killings; that is,

$$J^* = J \text{ on } D\times D, \quad J^*(\{a_i^*\}, dy) = J(K_i, dy) \text{ on } D, \quad J^*(\{a_i^*\}, \{a_j^*\}) = J(K_i, K_j) \text{ for } i \neq j, \tag{1.1}$$

$$\kappa^* = \kappa \quad \text{on } D \quad \text{ and } \quad \kappa^*(\{a_j^*\}) = \kappa(K_j) \text{ for } 1 \leq j \leq N. \tag{1.2}$$

We will show that such X^* always exists and is unique in law. This process X^* coincides with the Markov process with darning introduced in [2] under the assumption that X enters each K_j in a continuous way, that is, $X_{\tau_D -} \in F$ on $\{\tau_D < \zeta\}$; see [2, Theorem 7.7.3]. Here ζ is the lifetime of X and $\tau_D := \inf\{t \geq 0 : X_t \notin D\}$ is the first exit time from D by the process X. Thus we will call X^* the Markov process obtained from X by darning (or shorting) each K_j into a singleton a_j^*, or simply, Markov process with darning. Note that as a consequence of the m^*-symmetry assumption, X^* spends zero Lebesgue amount of time on $E^* \setminus D = \{a_1^*, \cdots, a_N^*\}$.

The (new) Markov process with darning X^* will be constructed from X via Dirichlet form method. Since in applications, $\mathcal{E}(u,u)$ can be interpreted as energy of a potential $u \in \mathcal{F}$, intuitively speaking, restricting $\mathcal{E}$ to those $u \in \mathcal{F}$ that are constant $\mathcal{E}$-q.e. on each K_j exactly represents shorting each K_j into a single point a_j^*. Our Theorem 3.3 of this paper shows that, after a suitable identification, this approach indeed works in great generality, without any additional assumptions. We will further show in Theorem 3.4 that it is unique in distribution. When X is a Brownian motion in $\mathbb{R}^n$ and $F = K$ is a compact set, the above Dirichlet form method of constructing X^* was carried out in [1, 5] and we call X^* Brownian motion with darning (BMD). When E is the exterior of the unit disk in $\mathbb{R}^2$, $F = \partial E$ and X is the reflecting Brownian motion on E, BMD X^* has the same law as the excursion reflected Brownian motion appeared in [24] in connection with the study of SLE in multiply connected planar domains. Planar BMD enjoys conformal invariance property, see [2]. In [5, 3, 4], BMD has been used to study chordal Komatu-Loewner equation and stochastic Komatu-Loewner equation in standard slit domains in upper half space.

The second goal of this paper is to present approximation schemes for general symmetric Markov processes with darning X^* in the finite dimensional sense, which can also be used to simulate the darning processes. We note that the construction of X^* either by Dirichlet form method as in this paper or by Poisson point process of excursions when the process X enters the holes in a continuous way as in [2, 6, 14] does not provide a practical way to simulate X^*. Our approach of this paper is to introduce additional jumps among each K_j with large intensity. Intuitively, when the jumping intensity for these additional jumps increases to infinity, the new process can no longer distinguish points among each K_j, which would result in shorting (or darning) each K_j into a single point a_j^*. To be precise, for each j, let μ_j be a probability smooth measure whose quasi-support is K_j and having bounded 1-potential $G_1\mu_j$ (that is, there is a bounded function $h \in \mathcal{F}$, denoted as $G_1\mu_j$ and is called the 1-potential of μ_j, so that $\mathcal{E}_1(h,v) = \int_E v(x)\mu_j(dx)$ for every $v \in \mathcal{F}$). Since the compact set K_j is of positive $\mathcal{E}_1$-capacity, such a probability smooth measure always exists; see [2, 13]. For each $\lambda > 0$, consider the following Dirichlet form $(\mathcal{E}^{(\lambda)}, \mathcal{F})$ on $L^2(E;m)$:

$$\mathcal{E}^{(\lambda)}(u,v) = \mathcal{E}(u,v) + \lambda \sum_{j=1}^{N} \int_{K_j \times K_j} (u(x) - u(y))(v(x) - v(y))\mu_j(dx)\mu_j(dy) \tag{1.3}$$

for $u, v \in \mathcal{F}$. It is easy to see that $(\mathcal{E}^{(\lambda)}, \mathcal{F})$ is a regular Dirichlet form on $L^2(E;m)$ and thus by [13], there is a m-symmetric Hunt process $X^{(\lambda)}$ associated with it. The process $X^{(\lambda)}$ is the superposition of X with jumps among points within each K_j. The process $X^{(\lambda)}$ can also be obtained from X by the following piecing together procedure. Let X^0 be the subprocess obtained from X through killing via measure $\lambda \sum_{j=1}^N \mu_j$. More precisely, let A_s^j be positive continuous additive functional (PCAF in abbreviation) of X with Revuz measure μ_j. Then the law of X^0 is determined by the following: for every positive function f on E,

$$\mathbb{E}_x \left[f(X_t^0) \right] = \mathbb{E}_x \left[e^{-\lambda \sum_{j=1}^N A_t^j} f(X_t) \right].$$

Denote by ζ^0 the lifetime of X^0. For each starting point $x \in E$, $X^{(\lambda)}$ can be obtained from X^0 through the following redistribution and patching procedure. Run a copy of X^0 starting from x and set $X_t^{(\lambda)} = X_t^0$ for $t \in [0, T_1)$, where $T_1 = \zeta_1^0$ is the lifetime of X^0 starting from x. If $\zeta_1^0 = \infty$ or $X_{T_1-}^{(\lambda)} = \partial$, then we define $X_t^{(\lambda)} = \partial$ for $t \geq T_1$. Otherwise, $X_{T_1-}^{(\lambda)} \in F$, say $X_{T_1-}^{(\lambda)} \in K_{j_1}$. Select $x_2 \in K_{j_1}$ according to the probability distribution μ_{j_1} and define $X_{T_1-}^{(\lambda)} = x_2$. Run an independent copy of X^0 starting from x_2, whose lifetime is denoted as ζ_2^0. Define $X_{T_1+t}^{(\lambda)} = X_t^0$ for $t \in [0, \zeta_2^0)$ and set $T_2 = T_1 + \zeta_2^0$. If $T_2 = \infty$ or $X_{T_2-}^{(\lambda)} = \partial$, then we define $X_t^{(\lambda)} = \partial$ for $t \geq T_2$. Otherwise, $X_{T_2-}^{(\lambda)} \in F$, say $X_{T_2-}^{(\lambda)} \in K_{j_2}$. Let $x_3 \in K_{j_2}$ according to the probability distribution μ_{j_2} and define $X_{T_2}^{(\lambda)} = x_3$, and so on. The above described patching together procedure is a particular case discussed in [17]. The resulting process is a Hunt process on E. It is easy to verify that the Dirichlet form associated with $X^{(\lambda)}$ is $(\mathcal{E}^{(\lambda)}, \mathcal{F})$.

When the intensity $\lambda \to \infty$, process $X^{(\lambda)}$ behaves like X outside F but can not distinguish points in each K_j. In other words, in the limit, each K_j is collapsed into a single point a_j^*. So if the limit exists, the limiting process should be Markov process with darning of X but up to a time change. This is because m is a symmetrizing measure for each $X^{(\lambda)}$ so under stationarity, each $X^{(\lambda)}$ spends time in F at a rate proportional to $m(F)$. Let Y be the Hunt process on E^* obtained from X^* through a time change via Revuz measure $\mu = m|_D + \sum_{j=1}^N m(K_j)\delta_{\{a_j^*\}}$. That is, Y is a sticky Markov process with darning, which spends $m(K_j)$-proportional Lebesgue amount of time at a_j^* over time duration. We show in this paper that as $\lambda \to \infty$, $X^{(\lambda)}$ converges to Y in the finite dimensional sense; see Theorem 4.3 below for a precise statement.

When X is a diffusion process on E and each compact set K_j is connected, it is possible to get the diffusion with darning X^* on E^*, up to a time change, by increasing the conductance on each K_j to infinity. This is illustrated by Theorem 4.4.

An effective way of establishing finite dimensional convergence for symmetric Markov processes is the Mosco convergence of Dirichlet forms [26]. However, the state space E^* of X^* is different from that of $X^{(\lambda)}$ – there is a sudden collapsing of the state space right at the limit $\lambda = \infty$. This is in stark contrast with cases considered in [7, 21, 26], where the weak converges and Mosco convergence are studied for processes and for Dirichlet forms on different state spaces. In these papers, the state spaces are changing in a continuous way as $n \to \infty$. From the Dirichlet form point of view, the domain $\mathcal{F}^*$ of the Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ associated with our X^*, viewed as a subspace of $L^2(E; m)$, is exactly those in $\mathcal{F}$ that are constant quasi-everywhere on each K_j. So it may not be dense in $L^2(E; m)$ in general while the domain of the Dirichlet form for $X^{(\lambda)}$ is $\mathcal{F}$ for every $\lambda > 0$. Thus the existing theory of Mosco convergence [26, 21, 7] can not be applied directly. In Section 2 of this paper, we extend the characterization of L^2-convergence of semigroups for Mosco convergence to closed symmetric forms whose domains may not be dense in L^2-space; see Theorem 2.3 and the remarks proceeding it on its connection to [16, Theorem 3.5] and [23, Theorem 2.1]. This result may be of independent interest. The approximation schemes mentioned above for Markov processes with darning are established by applying Theorem 2.3.

The idea and approach of this paper can also be used to study approximation for other darning related processes. In Section 5, we illustrate how to use the ideas of this paper to approximate Brownian motion in a plane with a very thin flag pole studied recently in [10] by Brownian motion in the plane with a vertical cylinder whose horizontal motion on the cylinder is a circular Brownian motion moving at fast speed.

2 Mosco convergence of general closed symmetric forms

One way to establish the finite dimensional convergence is via Mosco convergence [26]. However, the characterization of convergence of symmetric semigroups in [26] is formulated only for those closed symmetric forms whose domains of definition are dense in the L^2-spaces. In this section, we study Mosco convergence of general closed symmetric forms whose domains of definition may not necessarily be dense in the corresponding L^2-spaces.

Let E be a locally compact separable metric space and m a Radon measure on E with full support. Suppose $(\mathcal{E},\mathcal{F})$ is a closed symmetric form on $L^2(E;m)$; that is, $\mathcal{F}$ is a linear subspace of $L^2(E;m)$, $\mathcal{E}$ is a non-negative definite symmetric form defined on $\mathcal{F}\times\mathcal{F}$ such that $\mathcal{F}$ is a Hilbert space with inner product $\mathcal{E}_1$. Here for $\alpha>0$,

$$\mathcal{E}_\alpha(f,g):=\mathcal{E}(f,g)+\alpha(f,g)_{L^2(E;m)},\qquad f,g\in\mathcal{F}.$$

Note that here we do not assume $\mathcal{F}$ is dense in $L^2(E;m)$. Throughout this paper, we use the convention that we define $\mathcal{E}(f,f)=\infty$ for $f\notin\mathcal{F}$. Given a closed symmetric form $(\mathcal{E},\mathcal{F})$ on $L^2(E;m)$, by Riesz representation theorem, for every $f\in L^2(E;m)$ and $\alpha>0$, there is a unique $G_\alpha f\in\mathcal{F}$ such that

$$\mathcal{E}_\alpha(G_\alpha f,g)=(f,g)_{L^2(E;m)}\quad\text{for every } g\in\mathcal{F}. \tag{2.1}$$

These linear operators $\{G_\alpha,\alpha>0\}$ on $L^2(E;m)$ is called the resolvent of $(\mathcal{E},\mathcal{F})$. It is known that the resolvent $\{G_\alpha,\alpha>0\}$ of $(\mathcal{E},\mathcal{F})$ is strongly continuous (that is, $\lim_{\alpha\to\infty}\|\alpha G_\alpha f-f\|_{L^2(E;m)}=0$ for every $f\in L^2(E;m)$) if and only if $\mathcal{F}$ is dense in $L^2(E;m)$. If $\mathcal{F}$ is dense in $L^2(E;m)$, then there is a unique, strongly continuous semigroup $\{P_t,t\geq 0\}$ associated with the strongly continuous resolvent $\{G_\alpha,\alpha>0\}$, and hence with $(\mathcal{E},\mathcal{F})$.

If $\mathcal{F}$ is not dense in $L^2(E;m)$, denote by $\overline{\mathcal{F}}$ the closure of $\mathcal{F}$ in $L^2(E;m)$. Then $(\mathcal{E},\mathcal{F})$ is a closed symmetric form on $\overline{\mathcal{F}}$. The following facts are known; see [2, pp.2-4] or [25]. There is a unique strongly continuous contraction symmetric resolvent $\{\widehat{G}_\alpha;\alpha>0\}$ on $\overline{\mathcal{F}}$ associated with it:

$$\mathcal{E}_\alpha(\widehat{G}_\alpha f,g)=(f,g)_{L^2(E;m)}\quad\text{for every } g\in\mathcal{F}. \tag{2.2}$$

It in turn is associated with a unique strongly continuous contraction symmetric semigroup $\{\widehat{P}_t;t\geq 0\}$ on $\overline{\mathcal{F}}$ via

$$\widehat{G}_\alpha f=\int_0^\infty e^{-\alpha t}\widehat{P}_t f\,dt,\quad f\in\overline{\mathcal{F}}.$$

The correspondence between $(\mathcal{E}, \mathcal{F})$, $\{\widehat{G}_\alpha, \alpha > 0\}$ and $\{\widehat{P}_t, t \geq 0\}$ on $\overline{\mathcal{F}}$ are one-to-one. In particular,

$$\begin{aligned}
\mathcal{F} &= \left\{ u \in \overline{\mathcal{F}} : \lim_{t \to 0} \frac{1}{t} (u - \widehat{P}_t u, u)_{L^2(E;m)} < \infty \right\}, \\
\mathcal{E}(u, v) &= \lim_{t \to 0} \frac{1}{t} (u - \widehat{P}_t u, v)_{L^2(E;m)} \quad \text{for } u, v \in \mathcal{F}.
\end{aligned}$$

Denote by $(\widehat{\mathcal{L}}, \mathcal{D}(\widehat{\mathcal{L}}))$ the generator of $\{\widehat{P}_t; t \geq 0\}$ in the Hilbert space $\overline{\mathcal{F}}$ (equipped with the L^2-inner product from $L^2(E;m)$). Then $u \in \mathcal{D}(\widehat{\mathcal{L}})$ if and only $u \in \mathcal{F}$ and there is $f \in \overline{\mathcal{F}}$ so that

$$\mathcal{E}(u, v) = -(f, v)_{L^2(E;m)} \quad \text{for every } v \in \mathcal{F};$$

in this case, $\widehat{\mathcal{L}} u = f$. We have $G_\alpha(\overline{\mathcal{F}}) = \mathcal{D}(\mathcal{L})$ and $\widehat{P}_t(\overline{\mathcal{F}}) \subset \mathcal{D}(\mathcal{L})$.

Let Π be the orthogonal projection operator from $L^2(E;m)$ onto $\overline{\mathcal{F}}$. Then we have from (2.1) and (2.2) that

$$G_\alpha f = \widehat{G}_\alpha(\Pi f) \quad \text{for every } \alpha > 0 \text{ and } f \in L^2(E;m). \tag{2.3}$$

Definition 2.1 *A sequence of closed symmetric forms $\{(\mathcal{E}^n, \mathcal{F}^n)\}$ on $L^2(E;m)$ is said to be convergent to a closed symmetric form $(\mathcal{E}, \mathcal{F})$ on $L^2(E;m)$ in the sense of Mosco if*

(a) *For every sequence $\{u_n, n \geq 1\}$ in $L^2(E;m)$ that converges weakly to u in $L^2(E;m)$,*

$$\liminf_{n \to \infty} \mathcal{E}^n(u_n, u_n) \geq \mathcal{E}(u, u).$$

(b) *For every $u \in L^2(E;m)$, there is a sequence $\{u_n, n \geq 1\}$ in $L^2(E;m)$ converging strongly to u in $L^2(E;m)$ such that*

$$\limsup_{n \to \infty} \mathcal{E}^n(u_n, u_n) \leq \mathcal{E}(u, u).$$

Denote by $\{G_\alpha, \alpha > 0\}$ and $\{G^n_\alpha, \alpha > 0\}$ the corresponding resolvents of $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}^n, \mathcal{F}^n)$, respectively. When $\mathcal{F}$ and $\mathcal{F}^n$ are dense in $L^2(E;m)$, the associated semigroup will be denoted by $\{P_t, t \geq 0\}$ and $\{P^n_t, t \geq 0\}$, respectively.

The following result is known (see Theorem 2.4.1 and Corollary 2.6.1 of [26]).

Proposition 2.2 *Let $(\mathcal{E}, \mathcal{F})$ and $\{(\mathcal{E}^n, \mathcal{F}^n), n \geq 1)\}$ be a sequence of closed symmetric forms on $L^2(E;m)$. The following two statements are equivalent:*

(i) *$(\mathcal{E}^n, \mathcal{F}^n)$ converges to $(\mathcal{E}, \mathcal{F})$ in the sense of Mosco.*

(ii) *For every $\alpha > 0$ and $f \in L^2(E;m)$, $G^n_\alpha f$ converges to $G_\alpha f$ in $L^2(E;m)$ as $n \to \infty$;*

When $\mathcal{F}^n$ and $\mathcal{F}$ are all dense in $L^2(E;m)$, then (i) *is equivalent to the following:*

(iii) *For every $t > 0$ and $f \in L^2(E;m)$, $P^n_t f$ converges to $P_t f$ in $L^2(E;m)$ as $n \to \infty$.*

The next result addresses the case when $\mathcal{F}^n$ and $\mathcal{F}$ may not be dense in $L^2(E;m)$. It is possible to prove it by using [23, Theorem 2.1] and its part (i) is a particular case of [16, Theorem 3.5]. We give a direct alternative proof here for readers convenience.

Theorem 2.3 *Let $(\mathcal{E},\mathcal{F})$ and $\{(\mathcal{E}^n,\mathcal{F}^n), n\geq 1)\}$ be closed symmetric forms on $L^2(E;m)$. Let $\overline{\mathcal{F}}^n$ and $\overline{\mathcal{F}}$ be the closure of $\mathcal{F}^n$ and $\mathcal{F}$ in $L^2(E;m)$, respectively. Suppose that $\overline{\mathcal{F}}^n\supset\overline{\mathcal{F}}$ for every $n\geq 1$. Let $(\widehat{P}^n_t; t\geq 0\}$ and $(\widehat{P}_t; t\geq 0\}$ be the semigroups on $\overline{\mathcal{F}}^n$ and $\overline{\mathcal{F}}$ associated with $(\mathcal{E}^n,\mathcal{F}^n)$ and $(\mathcal{E},\mathcal{F})$, respectively. Then the following hold.*

(i) *If $(\mathcal{E}^n,\mathcal{F}^n)$ converges to $(\mathcal{E},\mathcal{F})$ in the sense of Mosco, then for every $t>0$ and $f\in\overline{\mathcal{F}}$, $\widehat{P}^n_t f$ converges to $\widehat{P}_t f$ in $L^2(E;m)$ as $n\to\infty$.*

(ii) *Suppose that the closed subspace $\overline{\mathcal{F}}^n$ converges to $\overline{\mathcal{F}}$ in $L^2(E;m)$ in the sense that*

$$\lim_{n\to\infty}\|\Pi^n f-\Pi f\|_{L^2(E;m)}=0 \quad \text{for every } f\in L^2(E;m),$$

where Π^n and Π denote the orthogonal projection operators of $L^2(E;m)$ onto $\overline{\mathcal{F}}^n$ and $\overline{\mathcal{F}}$, respectively. If $\widehat{P}^n_t f$ converges to $\widehat{P}_t f$ in $L^2(E;m)$ for every $t>0$ and $f\in\overline{\mathcal{F}}$, then $(\mathcal{E}^n,\mathcal{F}^n)$ converges to $(\mathcal{E},\mathcal{F})$ in the sense of Mosco.

Proof: Let $\{G^n_\alpha;\alpha>0\}$ and $(\widehat{G}^n_\alpha;\alpha>0\}$ be the resolvents on $L^2(E;m)$ and on $\overline{\mathcal{F}}^n$, respectively, associated with the closed symmetric form $(\mathcal{E}^n,\mathcal{F}^n)$ via (2.1) and (2.2). Similar notations $\{G_\alpha;\alpha>0\}$ and $(\widehat{G}_\alpha;\alpha>0\}$ will be used for $(\mathcal{E},\mathcal{F})$. We know from Proposition 2.2 that $(\mathcal{E}^n,\mathcal{F}^n)$ converges to $(\mathcal{E},\mathcal{F})$ in the sense of Mosco if and only if $G^n_\alpha f$ converges to $G_\alpha f$ in $L^2(E;m)$ for every $f\in L^2(E;m)$.

(i) Suppose that $(\mathcal{E}^n,\mathcal{F}^n)$ converges to $(\mathcal{E},\mathcal{F})$ in the sense of Mosco,. Then in view of (2.3) and the assumption that $\overline{\mathcal{F}}^n\supset\overline{\mathcal{F}}$, we have for every $\alpha>0$ and $f\in\overline{\mathcal{F}}$, $\widehat{G}^n_\alpha f$ converges to $\widehat{G}_\alpha f$ in $L^2(E;m)$. We claim this implies that $\widehat{P}^n_t f$ converges to $\widehat{P}_t f$ in $L^2(E;m)$ as $n\to\infty$ for every $t>0$ and $f\in L^2(E;m)$. The proof is similar to that for [20, Theorem IX.2.16]. For reader's convenience, we spell out the details here.

Denote by $\widehat{\mathcal{L}}^n$ and $\widehat{\mathcal{L}}$ the generators of the strongly continuous semigroups $\{\widehat{P}^n_t; t\geq 0\}$ and $\{\widehat{P}_t; t\geq 0\}$, respectively. Note that

$$\begin{aligned}\frac{d}{dt}\widehat{P}_t\widehat{G}_\alpha &= \widehat{P}_t\widehat{\mathcal{L}}\widehat{G}_\alpha=\widehat{P}_t(\alpha\widehat{G}_\alpha-I),\\ \frac{d}{dt}\widehat{P}^n_t\widehat{G}^n_\alpha &= \widehat{P}^n_t\widehat{\mathcal{L}}^n\widehat{G}^n_\alpha=\widehat{P}^n_t(\alpha\widehat{G}^n_\alpha-I).\end{aligned}$$

Thus in view of $\overline{\mathcal{F}}\subset\overline{\mathcal{F}}^n$, we have

$$\begin{aligned}\frac{d}{ds}\widehat{P}^n_{t-s}\widehat{G}^n_\alpha\widehat{P}_s\widehat{G}_\alpha &= -\widehat{P}^n_{t-s}(\alpha\widehat{G}^n_\alpha-I)\widehat{P}_s\widehat{G}_\alpha+\widehat{P}^n_{t-s}\widehat{G}^n_\alpha\widehat{P}_s(\alpha\widehat{G}_\alpha-I)\\ &= \widehat{P}^n_{t-s}(\widehat{P}_s\widehat{G}_\alpha-\widehat{G}^n_\alpha\widehat{P}_s)=\widehat{P}^n_{t-s}(\widehat{G}_\alpha-\widehat{G}^n_\alpha)\widehat{P}_s.\end{aligned}$$

Integrating in s over $[0,t]$ yields

$$\widehat{G}^n_\alpha\widehat{P}_t\widehat{G}_\alpha-\widehat{P}^n_t\widehat{G}^n_\alpha\widehat{G}_\alpha=\int_0^t\widehat{P}^n_{t-s}(\widehat{G}_\alpha-\widehat{G}^n_\alpha)\widehat{P}_s ds.$$

Hence for every $f\in\overline{\mathcal{F}}$,

$$\lim_{n\to\infty}\|\widehat{G}^n_\alpha(\widehat{P}_t-\widehat{P}^n_t)\widehat{G}_\alpha f\|_{L^2(E;m)}\leq\lim_{n\to\infty}\int_0^t\|\widehat{P}^n_{t-s}(\widehat{G}_\alpha-\widehat{G}^n_\alpha)\widehat{P}_s f\|_{L^2(E;m)}ds=0.$$

Since $\widehat{G}_\alpha(\overline{\mathcal{F}})$ is L^2-dense in $\overline{\mathcal{F}}$, we have for every $u \in \overline{\mathcal{F}}$,

$$\lim_{n\to\infty} \|\widehat{G}^n_\alpha(\widehat{P}_t - \widehat{P}^n_t)u\|_{L^2(E;m)} = 0.$$

On the other hand, by the L^2-contraction property of $\widehat{P}^n_t$ and that $\widehat{P}_t u \in \overline{\mathcal{F}}$, we have for $u \in \overline{\mathcal{F}} \subset \overline{\mathcal{F}}^n$, $\widehat{G}^n_\alpha \widehat{P}_t u - \widehat{P}_t \widehat{G}_\alpha u = (\widehat{G}^n_\alpha - \widehat{G}_\alpha)\widehat{P}_t u \to 0$ in $L^2(E;m)$ as $n \to \infty$, and $\widehat{G}^n_\alpha \widehat{P}^n_t u - \widehat{P}^n_t \widehat{G}_\alpha u = \widehat{P}^n_t(\widehat{G}^n_\alpha - \widehat{G}_\alpha)u \to 0$ in $L^2(E;m)$ as $n \to \infty$. It follows then

$$\lim_{n\to\infty} \|(\widehat{P}^n_t - \widehat{P}_t)\widehat{G}_\alpha u\|_{L^2(E;m)} = 0 \quad \text{for every } u \in \overline{\mathcal{F}}.$$

Since $\widehat{G}_\alpha(\overline{\mathcal{F}})$ is L^2-dense in $\overline{\mathcal{F}}$, we have $\lim_{n\to\infty} \|(\widehat{P}^n_t - \widehat{P}_t)u\|_{L^2(E;m)} = 0$ for every $u \in \overline{\mathcal{F}}$.

(ii) Conversely, assume $\overline{\mathcal{F}}^n$ converges to $\overline{\mathcal{F}}$ and $\lim_{n\to\infty} \|(\widehat{P}^n_t - \widehat{P}_t)u\|_{L^2(E;m)} = 0$ for every $u \in \overline{\mathcal{F}}$. Denote by Π^n and Π the orthogonal projection operator of $L^2(E;m)$ onto $\overline{\mathcal{F}}^n$ and $\overline{\mathcal{F}}$, respectively. We have by (2.3) and the L^2-contraction property of $\widehat{P}^n_t$ and $\widehat{P}_t$ that for every $\alpha > 0$ and $f \in L^2(E;m)$,

$$\begin{aligned}
\lim_{n\to\infty} \|G^n_\alpha f - G_\alpha f\|_{L^2(E;m)} &= \lim_{n\to\infty} \|\widehat{G}^n_\alpha(\Pi^n f) - \widehat{G}_\alpha(\Pi f)\|_{L^2(E;m)} \\
&\le \lim_{n\to\infty} \Big(\|(\widehat{G}^n_\alpha - \widehat{G}_\alpha)(\Pi f)\|_{L^2(E;m)} + \|\widehat{G}^n_\alpha(\Pi^n f - \Pi f)\|_{L^2(E;m)} \Big) \\
&\le \lim_{n\to\infty} \alpha^{-1} \|\Pi^n f - \Pi f\|_{L^2(E;m)} = 0.
\end{aligned}$$

It follows from Proposition 2.2 that $(\mathcal{E}^n, \mathcal{F}^n)$ converges to $(\mathcal{E}, \mathcal{F})$ in the sense of Mosco. □

3 Markov processes with darning

Suppose $(\mathcal{E}, \mathcal{F})$ is a regular symmetric Dirichlet form on $L^2(E;m)$. In particular, $\mathcal{F}$ is a dense linear subspace of $L^2(E;m)$. Let X be the Hunt process on E associated with $(\mathcal{E}, \mathcal{F})$. As we mentioned earlier, in this paper we use the convention that every $f \in \mathcal{F}$ is represented by its quasi-continuous version, which is unique up to an $\mathcal{E}$-polar set. Suppose that $K_1, \dots, K_N$ are disjoint, non-$\mathcal{E}$-polar compact subsets of E. Let $F = \cup_{j=1}^N K_j$ and $D = E \setminus F$. We short (or collapse) each K_j into a single point a^*_j. Formally, by identifying each K_j with a single point a^*_j, we can get an induced topological space $E^* := D \cup \{a^*_1, \dots, a^*_N\}$ from E, with a neighborhood of each a^*_j defined as $(U \cap D) \cup \{a^*_j\}$ for some neighborhood U of K_j in E. Let $m^* = m$ and D and set $m^*(F^*) = 0$, where $F^* := \{a^*_1, \dots, a^*_N\}$.

Definition 3.1 A strong Markov process on E^* is said to be a Markov process with darning obtained from X by shorting each K_j into a single point a^*_j, or simply a Markov process with darning, is an m^*-symmetric strong Markov process X^* on E^* such that

(i) the part process of X^* in D has the same law as the part process X in D for $\mathcal{E}$-q.e. starting point in D;

(ii) The jumping measure $J^*(dx, dy)$ and killing measure κ^* of X^* on E^* have the properties inherited from X without incurring additional jumps or killings, that is, they have the properties (1.1) and (1.2).

Remark 3.2 Note that if X^* is a Markov process with darning of X, it follows from Definition 3.1 that

$$\mathbb{P}_x(X^*_{\tau^*_D} = a^*_j) = \mathbb{P}_x(X_{\tau_D} \in K_j) \quad \text{for q.e. } x \in D, \tag{3.1}$$

where $\tau^*_D := \inf\{t > 0 : X^*_t \notin D\}$ and $\tau_D := \inf\{t > 0 : X_t \notin D\}$. Hence each $\{a^*_j$ is of positive capacity with respect to the process X^* because K_j is of positive $\mathcal{E}_1$-capacity. In particular, each $\{a^*_j\}$ is regular for itself; that is, $\mathbb{P}_{a^*_j}(\sigma_{a^*_j} = 0) = 1$, where $\sigma_{a^*_j} := \inf\{t > 0 : X^*_t = a^*_j\}$. This is due to the general fact that for any nearly Borel measurable set $A \subset E^*$, $A \setminus A^r$ is semipolar for process X^* and hence $\mathcal{E}^*$-polar. Here A^r denotes all the regular points for A with respect to the strong Markov process X^*.

We will show in this section that Markov process with darning X^* from X always exists and is unique in distribution.

For $1 \leq j \leq N$ and $\alpha > 0$, define

$$\varphi^{(j)}(x) := \mathbb{P}_x(X_{\sigma_F} \in K_j) \quad \text{and} \quad u^{(j)}_\alpha(x) := \mathbb{E}_x[e^{-\alpha\sigma_F}; X_{\sigma_F} \in K_j].$$

Here $\sigma_F := \inf\{t \geq 0 : X_t \in F\}$. Let $\mathcal{H}_\alpha$ be the linear span of $\{u^{(j)}_\alpha, j = 1, \cdots, N\}$, and $(\mathcal{E}, \mathcal{F}_D)$ the Dirichlet form for the part process X^D of X killed upon exiting D. Since each K_j is compact and $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form, there is a function $f_j \in C_c(E) \cap \mathcal{F}$ such that $f_j = 1$ on K_j and $f_j = 0$ on $\cup_{l:l\neq j} K_l$. Consequently, $u^{(j)}_\alpha(x) = \mathbb{E}_x[e^{-\alpha\sigma_F} f_j(X_{\sigma_F})]$ is the $\mathcal{E}_\alpha$-orthogonal projection of f_j to the complement of $\mathcal{F}_D$, where

$$\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha(u, v)_{L^2(E;m)} \quad \text{for } u, v \in \mathcal{F}.$$

So in particular, $u^{(j)}_\alpha \in \mathcal{F}$ for every $\alpha > 0$ and $1 \leq j \leq N$. Define

$$\mathcal{F}^* = \mathcal{F}_D \oplus \mathcal{H}_\alpha. \tag{3.2}$$

It is easy to see that the above definition of $\mathcal{F}^*$ is independent of $\alpha > 0$. The space $\mathcal{F}_D$ is exactly the collection of functions in $\mathcal{F}$ that vanish $\mathcal{E}$-quasi-everywhere ($\mathcal{E}$-q.e. in abbreviation) on D^c, while $u^{(j)}_\alpha = 1$ on $\mathcal{E}$-q.e. on K_j and vanishes $\mathcal{E}$-q.e. on K_l for $l \neq j$. Hence by regarding each $u^{(j)}_\alpha$ as a function defined on E^*, $\mathcal{F}^*$ can be viewed as a dense linear subspace of $L^2(E^*; m^*)$. Define

$$\mathcal{E}^*(u, v) = \mathcal{E}(u, v) \quad \text{for } u, v \in \mathcal{F}^*. \tag{3.3}$$

We will show in Theorem 3.3 below that $(\mathcal{E}^*, \mathcal{F}^*)$ is a regular Dirichlet form on $L^2(E^*; m^*)$. Consequently, it uniquely determines an m^*-symmetric Hunt process X^* on E^*.

As we saw from the above, $\mathcal{F}^* = \mathcal{F}_D \oplus \mathcal{H}_\alpha$ can be identified with functions in $\mathcal{F}$ that are constant $\mathcal{E}$-q.e. on each K_j. For $f \in \mathcal{F}$, define $\mathbf{H}^1_F f(x) = \mathbb{E}_x\left[e^{-\sigma_F} f(X_{\sigma_F})\right]$. Note that $f - \mathbf{H}^1_F f \in \mathcal{F}_D$ and $\mathbf{H}^1_F f$ is $\mathcal{E}_1$-orthogonal to $\mathcal{F}_D$.

Theorem 3.3 *$(\mathcal{E}^*, \mathcal{F}^*)$ is a regular symmetric Dirichlet form on $L^2(E^*; m^*)$ and its associated Hunt process X^* on E^* is a Markov process of darning obtained from X by shorting each K_j into a single point a^*_j.*

Proof: Let $\mathcal{C} = \{u \in \mathcal{F} \cap C_c(E) : u \text{ is constant on each } K_j\}$. By defining $u(a_j^*)$ to be the value of u on K_j, we can view $\mathcal{C}$ as a subspace of $\mathcal{F}^* \cap C_c(E^*)$. Since $\mathcal{C}$ is an algebra that separates points in E^*, $\mathcal{C}$ is uniformly dense in $C_\infty(E^*)$ by Stone-Weierstrass theorem. Next we show that $\mathcal{C}$ is $\mathcal{E}_1^*$-dense in $\mathcal{F}^*$. For this, it suffices to establish that each $u_j^{(1)}(x) := \mathbb{E}_x\left[e^{-\sigma_F}; X_{\sigma_F} \in K_j\right]$ can be $\mathcal{E}_1$-approximated by elements in $\mathcal{C}$. Let $f_j \in \mathcal{F} \cap C_c(E)$ so that $f_j = 1$ on K_j and $f_j = 0$ on K_i for $i \neq j$. Note that $u_j^{(1)} = \mathbf{H}_F^1 f_j = f_j - (f_j - \mathbf{H}_F^1 f_j)$ and $f_j - \mathbf{H}_F^1 f_j \in \mathcal{F}_D$. Since $(\mathcal{E}, \mathcal{F}_D)$ is a regular Dirichlet form on $L^2(D; m)$, there is a sequence $\{g_k, k \geq 1\} \subset \mathcal{F}_D \cap C_c(D)$ that is $\mathcal{E}_1$-convergent to $f_j - \mathbf{H}_F^1 f_j$. Let $v_k := f_j - g_k$, which is in $\mathcal{C}$ and $\mathcal{E}_1^*$-convergent to $u_j^{(1)}$. Thus we have established that $(\mathcal{E}^*, \mathcal{F}^*)$ is a regular Dirichlet form on $L^2(E^*; m^*)$.

Let X^* be the symmetric Hunt process on E^* associated with the regular Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(E^*; m^*)$. Clearly the part process of $X^{*,D}$ of X^* in D has the same distribution as the part process of X in D because the part Dirichlet forms of $(\mathcal{E}^*, \mathcal{F}^*)$ and $(\mathcal{E}, \mathcal{F})$ on D are the same. Denote by $J^*(dx, dy)$ and κ^* the jumping measure and killing measure of $(\mathcal{E}^*, \mathcal{F}^*)$. For every $f \in \mathcal{F}^*$, by the Beurling-Deny decomposition of $(\mathcal{E}^*, \mathcal{F}^*)$,

$$\mathcal{E}^*(f, f) = \mathcal{E}^{*c}(f, f) + \frac{1}{2} \int_{E^* \times E^*} (f(x) - f(y))^2 J^*(dx, dy) + \int_{E^*} f(x)^2 \kappa^*(dx),$$

where $\mathcal{E}^{*c}$ is a non-negative definite symmetric bilinear form on $\mathcal{F}^*$ that has strong local property. On the other hand, by (3.2), each $f \in \mathcal{F}^*$ can be regarded a function in $\mathcal{F}$ that is constant on each K_j and

$$\mathcal{E}^*(f, f) = \mathcal{E}(f, f) = \mathcal{E}^c(f, f) + \frac{1}{2} \int_{E \times E} (f(x) - f(y))^2 J(dx, dy) + \int_E f(x)^2 \kappa(dx).$$

Comparing the above two displays yields $\mathcal{E}^{*c}(f, f) = \mathcal{E}^c(f, f)$ and J^* and κ^* satisfy (1.1)-(1.2). This proves that X^* is a Markov process with darning for X. $\square$

The next result gives the uniqueness of the Markov process with darning for X.

Theorem 3.4 *Suppose X^* is a Markov process with darning for X in the sense of Definition 3.1. Then the Dirichlet form for X^* on $L^2(E^*; m^*)$ is the one $(\mathcal{E}^*, \mathcal{F}^*)$ given by* (3.2)-(3.3)*. Consequently, Markov process with darning for X is unique in distribution.*

Proof: Let $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ be the quasi-regular Dirichlet form of X^* on $L^2(E^*; m^*)$ (cf. [2, 12]). We want to show $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}) = (\mathcal{E}^*, \mathcal{F}^*)$. By Definition 3.1(i), $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}_D) = (\mathcal{E}, \mathcal{F}_D)$, where $\widetilde{\mathcal{F}}_D$ and $\mathcal{F}_D$ denote the part Dirichlet form of $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ and $(\mathcal{E}, \mathcal{F})$ on D, respectively. Let $F^* := \{a_1^*, \ldots, a_N^*\}$ and $\sigma^* := \inf\{t > 0 : X_t^* \in F^*\}$. By the $\widetilde{\mathcal{E}}_1$-orthogonal projection (see [2, Theorem 3.2.2]), for every $u \in \widetilde{\mathcal{F}}$, $\mathbf{H}_{F^*}^1 u(x) := \mathbb{E}_x\left[e^{-\sigma^*} u(X_{\sigma^*}^*)\right] \in \widetilde{\mathcal{F}}$ and $u - \mathbf{H}_{F^*}^1 u \in \widetilde{\mathcal{F}}_D = \mathcal{F}_D$. It follows from Definition 3.1 (cf. (3.1)) that for $x \in D$,

$$\mathbf{H}_{F^*}^1 u(x) = \sum_{j=1}^N u(a_j^*)\, \mathbb{E}_x\left[e^{-\sigma^*}; X_{\sigma^*}^* = a_j^*\right] = \sum_{j=1}^N u(a_j^*)\, \mathbb{E}_x\left[e^{-\sigma_F}; X_{\sigma_F} \in K_j\right] = \sum_{j=1}^N u(a_j^*)\, u_1^{(j)}(x).$$

As by Remark 3.2, each $\{a_j^*\}$ is of positive $\widetilde{\mathcal{E}}_1$-capacity, hence non-$\widetilde{\mathcal{E}}$-polar, we have

$$\left\{(u(a_1^*), \ldots, u(a_N^*));\, u \in \widetilde{\mathcal{F}}\right\} = \mathbb{R}^N$$

and so $\widetilde{\mathcal{F}} = \mathcal{F}^*$ by (3.2).

For $u \in \widetilde{\mathcal{F}} = \mathcal{F}^*$, let $\widetilde{\mu}^c_{\langle u\rangle}$ and $\mu^c_{\langle u\rangle}$ be the energy measure of u corresponding to the strongly local part $\widetilde{\mathcal{E}}^c$ and $\mathcal{E}^{*c}$ of the corresponding Dirichlet forms $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ and $(\mathcal{E}^*, \mathcal{F}^*)$, respectively. Since $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}_D) = (\mathcal{E}, \mathcal{F}_D) = (\mathcal{E}^*, \mathcal{F}^*_D)$, we have

$$\widetilde{\mu}^c_{\langle u\rangle}(D) = \mu^{*c}_{\langle u\rangle}(D).$$

On the other hand, for every bounded $u \in \widetilde{\mathcal{F}}$, since the energy measures $\widetilde{\mu}^c_{\langle u\rangle}$ and $\mu^{*c}_{\langle u\rangle}$ of u do not charge on level sets of u (cf. [2, Theorem 4.3.8]), we have

$$\widetilde{\mu}^c_{\langle u\rangle}(\{a^*_j\}) = 0 = \mu^{*c}_{\langle u\rangle}(\{a^*_j\}) \quad \text{for every } 1 \le j \le N.$$

Consequently,

$$\widetilde{\mu}^c_{\langle u\rangle}(E^* \setminus D) = \sum_{j=1}^N \widetilde{\mu}^c_{\langle u\rangle}(\{a^*_j\}) = 0 = \sum_{j=1}^N \mu^{*c}_{\langle u\rangle}(\{a^*_j\}) = \mu^{*c}_{\langle u\rangle}(E^* \setminus D).$$

We conclude from the above two displays that

$$\widetilde{\mathcal{E}}^c(u,u) = \frac{1}{2}\widetilde{\mu}^c_{\langle u\rangle}(E^*) = \frac{1}{2}\mu^{*c}_{\langle u\rangle}(E^*) = \mathcal{E}^{*c}(u,u)$$

for every bounded $u \in \widetilde{\mathcal{F}}$ and hence for every $u \in \widetilde{\mathcal{F}}$. Since X^* is a Markov process with darning for X, we have from Definition 3.1 that the jumping measure $\widetilde{J}$ and the killing measure $\widetilde{\kappa}$ of $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is the same as J^* and κ^* prescribed by (1.1)-(1.2). Hence by the Beurling-Deny decomposition of $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$, we have for every $u \in \widetilde{\mathcal{F}} = \mathcal{F}^*$,

$$\begin{aligned} \widetilde{\mathcal{E}}(u,u) &= \widetilde{\mathcal{E}}^c(u,u) + \frac{1}{2}\int_{E^*\times E^*} (f(x)-f(y))^2 \widetilde{J}(dx,dy) + \int_{E^*} f(x)^2 \widetilde{\kappa}(dx) \\ &= \mathcal{E}^{*c}(u,u) + \frac{1}{2}\int_{E^*\times E^*} (f(x)-f(y))^2 J^*(dx,dy) + \int_{E^*} f(x)^2 \kappa^*(dx) \\ &= \mathcal{E}^*(u,u). \end{aligned}$$

This proves that $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}) = (\mathcal{E}^*, \mathcal{F}^*)$. □

4 Approximation of Markov processes with darning

We continue to work under the setting of Section 3. Let X^* be the Markov process with darning obtained from X by shorting (or darning) each K_j into a single point a^*_j. In this section, we study its approximations, whose scheme can be used to simulate X^*. For this, we first need to introduce sticky Markov process with darning obtained from X^* by a time change to possibly prolong the time spent on each a^*_j.

Define $\mu = m^* + \sum_{j=1}^N m(K_j)\delta_{a^*_j}$, where δ_{a^*} is the Dirac measure concentrated at the point a^*_j. The smooth measure μ determines a positive continuous additive functional A^μ of X^*. In fact,

$$A^\mu_t = t \wedge \zeta^* + \sum_{j=1}^N m(K_j) L^{a^*_j}_t,$$

where ζ^* is the lifetime of X^* and $L^{a_j^*}$ is the local time of X^* at a_j^* having Revuz measure $\delta_{a_j^*}$. Let $\tau_t := \inf\{s > 0 : A_s^\mu > t\}$ and $Y_t = X^*_{\tau_t}$. Then the time-changed process $Y = \{Y_t; t \geq 0\}$ is μ-symmetric and has Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(E^*; \mu)$; see [2, 13]. The process Y is a sticky Markov process with darning, as it may spend positive amount of Lebesgue time at each a_j^*.

Conversely, starting with a sticky Markov process with darning Y on E^* associated with the regular Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(E^*; \mu)$, one can recover in distribution the Markov process with darning X^* on E^* through a time change as follows. Let $\widetilde{A}_t = \int_0^t 1_D(Y_s)ds$, which is a positive continuous additive functional of Y having Revuz measure m^*. Define its inverse $\widetilde{\tau}_t = \inf\{s > 0 : \widetilde{A}_s > t\}$. Then $\widetilde{X}_t := Y_{\widetilde{\tau}_t}$ is an m^*-symmetric strong Markov process on E^* whose associated Dirichlet form is $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(E; m^*)$ (cf. [2, 13]). In other words, $\widetilde{X}$ has the same distribution as X^*.

Let

$$\widetilde{\mathcal{F}} = \{f \in \mathcal{F} : f = \text{constant } \mathcal{E}\text{-q.e. on } K_j \text{ for each } 1 \leq j \leq N\}. \tag{4.1}$$

Note that $(\mathcal{E}, \widetilde{\mathcal{F}})$ is a closed symmetric Markovian bilinear form on $L^2(E; m)$ but $\widetilde{\mathcal{F}}$ is not dense in $L^2(E; m)$ in general since each K_j has positive $\mathcal{E}_1$-capacity. To emphasize its dependence on the domain of definition, we write $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ for $(\mathcal{E}, \widetilde{\mathcal{F}})$. The quadratic form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is a Dirichlet form in the wide sense; cf. [13, p. 29]. Denote by Π the orthogonal projection of $L^2(E; m)$ onto the closure $\overline{\widetilde{\mathcal{F}}}$ of $\widetilde{\mathcal{F}}$ in $L^2(E; m)$. Let $\{\widetilde{G}_\alpha, \alpha > 0\}$ be the resolvent associated with $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $L^2(E; m)$, and $\{\widehat{P}_t; t \geq 0\}$ and $(\widehat{G}_\alpha; \alpha > 0\}$ the semigroup and resolvent of the closed symmetric form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $\overline{\widetilde{\mathcal{F}}}$, respectively. We know from (2.3) that $\widetilde{G}_\alpha f = \widehat{G}_\alpha(\Pi f)$. We now identify $\widehat{P}_t$ and $\widehat{G}_\alpha$, as well as Π.

The following map T establishes a one-to-one and onto correspondence between the closed symmetric form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $\overline{\widetilde{\mathcal{F}}} \subset L^2(E; m)$ and the regular Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(E^*; \mu)$: for every $f \in \widetilde{\mathcal{F}}$,

$$Tf = f \ \text{ on } D \quad \text{ and } \quad Tf(a_j^*) = f(K_j) \ \text{ for } 1 \leq j \leq N. \tag{4.2}$$

(For $g \in \mathcal{F}^*$, $T^{-1}g(x) = g(x)$ for $x \in D$ and $T^{-1}g(x) = g(a_j^*)$ for $x \in K_j$.) The map T has the property that for every $f \in \widetilde{\mathcal{F}}$,

$$\widetilde{\mathcal{E}}(f, f) = \mathcal{E}^*(Tf, Tf) \quad \text{ and } \quad \|f\|_{L^2(E;m)} = \|Tf\|_{L^2(E^*;\mu)}. \tag{4.3}$$

In other words, T is an isometry between $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $\overline{\widetilde{\mathcal{F}}} \subset L^2(E; m)$ and $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(E^*; \mu)$ both in $\mathcal{E}$ and in the L^2 sense. Denote by $\{G_\alpha^*; \alpha > 0\}$ and $\{P_t^*; t \geq 0\}$ the resolvent and semigroup associated with the regular Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(E^*; \mu)$, and $\{\widehat{G}_\alpha; \alpha > 0\}$ and $\{\widehat{P}_t; t \geq 0\}$ the resolvent and semigroup on $\overline{\widetilde{\mathcal{F}}}$ associated with the Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $L^2(E; \mu)$ in the wide sense.

For every $f \in L^2(E; m)$, we can define a function f^* on E^* by setting $f^* = f$ on D and

$$f^*(a_j^*) := \begin{cases} \int_{K_j} f(y)m(dy)/m(K_j) & \text{when } m(K_j) > 0, \\ 0 & \text{when } m(K_j) = 0. \end{cases} \tag{4.4}$$

Clearly, $f^* \in L^2(E^*; \mu)$ for $f \in L^2(E; m)$ and $Tf = f^*$ for $f \in \overline{\widetilde{\mathcal{F}}}$. For an $\mathcal{E}^*$-quasi-continuous function g on E^*, we define

$$T^{-1}g(x) := \begin{cases} g(x) & \text{for } x \in D, \\ g(a_j^*) & \text{for } x \in K_j. \end{cases}$$

Clearly,

$$T \circ T^{-1} g = g \qquad \text{and} \qquad \|T^{-1} g\|_{L^2(E;m)} = \|g\|_{L^2(E^*;\mu)}.$$

Since T extends to be an isometry between $\overline{\widetilde{\mathcal{F}}} \subset L^2(E;m)$ and $L^2(E^*;\mu)$, the above defined T^{-1} extends to be an isometry between $L^2(E^*;\mu)$ and $\overline{\widetilde{\mathcal{F}}} \subset L^2(E;m)$. We conclude that

$$\overline{\widetilde{\mathcal{F}}} = T^{-1}(L^2(E^*;\mu)) = \{f \in L^2(E;m) : f \text{ is constant } m\text{-a.e. on each } K_i\}.$$

Theorem 4.1 (i) *For $f \in L^2(E;m)$, $\Pi f = T^{-1} f^*$ m-a.e.*

(ii) $\widetilde{G}_\alpha f = T^{-1} G^*_\alpha f^*$ *for $f \in L^2(E;m)$.*

(iii) *For $f \in \overline{\widetilde{\mathcal{F}}}$, $t > 0$ and $\alpha > 0$,*

$$\widehat{P}_t f = T^{-1} P^*_t(f^*) \quad \textit{and} \quad \widehat{G}_\alpha f = T^{-1} G^*_\alpha(f^*). \tag{4.5}$$

Proof: (i) Let $\mathcal{C}$ be the set defined in the proof of Theorem 3.3, which has shown to be $\mathcal{E}^*_1$-dense in $\mathcal{F}^*$ in $L^2(E^*;\mu)$. So in particular $\mathcal{C}$ is L^2-dense in $L^2(E^*;\mu)$. Consequently, $T^{-1}\mathcal{C}$ is $L^2(E;m)$-dense in $\overline{\widetilde{\mathcal{F}}}$. On the other hand, it is clear that for $f \in L^2(E;m)$,

$$(f, T^{-1} g)_{L^2(E;m)} = (T^{-1} f^*, T^{-1} g)_{L^2(E;m)} \quad \text{for every } g \in \mathcal{C}.$$

Thus we have $\Pi f = T^{-1} f^*$ m-a.e.

(ii) Let $f \in L^2(E;m)$. For every $\alpha > 0$ and $g \in \widetilde{\mathcal{F}}$, it follows from (4.3) that

$$\begin{aligned}
\widetilde{\mathcal{E}}_\alpha(\widetilde{G}_\alpha f, g) &= \int_E f(x) g(x) m(dx) = \int_{E^*} f^*(x) Tg(x) \mu(dx) \\
&= \mathcal{E}^*(G^*_\alpha f^*, Tg) + \alpha \int_{E^*} G^* f^*(x) Tg(x) \mu(dx) \\
&= \widetilde{\mathcal{E}}_\alpha(T^{-1} G^*_\alpha f^*, g).
\end{aligned}$$

Here in the last equality, we used (4.3). We thus conclude that $\widetilde{G}_\alpha f = T^{-1} G^*_\alpha f^*$.

(iii) This follows immediately from (i), (ii) and (2.3) that for $f \in \overline{\widetilde{\mathcal{F}}}$,

$$\widehat{G}_\alpha f = \widetilde{G}_\alpha f = T^{-1} G^*_\alpha f^*.$$

It is clear that $T_t f := T^{-1} P^*_t f^*$ defines a symmetric strongly continuous contraction semigroup on $\overline{\widetilde{\mathcal{F}}} \subset L^2(E;m)$, as $\{P^*_t; t \geq 0\}$ is a strongly continuous contraction semigroup on $L^2(E^*;m^*)$. Moreover, for every $\alpha > 0$, $\int_0^\infty e^{-\alpha t} T_t f dt = T^{-1} G^*_\alpha f^* = \widehat{G}_\alpha f$. Thus $T_t = \widehat{P}_t$. □

We now study an approximation scheme of Markov processes with darning by introducing additional jumps over each K_j with large intensity. For each j, let μ_j be a probability smooth measure whose quasi-support is K_j and having bounded 1-potential $G_1 \mu_j$, which always exists. For $\lambda > 0$, consider the symmetric regular Dirichlet form $(\mathcal{E}^{(\lambda)}, \mathcal{F})$ on $L^2(E;m)$ defined by (1.3). Observe that by [29] for every $u \in \mathcal{F}$,

$$\begin{aligned}
\sum_{j=1}^N \int_{K_j \times K_j} (u(x) - u(y))^2 \mu_j(dx) \mu_j(dy) &\leq \sum_{j=1}^N 4\mu_j(K_j) \int_E u(x)^2 \mu_j(dx) \\
&\leq \sum_{j=1}^N 4\mu_j(K_j) \|G_1 \mu_j\|_\infty \mathcal{E}_1(u, u).
\end{aligned}$$

Thus there is a constant $C_0 > 0$ so that

$$\mathcal{E}_1(u,u) \le \mathcal{E}^{(\lambda)}(u,u) \le (1+C_0\lambda)\mathcal{E}_1(u,u) \quad \text{for every } u \in \mathcal{F}.$$

It follows that for every $\lambda > 0$, $(\mathcal{E}^{(\lambda)}, \mathcal{F})$ is a regular Dirichlet form on $L^2(E;m)$.

Theorem 4.2 *For any increasing sequence $\{\lambda_n, n \ge 1\}$ of positive real numbers that converges to infinity, the Dirichlet form $(\mathcal{E}^{(\lambda_n)}, \mathcal{F})$ is Mosco convergent to the closed symmetric form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $L^2(E;m)$.*

Proof. Let $\{u_n, n \ge 1\}$ be a sequence in $L^2(E;m)$ that converges weakly to u in $L^2(E;m)$ with $\liminf_{n\to\infty} \mathcal{E}^{(\lambda_n)}(u_n,u_n) < \infty$. By taking a subsequence if necessary, we may and do assume that $\mathcal{E}^{(\lambda_n)}(u_n,u_n)$ converges, $\sup_{n\ge 1} \mathcal{E}_1^{(\lambda_n)}(u_n,u_n) < \infty$ and that the Cesàro mean sequence $\{v_n := \sum_{k=1}^n u_k/n; n \ge 1\}$ is $\mathcal{E}_1^{(1)}$-convergent to some $v \in \mathcal{F}$. (The last property follows from Banach-Saks Theorem, see, for example, Theorem A.4.1 of [2]). As in particular, v_n is $L^2(E;m)$-convergent to v, we must have $v = u$ m-a.e. on E. Hence $u \in \mathcal{F}$ has a quasi-continuous version which will still be denoted as u. Thus for every $k \ge 1$,

$$\begin{aligned}
\infty &> \liminf_{n\to\infty} \mathcal{E}^{(\lambda_n)}(u_n,u_n) \ge \lim_{n\to\infty} \mathcal{E}^{(\lambda_n)}(v_n,v_n) \ge \lim_{n\to\infty} \mathcal{E}^{(\lambda_k)}(v_n,v_n) = \mathcal{E}^{(\lambda_k)}(u,u) \\
&= \mathcal{E}(u,u) + \lambda_k \sum_{j=1}^N \int_{K_j\times K_j} (u(x)-u(y))^2 \mu_j(dx)\mu_j(dy). \qquad (4.6)
\end{aligned}$$

Letting $k \to \infty$ in above inequality, we conclude that for each $j = 1, \cdots, N$,

$$\int_{K_j\times K_j} (u(x)-u(y))^2 \mu_j(dx)\mu_j(dy) = 0.$$

This implies that u is constant μ_j-a.e. on K_j and hence by [2, Theorem 3.3.5] q.e. on K_j, because K_j is a quasi-support of μ_j. Thus $u \in \widetilde{\mathcal{F}}$ and by (4.6)

$$\liminf_{n\to\infty} \mathcal{E}^{(\lambda_n)}(u_n,u_n) \ge \widetilde{\mathcal{E}}(u,u),$$

which establishes part (a) for the Mosco convergence.

To show part (b) of the Mosco convergence, it suffices to establish it for $u \in \widetilde{\mathcal{F}}$ (for $u \notin \widetilde{\mathcal{F}}$, $\widetilde{\mathcal{E}}(u,u) = \infty$ and so the property holds automatically). Note that $\widetilde{\mathcal{F}} \subset \mathcal{F}$. We take $u_n = u$ for every $n \ge 1$. Then

$$\mathcal{E}^{(\lambda_n)}(u_n,u_n) = \mathcal{E}(u,u) = \widetilde{\mathcal{E}}(u,u) \quad \text{for every } n \ge 1.$$

This completes the proof of the theorem. □

Let $X^n = \{X_t^n, t \ge 0; \mathbb{P}_x^n, x \in E\}$ be the Hunt process associated with the regular Dirichlet form $(\mathcal{E}^{(\lambda_n)}, \mathcal{F})$ on $L^2(E;m)$. Recall that Y is the sticky Markov process with darning associated with the regular Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(E^*;\mu)$.

Theorem 4.3 *For every $0 = t_0 < t_1 < \cdots < t_k < \infty$ and bounded $\{f_j; 0 \le j \le k\} \subset \overline{\widetilde{\mathcal{F}}}$,*

$$\lim_{n\to\infty} \mathbb{E}_m^n \left[\prod_{j=0}^k f_j(X_{t_j}^n) \right] = \mathbb{E}_\mu^* \left[\prod_{j=0}^k f_j^*(Y_{t_j}) \right],$$

where f_j^* is defined by (4.4) with f_j in place of f.

Proof: For simplicity, we prove the theorem for $k = 2$; the other cases are similar. Note that the semigroup $\{P_t^n; t \geq 0\}$ associated with the regular Dirichlet form $(\mathcal{E}^{(\lambda_n)}, \mathcal{F})$ on $L^2(E; m)$ is given by $P_t^n f(x) = \mathbb{E}_x^n[f(X_t^n)]$, while, in view of Theorem 4.1, the semigroup $\{\widehat{P}_t; t \geq 0\}$ associated with the closed symmetric form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $\overline{\widetilde{\mathcal{F}}} \subset L^2(E; m)$ is given by

$$\widehat{P}_t f = T^{-1} P_t^* f^*, \quad \text{where } P_t^* f^*(x) = \mathbb{E}_x \left[f^*(Y_t) \right],$$

for $f \in \overline{\widetilde{\mathcal{F}}}$. By Theorems 4.2 and 2.3, $P_t^n f$ converges to $\widehat{P}_t f$ for every $f \in \overline{\widetilde{\mathcal{F}}}$ and $t > 0$. It follows that $f_1 P_{t_2 - t_1}^n f_2$ converges to $f_1 \widehat{P}_{t_2 - t_1} f_2$ in $L^2(E; m)$. Since $f_1 \widehat{P}_{t_2 - t_1} f_2 \in \overline{\widetilde{\mathcal{F}}}$, it follows

$$\begin{aligned}
& \lim_{n \to \infty} \| P_{t_1}^n (f_1 P_{t_2 - t_1}^n f_2) - \widehat{P}_{t_1} (f_1 \widehat{P}_{t_2 - t_1} f_2) \|_{L^2(E;m)} \\
\leq \; & \lim_{n \to \infty} \Big(\| P_{t_1}^n (f_1 P_{t_2 - t_1}^n f_2 - f_1 \widehat{P}_{t_2 - t_1} f_2) \|_{L^2(E;m)} + \| P_{t_1}^n (f_1 \widehat{P}_{t_2 - t_1} f_2) - \widehat{P}_{t_1} (f_1 \widehat{P}_{t_2 - t_1} f_2) \|_{L^2(E;m)} \Big) \\
\leq \; & \lim_{n \to \infty} \| f_1 P_{t_2 - t_1}^n f_2 - f_1 \widehat{P}_{t_2 - t_1} f_2 \|_{L^2(E;m)} = 0.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\lim_{n \to \infty} \mathbb{E}_m^n \left[\prod_{j=0}^k f_j(X_{t_j}^n) \right] &= \lim_{n \to \infty} \int_E f_0(x) P_{t_1}^n (f_1 P_{t_2 - t_1}^n f_2)(x) m(dx) \\
&= \int_E f_0(x) \widehat{P}_{t_1} (f_1 \widehat{P}_{t_2 - t_1} f_2)(x) m(dx) \\
&= \int_{E^*} f_0^*(x) P_{t_1}^* (f_1^* P_{t_2 - t_1}^* f_2^*)(x) \mu(dx) \\
&= \mathbb{E}_\mu^* \left[\prod_{j=k}^n f_j^*(Y_{t_j}) \right],
\end{aligned}$$

where in the third inequality we used Theorem 4.1(iii) and (4.3). □

Theorem 4.3 says that X^n converges to the sticky Markov process with darning Y in the finite dimensional sense for all the testing functions that are constant on each K_j.

When $(\mathcal{E}, \mathcal{F})$ is a local Dirichlet form (or, equivalently, when X is a diffusion on E) and each K_j is connected and has positive measure, it is possible to approximate sticky diffusions with darning by increasing the diffusion coefficients on each K_j to infinity. This provides a very intuitive picture for shorting of each K_j – achieved by increasing the conductance on K_j to infinity. We illustrate this by the following slightly more general example for which we allow $\lambda_n(x)$ to be a bounded function that tends to infinity on holes.

Suppose that $A(x) = (a_{ij}(x))_{1 \leq i,j \leq d}$ is a matrix-valued function on $\mathbb{R}^d$ that is uniformly elliptic and bounded, and ρ is a measurable function on $\mathbb{R}^d$ that is bounded between two positive constants. Define $\mathcal{F} = W^{1,2}(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d; dx) : \nabla u \in L^2(\mathbb{R}^d; dx) \right\}$ and

$$\mathcal{E}(u, v) = \frac{1}{2} \int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} \rho(x) dx, \quad u, v \in W^{1,2}(\mathbb{R}^d). \tag{4.7}$$

Then $(\mathcal{E},\mathcal{F})$ is a strongly local regular Dirichlet form on $L^2(\mathbb{R}^d;m)$, where $m(dx) := \rho(x)dx$. It uniquely determines an m-symmetric diffusion process X on $\mathbb{R}^d$ whose infinitesimal generator is

$$\mathcal{L} = \frac{1}{2\rho(x)} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(\rho(x) a_{ij}(x) \frac{\partial}{\partial x_j} \right).$$

Let $\{K_j; 1 \le j \le N\}$ be a finite number of disjoint compact sets which are the closure of non-empty connected open sets. Let $\{\lambda_n(x); n \ge 1\}$ be a sequence of bounded positive functions defined on $\cup_{j=1}^N K_j$ with $\lim_{n\to\infty} \lambda_n(x) = \infty$ a.e.. Define

$$\mathcal{E}^{(\lambda_n)}(u,v) = \mathcal{E}(u,v) + \sum_{l=1}^{N} \int_{K_l} (\nabla u(x) \cdot \nabla v(x))\, \lambda_n(x)\rho(x)dx, \quad u,v \in W^{1,2}(\mathbb{R}^d).$$

Clearly for every $n \ge 1$, $(\mathcal{E}^{(\lambda_n)},\mathcal{F})$ is a regular m-symmetric strongly local Dirichlet form on $L^2(\mathbb{R}^n;m)$ and so there is an m-symmetric diffusion process $X^{(n)}$ associated with it. Let $\widetilde{\mathcal{F}}$ be defined as in (4.1), and $\widetilde{\mathcal{E}} := \mathcal{E}$ on $\widetilde{\mathcal{F}}$.

Define $D = E \setminus \cup_{j=1}^N K_j$. We short (or collapse) each K_j into a single point a_j^*. Formally, by identifying each K_j with a single point a_j^*, we can get an induced topological space $E^* := D \cup \{a_1^*, \dots, a_N^*\}$ from E, with a neighborhood of each a_j^* defined as $(U \cap D) \cup \{a_j^*\}$ for some neighborhood U of K_j in E. We define a measure μ on E^* by setting $\mu = m$ and D and $\mu(\{a_j^*\}) = m(K_j)$. Let $(\mathcal{E}^*,\mathcal{F}^*)$ be defined from $(\mathcal{E},\mathcal{F})$ as in (3.2)-(3.3). Then $(\mathcal{E}^*,\mathcal{F}^*)$ is a regular Dirichlet form on $L^2(E^*;\mu)$. There is an associated diffusion process Y on E^*, which we call sticky diffusion process with darning. If we take m^* defined by $m^*(A) := \mu(A \cap D)$, the diffusion process X^* on E^* associated with the regular Dirichlet form $(\mathcal{E}^*,\mathcal{F}^*)$ on $L^2(E^*;m^*)$ is called diffusion process with darning. These two processes are related to each other by a time change.

Theorem 4.4 *Suppose $\{\lambda_n; n \ge 1\}$ is a sequence of bounded positive functions that converges to infinity a.e. on $\cup_{j=1}^N K_j$.*

(i) *The Dirichlet form $(\mathcal{E}^{(\lambda_n)},\mathcal{F})$ is Mosco convergent to the closed symmetric form $(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}})$ on $L^2(E;m)$.*

(ii) *Let $X^n = \{X_t^n, t \ge 0; \mathbb{P}_x^n, x \in E\}$ be the Hunt process associated with the regular Dirichlet form $(\mathcal{E}^{(\lambda_n)},\mathcal{F})$ on $L^2(E;m)$. Then X^n converges in the finite-dimensional distribution sense of Theorem 4.3 to the sticky diffusion with darning Y on E^*.*

Proof. In comparison with the proof of Theorem 4.2, additional care is needed in this proof since λ_n here is a function rather than a constant. Suppose $\{u_n, n \ge 1\}$ be a sequence in $L^2(E;m)$ that converges weakly to u in $L^2(E;m)$ with $\liminf_{n\to\infty} \mathcal{E}^{(\lambda_n)}(u_n,u_n) < \infty$. Fix an arbitrary $\lambda \ge 1$ and define

$$A_k(\lambda) = \left\{ x \in \cup_{j=1}^N K_j : \lambda_n(x) \ge \lambda \text{ for every } n \ge k \right\}.$$

Clearly $A_k(\lambda)$ is increasing in k and $\cup_{k\ge1} A_k(\lambda) = \cup_{j=1}^N K_j$ a.e. since $\lim_{n\to\infty} \lambda_n(x) = \infty$ a.e. on $\cup_{j=1}^N K_j$.

By taking a subsequence if necessary, we may and do assume that $\sup_{n\ge1} \mathcal{E}_1^{(\lambda_n)}(u_n,u_n) < \infty$, $\mathcal{E}^{(\lambda_n)}(u_n,u_n)$ and $\mathcal{E}(u_n,u_n)$ converge, $\int_{A_k(\lambda)} |\nabla u_n(x)|^2 \rho(x)dx$ converges for each $k \ge 1$, and that the Cesàro mean sequence $\{v_n := \sum_{k=1}^n u_k/n; n \ge 1\}$ is $\mathcal{E}_1$-convergent to some $v \in \mathcal{F}$. As in particular, v_n is $L^2(E;m)$-convergent to v, we must have $v = u$ m-a.e. on E. Hence $u \in \mathcal{F}$ has a quasi-continuous version which will still be denoted as u. Note that for each fixed $k \geq 1$,

$$\begin{aligned} \infty &> \liminf_{n\to\infty} \mathcal{E}^{(\lambda_n)}(u_n, u_n) \geq \lim_{n\to\infty} \mathcal{E}(u_n, u_n) + \lim_{n\to\infty} \lambda \int_{A_k(\lambda)} |\nabla u_n|^2 \rho(x)dx \\ &\geq \lim_{n\to\infty} \mathcal{E}(v_n, u_v) + \lim_{n\to\infty} \lambda \int_{A_k(\lambda)} |\nabla v_n|^2 \rho(x)dx \\ &\geq \mathcal{E}(u,u) + \lambda \int_{A_k(\lambda)} |\nabla u|^2 \rho(x)dx. \end{aligned}$$

Taking $k \to \infty$, we have from the above that for every $\lambda \geq 1$,

$$\infty > \liminf_{n\to\infty} \mathcal{E}^{(\lambda_n)}(u_n, u_n) \geq \mathcal{E}(u,u) + \lambda \int_{\cup_{j=1}^N K_j} |\nabla u|^2 \rho(x)dx. \tag{4.8}$$

This yields that $\nabla u = 0$ a.e. on K_j for each $j = 1, \ldots, N$. So u is constant a.e. in the interior of K_j. Since u is $\mathcal{E}$-quasi-continuous on $\mathbb{R}^d$, u is constant $\mathcal{E}$-q.e. on each K_j. Hence $u \in \widetilde{\mathcal{F}}$ and by (4.8)

$$\liminf_{n\to\infty} \mathcal{E}^{(\lambda_n)}(u_n, u_n) \geq \mathcal{E}(u,u) = \widetilde{\mathcal{E}}(u,u),$$

establishing part (a) for the Mosco convergence.

To show part (b) of the Mosco convergence, it suffices to establish it for $u \in \widetilde{\mathcal{F}}$. Note that $\widetilde{\mathcal{F}} \subset \mathcal{F}$. We take $u_n = u$. Then

$$\mathcal{E}^{(\lambda_n)}(u_n, u_n) = \mathcal{E}(u,u) = \widetilde{\mathcal{E}}(u,u) \quad \text{for every } n \geq 1.$$

This completes the proof that the Dirichlet form $(\mathcal{E}^{(\lambda_n)}, \mathcal{F})$ is Mosco convergent to the closed symmetric form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $L^2(E;m)$.

(ii) The proof is exactly the same as that for Theorem 4.3. $\square$

Remark 4.5 For simplicity, we took λ_n in Theorem 4.2 to be a constant. It is also possible to replace constants λ_n in Theorem 4.2 by bounded positive functions $\lambda_n(x)$ so that $\lim_{n\to\infty} \lambda_n(x) = \infty$ μ_j-a.e. on K_j for each $j = 1, \ldots, N$. In this case, the Dirichlet form $(\mathcal{E}^{(\lambda_n)}, \mathcal{F})$ is given by

$$\mathcal{E}^{(\lambda_n)}(u,v) = \mathcal{E}(u,v) + \sum_{j=1}^N \int_{K_j \times K_j} (u(x) - u(y))(v(x) - v(y))\lambda_n(x)\lambda_n(y)\mu_j(dx)\mu_j(dy). \tag{1.3'}$$

By a similar argument as that for Theorem 4.4 but by replacing $A_k(\lambda)$ with

$$A_k^{(j)}(\lambda) := \{x \in K_j : \lambda_n(x) \geq \lambda \text{ for every } n \geq k\},$$

one can show that $(\mathcal{E}^{(\lambda_n)}, \mathcal{F})$ is Mosco convergent to $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$, and, consequently, Theorem 4.3 holds for their corresponding processes.

Remark 4.6 In Theorem 4.2 (constant λ_n case), by taking a sub-subsequence, one can also use the Monotone convergence theorem from [28, Theorem S.14] (extended to non-dense domain case in [27, Proposition 2.2]) to show that the resolvents $\{G_\alpha^{(n)}, \alpha > 0\}$ of $(\mathcal{E}^{(\lambda_n)}, \mathcal{F})$ is L^2-convergent to the resolvent $\{G_\alpha; \alpha > 0\}$ of $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$. This implies by Proposition 2.2 the Mosco convergence of the corresponding Dirichlet forms. One can then apply the extended Mosco convergence theorem, Theorem 2.3, to get the strong L^2-convergence of the semigroups $\{P_t^{(n)}; t \geq 0\}$ to the associated semigroup $\{\widehat{P}_t; t \geq 0\}$ of the Dirichlet form in the wide sense $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$, and consequently the convergence of the processes in the sense of finite-dimensioanl distributions as stated in Theorem 4.3.

However, for variable $\lambda_n(x)$ in Theorem 4.4 and in Remark 4.5 for Theorem 4.2, the Monotone convergence theorem is not applicable in general.

5 Brownian motion on spaces with varying dimension

A simple example of spaces with varying dimension is a large square with a thin flag pole. Mathematically, it can be modeled by a plane with a vertical line installed on it:

$$\mathbb{R}^2 \cup \mathbb{R}_+ := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0 \text{ or } x_1 = x_2 = 0 \text{ and } x_3 > 0 \right\}. \tag{5.1}$$

Here $\mathbb{R}_+ := (0, +\infty)$. Spaces with varying dimension arise in many disciplines including statistics, physics and engineering (e.g. molecular dynamics, plasma dynamics). It is natural to study Brownian motion and "Laplacian operator" on such spaces. Intuitively, Brownian motion on space $\mathbb{R}^2 \cup \mathbb{R}_+$ should behave like a two-dimensional Brownian motion when it is on the plane, and like a one-dimensional Brownian motion when it is on the vertical line (flag pole). However the space $\mathbb{R}^2 \cup \mathbb{R}$ is quite singular in the sense that the base O of the flag pole where the plane and the vertical line meet is a singleton. A singleton would never be visited by a two-dimensional Brownian motion, which means Brownian motion starting from a point on the plane will never visit O. Hence there is no chance for such a process to climb up the flag pole. The solution is to collapse or short (imagine putting an infinite conductance on) a small closed disk $\overline{B(0, \varepsilon)} \subset \mathbb{R}^2$ centered at the origin into a point a^* and consider the resulting Brownian motion with darning on the collapsed plane, for which a^* will be visited. Through a^* a vertical pole can be installed and one can construct Brownian motion with varying dimension (BMVD) on $\mathbb{R}^2 \cup \mathbb{R}_+$ by joining together the Brownian motion with darning on the plane and the one-dimensional Brownian motion along the pole. This is done in [10] through a Dirichlet form method.

To be more precise, the state space of BMVD is defined as follows. Fix $\varepsilon > 0$ and $p > 0$. Let $D_0 = \mathbb{R}^2 \setminus \overline{B}(0, \varepsilon)$. By identifying the closed ball $\overline{B(0, \varepsilon)}$ with a singleton denoted by a^*, we can introduce a topological space $E^* := D_0 \cup \{a^*\} \cup \mathbb{R}_+$, with the origin of $\mathbb{R}_+$ identified with a^* and with the topology on E^* induced from that of $\mathbb{R}^2 \cup \mathbb{R}_+$. Let m_p^* be the measure on E^* whose restriction on $\mathbb{R}_+$ and D_0 is the Lebesgue measure multiplied by p and 1, respectively.

Definition 5.1 Let $\varepsilon > 0$ and $p > 0$. A Brownian motion with varying dimension (BMVD in abbreviation) on E^* with parameters (ε, p) on E^* is an m_p^*-symmetric diffusion X^* on E such that

(i) its part process in $\mathbb{R}_+$ (respectively, in D_0) has the same law as the part process of a standard Brownian motion killed upon leaving $\mathbb{R}_+$ (respectively, D_0).

(ii) it admits no killings at a^*.

It follows from the m_p^*-symmetry of X^* and the fact $m_p^*(\{a^*\}) = 0$ that BMVD X^* spends zero Lebesgue amount of time at a^*. It is shown in [10, Theorem 1.2] that for every $\varepsilon > 0$ and $p > 0$, BMVD with parameters (ε, p) exists and is unique in law. In fact, BMVD on E^* can be constructed as the m_p^*-symmetric Hunt process associated with the regular Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(E^*; m_p^*)$ given by

$$\begin{aligned} \mathcal{F}^* &= \left\{ f : f|_{D_0} \in W^{1,2}(D_0),\ f|_{\mathbb{R}_+} \in W^{1,2}(\mathbb{R}_+),\ \text{and } f(x) = f(0) \text{ q.e. on } \partial D_0 \right\}, && (5.2) \\ \mathcal{E}^*(f,g) &= \frac{1}{2}\int_{D_0} \nabla f(x) \cdot \nabla g(x) dx + \frac{p}{2}\int_{\mathbb{R}_+} f'(x) g'(x) dx. && (5.3) \end{aligned}$$

Here for an open set $U \subset \mathbb{R}^d$, $W^{1,2}(U)$ is the Sobolev space on U of order $(1,2)$; that is,

$$W^{1,2}(U) = \left\{ f \in L^2(U; dx) : \nabla f \in L^2(U; dx) \right\}.$$

Sample path properties of X^* including that at the base point and the two-sided transition density function estimates have been studied in [10]. Roughly speaking, when BMVD X^* is at the base point a^*, it enters the pole with probability $\frac{p}{2\pi\varepsilon + p}$ and enters the punched plane D_0 with probability $\frac{2\pi\varepsilon}{2\pi\varepsilon + p}$; see [10, Proposition 4.3].

We will show in this section that BMVD on E^* can be approximated by Brownian motion in the plane with a vertical cylinder whose horizontal motion on the cylinder is a circular Brownian motion moving at fast speed. Let

$$E := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \geq \varepsilon^2 \text{ and } x_3 = 0 \text{ or } x_1^2 + x_2^2 = \varepsilon^2 \text{ and } x_3 > 0 \right\}. \qquad (5.4)$$

That is, E is $D_0 \times \{0\}$ with a vertical cylinder with base radius ε sitting on top of $(\partial D_0) \times \{0\}$. Let $\overline{m}_p$ be the measure on E whose restriction on $D_0 \times \{0\}$ is the two-dimensional Lebesgue measure and its restriction to the cylinder $\partial D_0 \times [0, \infty)$ is the Lebesgue surface measure multiplied by $p/(2\pi\varepsilon)$. When there is no danger of confusion, we identify D_0 with $D_0 \times \{0\}$. The space E is a two-dimensional Lipschitz manifold. For every $\lambda > 0$, we can run an $\overline{m}_p$-symmetric diffusion $X^{(\lambda)}$ on E that behaves as Brownian motion on $D_0 \times \{0\}$ and behaves like $(B_{\lambda t}, W_t)$ while on the cylinder $\partial D_0 \times (0, \infty)$. Here B_t is a standard circular Brownian motion on ∂D_0 and W_t is a one-dimensional Brownian motion. We will show that as $\lambda \to \infty$, $X^{(\lambda)}$ converges in the finite-dimensional distribution sense, after a suitable identification, to the BMVD X^* on E^*; see Theorem 5.3 for a precise statement. Note that the state space E of $X^{(\lambda)}$, which is of dimension two, is different from the state space E^* of X^* with varying dimension. The space E^* can be viewed as E with the cylinder $\partial D_0 \times [0, \infty)$ collapsed into one single half line $\{a^*\} \times [0, \infty)$. The main difference, when compared with Brownian motion with darning in Section 3, is that here we collapse every circle $\partial D_0 \times \{z\}$ into one point $\{a^*\} \times \{z\}$ and there are a continuum of such circles to collapse. However the ideas developed in Section 4 can be adapted to establish the convergence of $X^{(\lambda)}$ to BMVD X^*, and we spell out the details in what follows.

First we give a precise construction of $X^{(\lambda)}$ via a Dirichlet form approach. First, we introduce Sobolve space $W^{1,2}(E)$ of order $(1,2)$ on E. For convenience, let $S := \partial D_0 \times (0, \infty)$. The space S can be identified with the infinite rectangle $[0, 2\pi\varepsilon] \times (0, \infty)$ with points $(0, z)$ and $(2\pi\varepsilon, z)$ identified. We denote the pull back measure on S of the Lebesgue measure on $[0, 2\pi\varepsilon] \times (0, \infty)$ by m. Functions on S can be parametrized by $(t, z) \in [0, 2\pi\varepsilon) \times [0, \infty)$. We define

$$W^{1,2}(S) = \left\{ f = f(t,z) \in L^2(S; m) : |\partial_t f| + |\partial_z f| \in L^2(S; m) \right\}.$$

Note that $W^{1,2}(D_0)$ and $W^{1,2}(S)$ are the Dirichlet spaces for the reflecting Brownian motion on D_0 and on the cylinder S, respectively. So for every $f \in W^{1,2}(D_0)$ and $g \in W^{1,2}(S)$, their quasi-continuous versions are well defined on $\partial D_0 = \partial S$ quasi-everywhere, which we call the trace on the circle ∂D_0 and we denote them by $f|_{\partial D_0}$ and $g|_{\partial D_0}$, respectively. Define

$$W^{1,2}(E) := \left\{ f : f_1 := f|_{D_0} \in W^{1,2}(D_0), f_2 = f|_S \in W^{1,2}(S) \text{ and } f_1|_{\partial D_0} = f_2|_{\partial D_0} \text{ q.e.} \right\}$$

For $f \in W^{1,2}(E)$, define its norm $\|f\|_{1,2}$ by

$$\|f\|_{1,2}^2 = \int_{D_0} |\nabla f(x)|^2 dx + \int_S \left(|\partial_t f(t,z)|^2 + |\partial_z f(t,z)^2 \right) m(dtdz).$$

It is easy to see that $W^{1,2}(E)$ is the $\|\cdot\|_{1,2}$-completion of the following subspace of continuous functions on E:

$$\left\{ f \in C(E) : \ f|_{D_0} \in W^{1,2}(D_0), \ f|_S \in W^{1,2}(S) \right\}$$

Now for every $\lambda > 0$, define $\mathcal{F}^{(\lambda)} = W^{1,2}(E)$ and for $f \in \mathcal{F}^{(\lambda)}$,

$$\begin{aligned} \mathcal{E}^{(\lambda)}(f,f) &= \frac{1}{2} \int_{D_0} |\nabla f(x)|^2 dx + \frac{\lambda p}{4\pi\varepsilon} \int_0^\infty \left(\int_0^{2\pi\varepsilon} |\partial_t f(t,z)|^2 dt \right) dz \\ &\quad + \frac{p}{4\pi\varepsilon} \int_0^{2\pi\varepsilon} \left(\int_0^\infty |\partial_z f(t,z)|^2 dz \right) dt. \end{aligned} \tag{5.5}$$

The last two terms in the right hand side of (5.5) represent the $\mathcal{E}^{(\lambda)}$-energy of f on the cylinder S. It is easy to check that $(\mathcal{E}^{(\lambda)}, \mathcal{F}^{(\lambda)})$ is a symmetric regular strongly local Dirichlet form on $L^2(E; \overline{m}_p)$ and so it uniquely determines a symmetric Hunt process $X^{(\lambda)}$ on E. Using the part Dirichlet form of $(\mathcal{E}^{(\lambda)}, \mathcal{F}^{(\lambda)})$ on D_0 and S, respectively, it is easy to see [2, 13] that the part process of $X^{(\lambda)}$ in D_0 and S are the part process of two dimension Brownian motion in D_0 and $(B_{\lambda t}, W_t)$ on S, respectively. Here B_t is the circular Brownian motion on ∂D_0 and W_t is a Brownian motion on $(0, \infty)$ independent of B_t.

Let

$$\widetilde{\mathcal{F}} = \{ f \in W^{1,2}(E) : f = \text{constant } \mathcal{E}\text{-q.e. on } \partial D_0 \times \{z\} \text{ for each } z \geq 0 \}. \tag{5.6}$$

Note that since each circle $\partial D_0 \times \{z\}$ is of positive $\mathcal{E}_1^{(\lambda)}$-capacity for every $\lambda > 0$, $(\mathcal{E}^{(\lambda)}, \widetilde{\mathcal{F}})$ is a closed symmetric Markovian bilinear form on $L^2(E; \overline{m}_p)$ but $\widetilde{\mathcal{F}}$ is not dense in $L^2(E; m)$. Note that $\mathcal{E}^{(\lambda)} = \mathcal{E}^{(1)}$ on $\widetilde{\mathcal{F}}$ for every $\lambda > 0$. To emphasize its dependence on the domain of definition, we write $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ for $(\mathcal{E}^{(1)}, \widetilde{\mathcal{F}})$. Denote by Π the orthogonal projection of $L^2(E; \overline{m}_p)$ onto the closure $\overline{\widetilde{\mathcal{F}}}$ of $\widetilde{\mathcal{F}}$ in $L^2(E; \overline{m}_p)$. Let $\{\widetilde{G}_\alpha, \alpha > 0\}$ be the resolvent associated with $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $L^2(E; m)$, and $\{\widehat{P}_t; t \geq 0\}$ and $(\widehat{G}_\alpha; \alpha > 0\}$ the strongly continuous semigroup and resolvent of the closed symmetric form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $\overline{\widetilde{\mathcal{F}}}$, respectively. We know from (2.3) that $\widetilde{G}_\alpha f = \widehat{G}_\alpha(\Pi f)$. We next identify $\widehat{P}_t$ and $\widehat{G}_\alpha$, as well as Π.

The following map T establishes a one-to-one and onto correspondence between $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $\overline{\widetilde{\mathcal{F}}} \subset L^2(E; \overline{m}_P)$ and $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(E^*; m_p^*)$: for every $f \in \widetilde{\mathcal{F}}$,

$$Tf = f \quad \text{on } D_0 \qquad \text{and} \qquad Tf(a^*, z) = f(\partial D_0 \times \{z\}) \quad \text{for } z \geq 0. \tag{5.7}$$

(For $g \in \mathcal{F}^*$, $T^{-1}g(x) = g(x)$ for $x \in D_0$ and $T^{-1}g(t,z) = g(a^*, z)$ for $(t,z) \in S$.) The map T has the property that for every $f \in \widetilde{\mathcal{F}}$,

$$\widetilde{\mathcal{E}}(f,f) = \mathcal{E}^*(Tf, Tf) \quad \text{and} \quad \|f\|_{L^2(E;\overline{m}_p)} = \|Tf\|_{L^2(E^*;m_p^*)}. \tag{5.8}$$

In other words, T is an isometry between $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $\overline{\widetilde{\mathcal{F}}} \subset L^2(E;\overline{m}_p)$ and $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(E^*; m_p^*)$ both in $\mathcal{E}$ and in the L^2 sense. Denote by $\{G_\alpha^*; \alpha > 0\}$ and $\{P_t^*; t \geq 0\}$ the resolvent and semigroup associated with the regular Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(E^*; m_p^*)$, and $\{\widehat{G}_\alpha; \alpha > 0\}$ and $\{\widehat{P}_t; t \geq 0\}$ the resolvent and semigroup on $\overline{\widetilde{\mathcal{F}}}$ associated with the closed symmetric form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $L^2(E;\overline{m}_p)$.

For every $f \in L^2(E;\overline{m}_p)$, we can define a function f^* on E^* by setting $f^* = f$ on D_0 and

$$f^*(a^*, z) = \frac{1}{2\pi\varepsilon} \int_{\partial D_0} f(t,z)dt \quad \text{for } z > 0. \tag{5.9}$$

Note that by Fubini theorem, $f^*(a^*, z)$ is well defined for a.e. $z \in (0,\infty)$. For an $\mathcal{E}^*$-quasi-continuous function g defined on E^*, we define

$$T^{-1}g = g(x) \quad \text{on } D_0 \qquad \text{and} \qquad T^{-1}g(t,z) = g(a^*, z) \quad \text{for } (t,z) \in \partial D_0 \times [0,\infty).$$

Clearly,

$$(T^{-1}g)^* = g \qquad \text{and} \qquad \|g\|_{L^2(E^*;m_p^*)} = \|T^{-1}g\|_{L^2(E;\overline{m}_p)}. \tag{5.10}$$

Since T extends to be an isometry between $\overline{\widetilde{\mathcal{F}}} \subset L^2(E;\overline{m}_p)$ and $L^2(E^*; m_p^*)$, the above defined T^{-1} extends to be an isometry between $L^2(E^*; m_p^*)$ and $\overline{\widetilde{\mathcal{F}}} \subset L^2(E;\overline{m}_p)$.

Theorem 5.2 (i) *For $f \in L^2(E;\overline{m}_p)$, $\Pi f = T^{-1}f^*$ $\overline{m}_p$-a.e. on E.*

(ii) *$\widehat{G}_\alpha f = T^{-1}G_\alpha^* f^*$ for $f \in L^2(E;\overline{m}_p)$.*

(iii) *For $f \in \overline{\widetilde{\mathcal{F}}}$, $t > 0$ and $\alpha > 0$,*

$$\widehat{P}_t f = T^{-1}P_t^*(f^*) \quad \textit{and} \quad \widehat{G}_\alpha f = T^{-1}G_\alpha^*(f^*). \tag{5.11}$$

Proof: (i) Let $\mathcal{C} = \mathcal{F}^* \cap C_c(E^*)$, which is a core of the regular Dirichlet form $(\mathcal{E}^*, \mathcal{F}^*)$ on $L^2(E^*; m_p^*)$. In particular, $\mathcal{C}$ is L^2-dense in $L^2(E^*; m_p^*)$. It follows from (5.8) and (5.10) that $T^{-1}\mathcal{C}$ is $L^2(E;\overline{m}_p)$-dense in $\overline{\widetilde{\mathcal{F}}}$ and by Fubini's theorem,

$$\overline{\widetilde{\mathcal{F}}} = \left\{ f \in L^2(E;\overline{m}_p) : f \text{ is constant a.e. on } \partial D_0 \times \{z\} \text{ for a.e. } z > 0 \right\}.$$

Thus for every $f \in L^2(E;\overline{m}_p)$, $f^* \in \overline{\widetilde{\mathcal{F}}}$. On the other hand, it is clear that for $f \in L^2(E;\overline{m}_p)$,

$$(f, T^{-1}g)_{L^2(E;\overline{m}_p)} = (T^{-1}f^*, T^{-1}g)_{L^2(E;\overline{m}_p)} \quad \text{for every } g \in \mathcal{C}.$$

Thus we conclude that $\Pi f = T^{-1}f^*$ $\overline{m}_p$-a.e.

(ii) Let $f \in L^2(E;\overline{m}_p)$. For every $\alpha > 0$ and $g \in \widetilde{\mathcal{F}}$, it follows from (5.8) that

$$\begin{aligned}
\widetilde{\mathcal{E}}_\alpha(\widetilde{G}_\alpha f, g) &= \int_E f(x)g(x)\overline{m}_p(dx) = \int_{E^*} f^*(x)Tg(x)m_p^*(dx) \\
&= \mathcal{E}^*(G^*_\alpha f^*, Tg) + \alpha \int_{E^*} G^* f^*(x) Tg(x) m_p^*(dx) \\
&= \widetilde{\mathcal{E}}_\alpha(T^{-1}G^*_\alpha f^*, g).
\end{aligned}$$

We thus have $\widetilde{G}_\alpha f = T^{-1}G^*_\alpha f^*$.

(iii) This follows immediately from (i), (ii) and (2.3) that for $f \in \overline{\widetilde{\mathcal{F}}}$,

$$\widehat{G}_\alpha f = \widetilde{G}_\alpha f = T^{-1}G^*_\alpha f^*.$$

It is clear that $T_t f := T^{-1}P_t^* f^*$ defines a symmetric strongly continuous contraction semigroup on $\overline{\widetilde{\mathcal{F}}} \subset L^2(E;\overline{m}_p)$, as $\{P_t^*; t \geq 0\}$ is a strongly continuous contraction semigroup on $L^2(E^*; m_p^*)$. Moreover, for every $\alpha > 0$, $\int_0^\infty e^{-\alpha t} T_t f dt = T^{-1}G^*_\alpha f^* = \widehat{G}_\alpha f$. We thus conclude that $T_t = \widehat{P}_t$. $\square$

Theorem 5.3 *Suppose $\{\lambda_n; n \geq 1\}$ is an increasing sequence of positive numbers that increases to infinity.*

(i) *The Dirichlet form $(\mathcal{E}^{(\lambda_n)}, \mathcal{F})$ is Mosco convergent to the closed symmetric form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $L^2(E;\overline{m}_p)$.*

(ii) *Let $X^n = \{X_t^n, t \geq 0; \mathbb{P}_x^n, x \in E\}$ be the Hunt process associated with the regular Dirichlet form $(\mathcal{E}^{(\lambda_n)}, \mathcal{F})$ on $L^2(E;\overline{m}_p)$. Then X^n converges in the finite dimensional distribution sense of Theorem 4.3 to the BMVD X^* on E^*.*

Proof. Let $\{u_n, n \geq 1\}$ be a sequence in $L^2(E;\overline{m}_p)$ that converges weakly to u in $L^2(E;\overline{m}_p)$ with $\liminf_{n\to\infty} \mathcal{E}^{(\lambda_n)}(u_n,u_n) < \infty$. By taking a subsequence if necessary, we may and do assume that $\mathcal{E}^{(\lambda_n)}(u_n,u_n)$ converges, $\sup_{n\geq 1} \mathcal{E}_1^{(\lambda_n)}(u_n,u_n) < \infty$ and that the Cesàro mean sequence $\{v_n := \sum_{k=1}^n u_k/n; n \geq 1\}$ is $\mathcal{E}_1^{(1)}$-convergent to some $v \in \mathcal{F}$. Since v_n is $L^2(E;\overline{m}_p)$-convergent to v, we must have $v = u$ $\overline{m}_p$-a.e. on E. Hence u has a quasi-continuous version which will still be denoted as u. Thus for every $k \geq 1$,

$$\begin{aligned}
\infty &> \lim_{n\to\infty} \mathcal{E}^{(\lambda_n)}(u_n,u_n) \geq \lim_{n\to\infty} \mathcal{E}^{(\lambda_n)}(v_n,v_n) \geq \lim_{n\to\infty} \mathcal{E}^{(\lambda_k)}(v_n,v_n) = \mathcal{E}^{(\lambda_k)}(u,u) \\
&= \frac{1}{2}\int_{D_0} |\nabla u(x)|^2 dx + \frac{\lambda_k p}{4\pi\varepsilon}\int_0^\infty \left(\int_0^{2\pi\varepsilon} |\partial_t u(t,z)|^2 dt \right) dz \\
&\quad + \frac{p}{4\pi\varepsilon}\int_0^{2\pi\varepsilon} \left(\int_0^\infty |\partial_z u(t,z)|^2 dz \right) dt. \qquad (5.12)
\end{aligned}$$

Letting $k \to \infty$ in above inequality, we conclude that there is a subset $\mathcal{N} \subset (0,\infty)$ having zero Lebesgue measure so that for every $z \in (0,\infty) \setminus \mathcal{N}$, $\int_0^{2\pi\varepsilon} |\partial_t u(t,z)|^2 dt = 0$. This implies that for every $z \in (0,\infty)\setminus\mathcal{N}$, $t \mapsto u(t,z)$ is equal to a constant $u(z)$ a.e. and hence $\mathcal{E}^{(1)}$-q.e. on $[0,2\pi) \times \{z\}$. For $0 < z_1 < z_2$ in $(0,\infty) \setminus \mathcal{N}$, by Cauchy-Schwarz inequality,

$$\begin{aligned}
|u(z_2) - u(z_1)| &= \frac{1}{2\pi\varepsilon}\left| \int_0^{2\pi\varepsilon} (u(t,z_2) - u(t,z_1)) dt \right| = \frac{1}{2\pi\varepsilon}\left| \int_0^{2\pi\varepsilon}\int_{z_1}^{z_2} \partial_z u(t,z) dz dt \right| \\
&\leq \frac{1}{\sqrt{2\pi\varepsilon}} \left(\int_0^{2\pi\varepsilon}\int_{z_1}^{z_2} |\partial_z u(t,z)|^2 dz dt \right) |z_2 - z_1|^{1/2}.
\end{aligned}$$

This shows that $u(z)$ is a Hölder continuous function on $[0,\infty)$. Since each horizontal circle and each vertical line on the cylinder is of positive $\mathcal{E}_1^{(1)}$-capacity and u is $\mathcal{E}^{(1)}$-quasi-continuous on E, it follows that an $\mathcal{E}^{(1)}$-quasi-continuous version of u can be taken so that $u(t,z)=u(z)$ for every $z\geq 0$ and $t\in[0,2\pi)$ (such defined function is continuous on the cylinder S). Hence $u\in\widetilde{\mathcal{F}}$ and by (5.12)

$$\liminf_{n\to\infty}\mathcal{E}^{(\lambda_n)}(u_n,u_n)\geq\widetilde{\mathcal{E}}(u,u),$$

which establishes (a) for the Mosco convergence.

To show (b) of the Mosco convergence, it suffices to establish it for $u\in\widetilde{\mathcal{F}}$. Note that $\widetilde{\mathcal{F}}\subset\mathcal{F}^{(\lambda)}$ for every $\lambda>0$. We take $u_n=u$. Then

$$\mathcal{E}^{(\lambda_n)}(u_n,u_n)=\widetilde{\mathcal{E}}(u,u)\quad\text{for every } n\geq 1.$$

This proves that the Dirichlet form $(\mathcal{E}^{(\lambda_n)},\mathcal{F})$ is Mosco convergent to $(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}})$ on $L^2(E;\overline{m}_p)$.

(ii) The proof is similar to that for Theorem 4.3 except using Theorem 5.2 instead of Theorem 4.1. We omit its details here. □

We remark that, since each horizontal circle on the cylinder that is to be collapsed into one single point has zero $\overline{m}_p$ measure, so the limiting process of X^n is just the BMVD X^* on E^*, not a sticky one.

For other related work and approaches on Markov processes living on spaces with possibly different dimensions, we refer the reader to [11, 15, 22] and the references therein.

6 Examples

In this section, we give some examples of the Dirichlet forms $(\mathcal{E},\mathcal{F})$, or equivalently symmetric Markov processes, for which the main results in Section 4 are applicable.

Example 6.1 (Sticky diffusion process with darning) Let $(\mathcal{E},\mathcal{F})$ be the strongly local regular Dirichlet form on $L^2(\mathbb{R}^d;m)$ defined by (4.7), where $m(dx)=\rho(x)dx$. Suppose that $K_1,\dots,K_N$ are separated, non-$\mathcal{E}$-polar compact (possibly disconnected) subsets of E. Let $F=\cup_{j=1}^N K_j$ and $D=\mathbb{R}^d\setminus F$. We short (or collapse) each K_j into a single point a_j^*. By identifying each K_j with a single point a_j^*, we can get an induced topological space $E^*:=D\cup\{a_1^*,\dots,a_N^*\}$ from E, with a neighborhood of each a_j^* defined as $(U\cap D)\cup\{a_j^*\}$ for some neighborhood U of K_j in E. Let $\mu=m$ on D and $\mu(a_j^*)=m(K_j)$. Let $(\mathcal{E}^*,\mathcal{F}^*)$ be defined as in (3.2)-(3.3). Then it is a regular Dirichlet form on $L^2(E^*;\mu)$. There is a unique diffusion process Y on E^* associated with it, which we call sticky diffusion process with darning. When $(a_{ij}(x))\equiv I$, the identity matrix, and $\rho\equiv 1$, Y is called sticky Brownian motion with darning. For each $1\leq j\leq N$, take a probability smooth measure μ_j whose quasi-support is K_j and having bounded 1-potential $G_1\mu_j$. For each $\lambda>0$, let $\mathcal{E}^{(\lambda)}$ be defined by (1.3). $(\mathcal{E}^{(\lambda)},\mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d;m)$ and it determines a diffusion process with jumps $X^{(\lambda)}$. By Theorem 4.3, for any increasing sequence $\{\lambda_n;n\geq 1\}$ that increases to infinity, $X^{(\lambda_n)}$ converges in the finite dimensional distribution in the sense of Theorem 4.3 to the sticky diffusion process with darning Y on E^*.

Example 6.2 (Sticky stable process with darning) Suppose the metric measure space (E, ρ, m) is a d-set; that is, there are positive constants c_1, c_2 so that

$$c_1 r^d \leq m(B(x,r)) \leq c_2 r^d \quad \text{for every } x \in E \text{ and } 0 < r < 1.$$

Here $B(x,r) := \{y \in E : \rho(y,x) < r\}$ is the open ball centered at x with radius r. Suppose $c(x,y)$ is a symmetric function on $E \times E$ that is bounded between two positive constants, and $0 < \alpha < 2$. Define

$$\mathcal{E}(f,f) = \int_{E\times E} (f(x) - f(y))^2 \frac{c(x,y)}{\rho(x,y)^{d+\alpha}} m(dx)m(dy),$$

and let $\mathcal{F}$ be the closure of Lipschitz functions on E with compact support under $\mathcal{E}_1$, where $\mathcal{E}_1(f,f) := \mathcal{E}(f,f) + \int_E f(x)^2 m(dx)$. The bilinear form $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(E;m)$. Its associated Hunt process X is called α-stable-like process on E (cf. [8, 9]). Suppose that $K_1, \ldots, K_N$ are separated, non-$\mathcal{E}$-polar compact (possibly disconnected) subsets of E. Let $F = \cup_{j=1}^N K_j$ and $D = E \setminus F$. We short (or collapse) each K_j into a single point a_j^*. By identifying each K_j with a single point a_j^*, we can get an induced topological space $E^* := D \cup \{a_1^*, \ldots, a_N^*\}$ from E, with a neighborhood of each a_j^* defined as $(U \cap D) \cup \{a_j^*\}$ for some neighborhood U of K_j in E. Let $\mu = m$ and D and $\mu(a_j^*) = m(K_j)$. Let $(\mathcal{E}^*, \mathcal{F}^*)$ be defined as in (3.2)-(3.3). Then it is a regular Dirichlet form on $L^2(E^*;\mu)$. There is a unique Hunt process Y on E^* associated with it, which we call sticky α-stable-like process with darning. For each $\lambda > 0$, let $\mathcal{E}^{(\lambda)}$ be defined by (1.3). $(\mathcal{E}^{(\lambda)}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d;m)$ and it determines a jump diffusion $X^{(\lambda)}$. By Theorem 4.3, for any increasing sequence $\{\lambda_n; n \geq 1\}$ that increases to infinity, $X^{(\lambda_n)}$ converges in the finite dimensional distribution in the sense of Theorem 4.3 to the sticky α-stable-like process with darning Y on E^*.

Similarly, we can consider darning of symmetric diffusions with jumps studied in [7] and their approximation by introducing additional jumps over the hulls K_j.

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