Perturbation by Non-Local Operators

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Abstract

Suppose that $d \geq 1$ and $0 < \beta < \alpha < 2$. We establish the existence and uniqueness of the fundamental solution $q^b(t,x,y)$ to a class of (possibly nonsymmetric) non-local operators $\mathcal{L}^b = \Delta^{\alpha/2} + \mathcal{S}^b$, where

$$\mathcal{S}^b f(x) := \mathcal{A}(d,-\beta) \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - \nabla f(x) \cdot z \mathbb{1}_{\{|z|\leq 1\}} \right) \frac{b(x,z)}{|z|^{d+\beta}} dz$$

and $b(x,z)$ is a bounded measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ with $b(x,z) = b(x,-z)$ for $x,z \in \mathbb{R}^d$. Here $\mathcal{A}(d,-\beta)$ is a normalizing constant so that $\mathcal{S}^b = \Delta^{\beta/2}$ when $b(x,z) \equiv 1$. We show that if $b(x,z) \geq -\frac{\mathcal{A}(d,-\alpha)}{\mathcal{A}(d,-\beta)} |z|^{\beta-\alpha}$, then $q^b(t,x,y)$ is a strictly positive continuous function and it uniquely determines a conservative Feller process X^b, which has strong Feller property. The Feller process X^b is the unique solution to the martingale problem of $(\mathcal{L}^b, \mathcal{S}(\mathbb{R}^d))$, where $\mathcal{S}(\mathbb{R}^d)$ denotes the space of tempered functions on $\mathbb{R}^d$. Furthermore, sharp two-sided estimates on $q^b(t,x,y)$ are derived. In stark contrast with the gradient perturbations, these estimates exhibit different behaviors for different types of $b(x,z)$. The model considered in this paper contains the following as a special case. Let Y and Z be (rotationally) symmetric α-stable process and symmetric β-stable processes on $\mathbb{R}^d$, respectively, that are independent to each other. Solution to stochastic differential equations $dX_t = dY_t + c(X_{t-})dZ_t$ has infinitesimal generator $\mathcal{L}^b$ with $b(x,z) = |c(x)|^\beta$.

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1 Introduction

Let $d \geq 1$ be an integer and $0 < \beta < \alpha < 2$. For integer $k \geq 1$, denote by $C_b^k(\mathbb{R}^d)$ (resp. $C_c^k(\mathbb{R}^d)$) the space of continuous functions on $\mathbb{R}^d$ that have bounded continuous partial derivatives up to order k (resp. the space of continuous functions on $\mathbb{R}^d$ with compact support that have continuous partial derivatives up to order k). Recall that a stochastic process $Y = (Y_t, \mathbb{P}_x, x \in \mathbb{R}^d)$ on $\mathbb{R}^d$ is called a (rotationally) symmetric α-stable process on $\mathbb{R}^d$ if it is a Lévy process having

$$\mathbb{E}_x \left[e^{i\xi\cdot(Y_t - Y_0)} \right] = e^{-t|\xi|^\alpha} \qquad \text{for every } x, \xi \in \mathbb{R}^d.$$

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Let $\widehat{f}(\xi) := \int_{\mathbb{R}^d} e^{i\xi\cdot x} f(x)dx$ denote the Fourier transform of a function f on $\mathbb{R}^d$. The fractional Laplacian $\Delta^{\alpha/2}$ on $\mathbb{R}^d$ is defined as

$$\Delta^{\alpha/2} f(x) = \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - \nabla f(x) \cdot z \mathbb{1}_{\{|z|\le 1\}} \right) \frac{\mathcal{A}(d,-\alpha)}{|z|^{d+\alpha}} dz \tag{1.1}$$

for $f \in C_b^2(\mathbb{R}^d)$. Here $\mathcal{A}(d,-\alpha)$ is the normalizing constant so that $\widehat{\Delta^{\alpha/2} f}(\xi) = -|\xi|^\alpha \widehat{f}(\xi)$. Hence $\Delta^{\alpha/2}$ is the infinitesimal generator for the symmetric α-stable process on $\mathbb{R}^d$.

Throughout this paper, $b(x,z)$ is a real-valued bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying

$$b(x,z) = b(x,-z) \qquad \text{for every } x, z \in \mathbb{R}^d. \tag{1.2}$$

This paper is concerned with the existence, uniqueness and sharp estimates on the "fundamental solution" of the following non-local operator on $\mathbb{R}^d$,

$$\mathcal{L}^b f(x) = \Delta^{\alpha/2} f(x) + \mathcal{S}^b f(x), \quad f \in C_b^2(\mathbb{R}^d),$$

where

$$\mathcal{S}^b f(x) := \mathcal{A}(d,-\beta) \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - \nabla f(x) \cdot z \mathbb{1}_{\{|z|\le 1\}} \right) \frac{b(x,z)}{|z|^{d+\beta}} dz. \tag{1.3}$$

We point out that since $b(x,z)$ satisfies condition (1.2), the truncation $|z| \le 1$ in (1.3) can be replaced by $|z| \le \lambda$ for any $\lambda > 0$; that is, for every $\lambda > 0$,

$$\mathcal{S}^b f(x) = \mathcal{A}(d,-\beta) \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - \langle \nabla f(x), z \rangle \mathbb{1}_{\{|z|\le \lambda\}} \right) \frac{b(x,z)}{|z|^{d+\beta}} dz. \tag{1.4}$$

In fact, under condition (1.2),

$$\begin{aligned} \mathcal{S}^b f(x) &= \mathcal{A}(d,-\beta) \text{ p.v.} \int_{\mathbb{R}^d} (f(x+z) - f(x)) \frac{b(x,z)}{|z|^{d+\beta}} dz \\ &:= \mathcal{A}(d,-\beta) \lim_{\varepsilon \to 0} \int_{\{z\in\mathbb{R}^d : |z|>\varepsilon\}} (f(x+z) - f(x)) \frac{b(x,z)}{|z|^{d+\beta}} dz. \end{aligned} \tag{1.5}$$

Operator $\mathcal{L}^b$ is typically non-symmetric.

Condition (1.2) allows us to reduce general bounded measurable function b on $\mathbb{R}^d \times \mathbb{R}^d$ to the situation where $\|b\|_\infty$ is sufficient small through a scaling argument (see (3.18) and Lemma 3.5). The operator $\mathcal{L}^b$ is in general non-symmetric. Clearly, $\mathcal{L}^b = \Delta^{\alpha/2}$ when $b \equiv 0$ and $\mathcal{L}^b = \Delta^{\alpha/2} + \Delta^{\beta/2}$ when $b \equiv 1$.

We are led to the study of this non-local operator $\mathcal{L}^b$ by the consideration of the following stochastic differential equation (SDE) on $\mathbb{R}^d$:

$$dX_t = dY_t + c(X_{t-})dZ_t, \tag{1.6}$$

where Y is a symmetric α-stable process on $\mathbb{R}^d$ and Z is an independent symmetric β-stable process with $0 < \beta < \alpha$. Such SDE arises naturally in applications when there are more than one sources of random noises. When c is a bounded Lipschitz function on $\mathbb{R}^d$, it is easy to show using Picard's iteration method that for every $x \in \mathbb{R}^d$, SDE (1.6) has a unique strong solution with $X_0 = x$. We denote the law of such a solution by $\mathbb{P}_x$. The collection of the solutions $(X_t, \mathbb{P}_x, x \in \mathbb{R}^d)$ forms a strong Markov process X on $\mathbb{R}^d$. Using Ito's formula, one concludes that the infinitesimal generator of X is $\mathcal{L}^b$ with $b(x,z)=|c(x)|^\beta$ and so in this case X solves the martingale problem for $(\mathcal{L}^b, C_b^2(\mathbb{R}^d))$. The following questions arise naturally: does the Markov process X have a transition density function? If so, what is its sharp two-sided estimates? Is there a solution to the martingale problem for $\Delta^{\alpha/2}+|c(x)|^\beta\Delta^{\beta/2}$ when c is not Lipschitz continuous? We will address these questions for the more general operator $\mathcal{L}^b$ in this paper.

For $a\geq 0$, denote by $p_a(t,x,y)$ the fundamental function of $\Delta^{\alpha/2}+a\Delta^{\beta/2}$ (or equivalently, the transition density function of the Lévy process $Y_t+a^{1/\beta}Z_t$). Clearly, $p_a(t,x,y)$ is a function of t and $x-y$, so sometimes we also write it as $p_a(t,x-y)$. It is known (see (2.3) of Section 2 for details) that on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$,

$$p_0(t,x,y) \asymp t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}, \tag{1.7}$$

$$p_a(t,x,y) \asymp \left(t^{-d/\alpha}\wedge(at)^{-d/\beta}\right)\wedge\left(\frac{t}{|x-y|^{d+\alpha}}+\frac{at}{|x-y|^{d+\beta}}\right). \tag{1.8}$$

Here for two non-negative functions f and g, the notation $f\asymp g$ means that there is a constant $c\geq 1$ so that $c^{-1}f\leq g\leq cf$ on their common domain of definitions. For real numbers $a,c\in\mathbb{R}$, we use $a\vee c$ and $a\wedge c$ to denote $\max\{a,c\}$ and $\min\{a,c\}$, respectively. We point out that the comparison constants in (1.8) is independent of $a>0$; see (2.3) in Section 2. Using the observation that $a\wedge b\asymp\frac{ab}{a+b}$, one concludes from (1.7) that

$$p_0(t,x,y)\asymp\frac{t}{(t^{1/\alpha}+|x-y|)^{d+\alpha}} \qquad \text{on } (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d. \tag{1.9}$$

Note that $(at)^{-d/\beta}\geq t^{-d/\alpha}$ whenever $0<t\leq a^{-\alpha/(\alpha-\beta)}$. Thus for every $k>0$,

$$p_a(t,x,y)\asymp t^{-d/\alpha}\wedge\left(\frac{t}{|x-y|^{d+\alpha}}+\frac{at}{|x-y|^{d+\beta}}\right) \qquad \text{on } (0,ka^{-\alpha/(\alpha-\beta)}]\times\mathbb{R}^d\times\mathbb{R}^d, \tag{1.10}$$

with the comparison constants depending only on d, α, β and k.

Since $\mathcal{L}^b=\Delta^{\alpha/2}+\mathcal{S}^b$ is a lower order perturbation of $\Delta^{\alpha/2}$ by $\mathcal{S}^b$, heuristically the fundamental solution (or kernel) $q^b(t,x,y)$ of $\mathcal{L}^b$ should satisfy the following Duhamel's formula:

$$q^b(t,x,y)=p_0(t,x,y)+\int_0^t\int_{\mathbb{R}^d}q^b(t-s,x,z)\mathcal{S}_z^b p_0(s,z,y)dzds \tag{1.11}$$

for $t>0$ and $x,y\in\mathbb{R}^d$. Here the notation $S_z^b p_0(s,z,y)$ means the non-local operator $\mathcal{S}^b$ is applied to the function $z\mapsto p_0(s,z,y)$. Similar notation will also be used for other operators, for example, $\Delta_z^{\alpha/2}$. Applying (1.11) recursively, it is reasonable to conjecture that $\sum_{n=0}^\infty q_n^b(t,x,y)$, if convergent, is a solution to (1.11), where $q_0^b(t,x,y):=p_0(t,x,y)$ and

$$q_n^b(t,x,y):=\int_0^t\int_{\mathbb{R}^d}q_{n-1}^b(t-s,x,z)\mathcal{S}_z^b p_0(s,z,y)dzds \quad \text{for } n\geq 1. \tag{1.12}$$

For each bounded function $b(x,z)$ on $\mathbb{R}^d\times\mathbb{R}^d$ and $\lambda>0$, define

$$m_{b,\lambda}=\operatorname{essinf}_{x,z\in\mathbb{R}^d,|z|>\lambda}b(x,z) \quad \text{and} \quad M_{b,\lambda}=\operatorname{esssup}_{x,z\in\mathbb{R}^d,|z|>\lambda}|b(x,z)|. \tag{1.13}$$

The followings are the main results of this paper.

Theorem 1.1. *For every bounded function b on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying condition (1.2), there is a unique continuous function $q^b(t,x,y)$ on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ that satisfies (1.11) on $(0,\varepsilon]\times\mathbb{R}^d\times\mathbb{R}^d$ with $|q^b(t,x,y)| \le cp_1(t,x,y)$ on $(0,\varepsilon]\times\mathbb{R}^d\times\mathbb{R}^d$ for some $\varepsilon, c>0$, and that*

$$\int_{\mathbb{R}^d} q^b(t,x,y)q^b(s,y,z)dy = q^b(t+s,x,z) \quad \text{for every } t,s>0 \text{ and } x,z\in\mathbb{R}^d. \tag{1.14}$$

Moreover, the following holds.

(i) *There is a constant $A_0 = A_0(d,\alpha,\beta)>0$ so that $q^b(t,x,y) = \sum_{n=0}^\infty q_n^b(t,x,y)$ on $(0,(A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}]\times\mathbb{R}^d\times\mathbb{R}^d$, where $q_n^b(t,x,y)$ is defined by (1.12).*

(ii) *$q^b(t,x,y)$ satisfies the Duhamel's formula (1.11) for all $t>0$ and $x,y\in\mathbb{R}^d$. Moreover, $\mathcal{S}_x^b q^b(t,x,y)$ exists pointwise in the sense of (1.5) and*

$$q^b(t,x,y) = p_0(t,x,y) + \int_0^t \int_{\mathbb{R}^d} p_0(t-s,x,z)\mathcal{S}_z^b q^b(s,z,y)dzds \tag{1.15}$$

for $t>0$ and $x,y\in\mathbb{R}^d$.

(iii) *For each $t>0$ and $x\in\mathbb{R}^d$, $\int_{\mathbb{R}^d} q^b(t,x,y)dy = 1$.*

(iv) *For every $f\in C_b^2(\mathbb{R}^d)$,*

$$T_t^b f(x) - f(x) = \int_0^t T_s^b \mathcal{L}^b f(x)ds,$$

where $T_t^b f(x) = \int_{\mathbb{R}^d} q^b(t,x,y)f(y)dy$.

(v) *Let $A>0$ and $\lambda>0$. There is a positive constant $C = C(d,\alpha,\beta,A,\lambda)\ge 1$ so that for any b satisfying (1.2) with $\|b\|_\infty \le A$,*

$$|q^b(t,x,y)| \le Ce^{Ct}p_{M_{b,\lambda}}(t,x,y) \quad \text{on } (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d. \tag{1.16}$$

We remark that estimate (1.16) allows one to get sharper bound on $|q^b(t,x,y)|$ by selecting optimal $\lambda>0$. When Z_t is the deterministic process t and c is an $\mathbb{R}^d$-valued bounded Lipschitz function on $\mathbb{R}^d$, the solution of (1.6) is a symmetric α-stable process with drift. Its infinitesimal generator is $\Delta^{\alpha/2} + c(x)\nabla$. Existence of integral kernel to $\Delta^{\alpha/2}+c(x)\nabla$ and its estimates have been studied recently in [6] (in fact, c there can be an $\mathbb{R}^d$-valued function in certain Kato class).

Unlike the gradient perturbation for $\Delta^{\alpha/2}$, in general the kernel $q^b(t,x,y)$ in Theorem 1.1 can take negative values. For example, this is the case when $b\equiv -1$, that is, when $\mathcal{L}^b = \Delta^{\alpha/2} - \Delta^{\beta/2}$, according to the next theorem. Observe that

$$\mathcal{L}^b f(x) = \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - \langle \nabla f(x), z\rangle \mathbb{1}_{\{|z|\le 1\}}\right) j^b(x,z)dz,$$

where

$$j^b(x,z) = \frac{\mathcal{A}(d,-\alpha)}{|z|^{d+\alpha}}\left(1 + \frac{\mathcal{A}(d,-\beta)}{\mathcal{A}(d,-\alpha)}\, b(x,z)\,|z|^{\alpha-\beta}\right). \tag{1.17}$$

The next result gives a necessary and sufficient condition for the kernel $q^b(t,x,y)$ in Theorem 1.1 to be non-negative when $b(x,z)$ is continuous in x for a.e. z.

Theorem 1.2. *Let b be a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ that satisfies* (1.2) *and that*

$$x \mapsto b(x,z) \text{ is continuous for a.e. } z \in \mathbb{R}^d. \tag{1.18}$$

Then $q^b(t,x,y) \geq 0$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ if and only if for each $x \in \mathbb{R}^d$, $j^b(x,z) \geq 0$ for a.e. $z \in \mathbb{R}^d$; that is, if and only if

$$b(x,z) \geq -\frac{\mathcal{A}(d,-\alpha)}{\mathcal{A}(d,-\beta)} |z|^{\beta-\alpha} \quad \text{for a.e. } z \in \mathbb{R}^d. \tag{1.19}$$

In particular, if $b(x,z) = b(x)$ is a function of x only, then $q^b(t,x,y) \geq 0$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ if and only if $b(x) \geq 0$ on $\mathbb{R}^d$.

Next theorem drops the assumption (1.18), gives lower bound estimates and refines upper bound estimates on $q^b(t,x,y)$ for $b(x,z)$ satisfying condition (1.19) and makes connections to the martingale problem for $\mathcal{L}^b$. To state it, we need first to recall some definitions.

Let $\mathbb{D}([0,\infty),\mathbb{R}^d)$ be the space of right continuous $\mathbb{R}^d$-valued functions having left limits on $[0,\infty)$, equipped with Skorokhod topology. Denote by X_t the projection coordinate map on $\mathbb{D}([0,\infty),\mathbb{R}^d)$. Let $\mathcal{C}$ be a subspace of $C_b^2(\mathbb{R}^d)$. A probability measure Q on the Skorokhod space $\mathbb{D}([0,\infty),\mathbb{R}^d)$ is said to to be a solution to the martingale problem for $(\mathcal{L}^b,\mathcal{C})$ with initial value $x \in \mathbb{R}^d$ if $Q(X_0 = x) = 1$ and for every $f \in \mathcal{C}$,

$$M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{L}^b f(X_s) ds$$

is a Q-martingale. The martingale problem $(\mathcal{L}^b,\mathcal{C})$ with initial value $x \in \mathbb{R}^d$ is said to be well-posed if it has a unique solution.

Let $C_\infty(\mathbb{R}^d)$ be the space of continuous functions on $\mathbb{R}^d$ that vanish at infinity, equipped with supremum norm. Set

$$C_\infty^2(\mathbb{R}^d) = \left\{ f \in C_\infty(\mathbb{R}^d) : \text{the first and second derivatives of } f \text{ are all in } C_\infty(\mathbb{R}^d) \right\}.$$

A Markov process on $\mathbb{R}^d$ is called a Feller process if its transition semigroup is a strongly continuous semigroup in $C_\infty(\mathbb{R}^d)$. Feller processes is a class of nice strong Markov processes, called Hunt processes (see [15]). Let $\overline{p}_0(t,x,y)$ be the fundamental solution of the truncated operator

$$\overline{\Delta}^{\alpha/2} f(x) = \int_{|z|\leq 1} \left(f(x+z) - f(x) - \nabla f(x) \cdot z \mathbb{1}_{\{|z|\leq 1\}} \right) \frac{\mathcal{A}(d,-\alpha)}{|z|^{d+\alpha}} dz;$$

or, equivalently, $\overline{p}_0(t,x,y)$ is the transition density function for the finite range α-stable (Lévy) process with Lév measure $\mathcal{A}(d,-\alpha)|z|^{-(d+\alpha)} \mathbb{1}_{\{|z|\leq 1\}}$. It is established in [8] that $\overline{p}_0(t,x,y)$ is jointly continuous and enjoys the following two sided estimates:

$$\overline{p}_0(t,x,y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \tag{1.20}$$

for $t \in (0,1]$ and $|x-y| \leq 1$, and there are constants $c_k = c_k(d,\alpha) > 0$, $k = 1,2,3,4$ so that

$$c_1 \left(\frac{t}{|x-y|} \right)^{c_2|x-y|} \leq \overline{p}_0(t,x,y) \leq c_3 \left(\frac{t}{|x-y|} \right)^{c_4|x-y|} \tag{1.21}$$

for $t \in (0,1]$ and $|x-y| > 1$.

Define $b^+(x,z) = \max\{b(x,z), 0\}$.

Theorem 1.3. *For every $A > 0$ and $\lambda > 0$, there are positive constants $C_k = C_k(d, \alpha, \beta, A)$, $k = 1, 2$, and $C_3 = C_3(d, \alpha, \beta, A, \lambda)$ such that for any bounded b satisfying (1.2) and (1.19) with $\|b\|_\infty \le A$,*

$$C_1 \overline{p}_0(t, C_2 x, C_2 y) \le q^b(t, x, y) \le C_3 p_{M_{b^+,\lambda}}(t, x, y) \quad \text{for } t \in (0, 1] \text{ and } x, y \in \mathbb{R}^d. \tag{1.22}$$

Moreover, for every $\varepsilon > 0$, there is a positive constant $C_4 = C_4(d, \alpha, \beta, A, \lambda, \varepsilon)$ such that for any b on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2) with $\|b\|_\infty \le A$ so that

$$j^b(x, z) \ge \varepsilon \, |z|^{-(d+\alpha)} \quad \text{for a.e. } x, z \in \mathbb{R}^d \tag{1.23}$$

we have

$$C_4 p_{m_{b^+,\lambda}}(t, x, y) \le q^b(t, x, y) \le C_3 p_{M_{b^+,\lambda}}(t, x, y) \quad \text{for } t \in (0, 1] \text{ and } x, y \in \mathbb{R}^d. \tag{1.24}$$

The kernel $q^b(t, x, y)$ uniquely determines a Feller process $X^b = (X^b_t, t \ge 0, \mathbb{P}_x, x \in \mathbb{R}^d)$ on the canonical Skorokhod space $\mathbb{D}([0, \infty), \mathbb{R}^d)$ such that

$$\mathbb{E}_x \left[f(X^b_t) \right] = \int_{\mathbb{R}^d} q^b(t, x, y) f(y) dy$$

for every bounded continuous function f on $\mathbb{R}^d$. The Feller process X^b is conservative and has a Lévy system $(J^b(x, y)dy, t)$, where $J^b(x, y) = j^b(x, y - x)$.

$$J^b(x, y) = j^b(x, y - x) = \frac{\mathcal{A}(d, -\alpha)}{|x - y|^{d+\alpha}} + \frac{\mathcal{A}(d, -\beta)\, b(x, y - x)}{|x - y|^{d+\beta}}. \tag{1.25}$$

Moreover, for each $x \in \mathbb{R}^d$, $(X^b, \mathbb{P}_x)$ is the unique solution to the martingale problem $(\mathcal{L}^b, \mathcal{S}(\mathbb{R}^d))$ with initial value x. Here $\mathcal{S}(\mathbb{R}^d)$ denotes the space of tempered functions on $\mathbb{R}^d$.

Here we say $(J^b(x, y)dy, t)$ is a Lévy system for X^b if for any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, y, y) = 0$ for all $y \in \mathbb{R}^d$, any stopping time T (with respect to the filtration of X^b) and any $x \in \mathbb{R}^d$,

$$\mathbb{E}_x \left[\sum_{s \le T} f(s, X^b_{s-}, X^b_s) \right] = \mathbb{E}_x \left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, X^b_s, y) J^b(X^b_s, y) dy \right) ds \right]. \tag{1.26}$$

A Lévy system for X^b describes the jumps of the process X^b. A Markov process on $\mathbb{R}^d$ is said to have strong Feller property if its transition semigroup maps bounded measurable functions on $\mathbb{R}^d$ into bounded continuous functions on $\mathbb{R}^d$. Since $q^b(t, x, y)$ is a continuous function, one has by Theorem 1.1 and the dominated convergence theorem that the Feller process X^b of Theorem 1.3 has strong Feller property.

Condition (1.23) is always satisfied if $b(x, z)$ is nonnegative. We emphasize the $m_{b^+,\lambda}$ and $M_{b^+,\lambda}$ terms appeared in the estimates in Theorem 1.3. Under condition (1.23) and the assumption that $\|b\|_\infty \le A$, the value of $b(x, z)$ on $\mathbb{R}^d \times \{z \in \mathbb{R}^d : |z| \le \lambda\}$ is irrelevant in the estimates of $q^b(t, x, y)$ in (1.24). By selecting suitable $\lambda > 0$ in (1.24), one can get optimal two-sided estimates on $q^b(t, x, y)$. The following follows immediately from Theorem 1.3 by taking a suitable $\lambda > 0$.

Corollary 1.4. *Let $A \geq 0$ and $\varepsilon > 0$. There is a positive constant $C = C(d, \alpha, \beta, A, \varepsilon) \geq 1$ so that for any bounded b satisfying* (1.2) *with $\|b\|_\infty \leq A$ and*

$$j^b(x,z) \geq \varepsilon \left(\frac{1}{|z|^{d+\alpha}} + \frac{1}{|z|^{d+\beta}} \right) \qquad \text{for a.e. } x, z \in \mathbb{R}^d,$$

we have

$$C^{-1} p_1(t,x,y) \leq q^b(t,x,y) \leq C p_1(t,x,y) \qquad \text{for } t \in (0,1] \text{ and } x, y \in \mathbb{R}^d.$$

Theorem 1.3 in particular implies that if $b(x, \cdot)$ is a bounded function satisfying (1.2) and (1.19) so that $b(x,z) = 0$ for every $x \in \mathbb{R}^d$ and $|z| \geq R$ for some $R > 0$; or, equivalently if $\mathcal{L}^b = \Delta^{\alpha/2} + \mathcal{S}^b$ is a lower order perturbation of $\Delta^{\alpha/2}$ by finite range non-local operator $\mathcal{S}^b$, then the upper bound of the kernel $q^b(t,x,y)$ is dominated by $p_0(t,x,y)$ for each $(t,x,y) \in (0,1] \times \mathbb{R}^d \times \mathbb{R}^d$. In fact, we have the following more general result.

Theorem 1.5. *For every $A > 0$ and $M \geq 1$, there is a constant $C_5 = C_5(d, \alpha, \beta, A, M) \geq 1$ such that for any bounded b satisfying* (1.2) *with $\|b\|_\infty \leq A$ and*

$$M^{-1} |z|^{-(d+\alpha)} \leq j^b(x,z) \leq M |z|^{-(d+\alpha)} \quad \text{for a.e. } x, z \in \mathbb{R}^d, \tag{1.27}$$

or equivalently,

$$-(1 - M^{-1}) \frac{\mathcal{A}(d,-\alpha)}{\mathcal{A}(d,-\beta)} |z|^{\beta-\alpha} \leq b(x,z) \leq (M-1) \frac{\mathcal{A}(d,-\alpha)}{\mathcal{A}(d,-\beta)} |z|^{\beta-\alpha} \quad \text{for a.e. } x, z \in \mathbb{R}^d, \tag{1.28}$$

we have

$$C_5^{-1} p_0(t,x,y) \leq q^b(t,x,y) \leq C_5 p_0(t,x,y) \quad \text{for } t \in (0,1] \text{ and } x, y \in \mathbb{R}^d. \tag{1.29}$$

Remark 1.6. (i) In general, we can not expect q^b to have comparable lower and upper bound estimates. The estimates in (1.22) and (1.24) are sharp in the sense that $q^b(t,x,y) = p_0(t,x,y)$ when $b \equiv 0$, $q^b(t,x,y) = p_1(t,x,y)$ when $b \equiv 1$, and $q^b(t,x,y) = \overline{p}_0(t,x,y)$ when $b(x,z) = 0$ for $|z| \leq 1$ and $b(x,z) = -\frac{\mathcal{A}(d,-\alpha)}{\mathcal{A}(d,-\beta)} |z|^{\beta-\alpha}$ for $|z| \geq 1$. Clearly, by (1.7)-(1.8), $p_0(t,x,y)$ and $p_1(t,x,y)$ are not comparable on $(0,1] \times \mathbb{R}^d \times \mathbb{R}^d$. We point out that it follows from (1.8) and (1.24) that every $A \geq 1$, there is a constant $\widetilde{C} = \widetilde{C}(d, \alpha, \beta, A) \geq 1$ so that for any non-negative b on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2) with $1/A \leq b(x,z) \leq A$ a.e.

$$(1/\widetilde{C})\, p_1(t,x,y) \leq q^b(t,x,y) \leq \widetilde{C}\, p_1(t,x,y) \quad \text{for } t \in (0,1] \text{ and } x, y \in \mathbb{R}^d. \tag{1.30}$$

(ii) Heat kernel estimates for discontinuous Markov processes have been under intense study recently. Most results obtained so far are mainly for symmetric Markov processes. See [7] for a recent survey. Results of this paper can also be viewed as an attempt in establishing heat kernel estimates for non-symmetric discontinuous Markov processes. For example, Theorem 1.5 and Corollary 1.4 can be viewed as the non-symmetric analogy, though in a restricted setting, of the two-sided heat kernel estimates for symmetric stable-like processes and mixed stable-like processes established in [11] and [12], respectively.

(iii) Heat kernel estimates for fractional Laplacian $\Delta^{\alpha/2}$ under gradient perturbation and (possibly non-local) Feynman-Kac perturbation have recently been studied in [6, 9, 10, 28]. In both of these cases, under a Kato class condition on the coefficients, the fundamental solution of the perturbed operator is always strictly positive and is comparable to the fundamental solution $p_0(t,x,y)$ of the fractional Laplacian $\Delta^{\alpha/2}$ on $(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$. Our Theorems 1.2 and 1.3 reveal that the fractional Laplacian $\Delta^{\alpha/2}$ under non-local perturbation $\mathcal{S}^b$ is in stark contrast with $\Delta^{\alpha/2}$ under either gradient (local) perturbations or (possibly non-local) Feynman-Kac perturbations. However, Theorem 1.5 in particular indicates that the heat kernel estimate for $\Delta^{\alpha/2}$ is stable under finite range lower order perturbation.

(iv) Martingale problem for non-local operators (with or without elliptic differential operator component) has been studied by many authors. See, e.g., [4, 5, 18, 19, 21, 22, 25, 27] and the references therein. In particular, Komatsu [19] and Mikulevicious-Pragarauskas [21] considered martingale problem for a class of non-local operators that is directly related to $\mathcal{L}^b$. In fact, the uniqueness of the martingale problem for $(\mathcal{L}^b,\mathcal{S}(\mathbb{R}^d))$ stated in Theorem 1.3 above is a direct consequence of [19, Theorem 3], while it follows from [21, Theorem 5] that for any bounded b satisfying (1.2) and (1.19), there is a unique solution to the martingale problem $(\mathcal{L}^b, C_c^\infty(\mathbb{R}^d))$. We also refer the reader to [17, 23] for more information on the connection between pseudo-differential operators and discontinuous Markov processes. The main contribution of Theorem 1.3 is on the two-sided transition density function estimates for the martingale problem solution X_t^b. We also mention that the well-posedness of martingale problem for $(\Delta^{\alpha/2}+b(x)\cdot\nabla, C_c^\infty(\mathbb{R}^d))$ with $b(x)$ an $\mathbb{R}^d$-valued Kato class function has recently been established in [13]. $\Box$

We can restate some of results from Theorems 1.1, 1.2, 1.3 and 1.5 as follows.

Theorem 1.7. *Let $b(x,z)$ be a bounded function on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying (1.2) and (1.19). For each $x\in\mathbb{R}^d$, the martingale problem for $(\mathcal{L}^b,\mathcal{S}(\mathbb{R}^d))$ with initial value x is well-posed. These martingale problem solutions $\{\mathbb{P}_x, x\in\mathbb{R}^d\}$ form a strong Markov process X^b, which has infinite lifetime and possesses a jointly continuous transition density function $q^b(t,x,y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Moreover, the following holds.*

(i) *The transition density function $q^b(t,x,y)$ can be explicitly constructed as follows. Define $q_0^b(t,x,y):=p_0(t,x,y)$ and*

$$q_n^b(t,x,y):=\int_0^t\int_{\mathbb{R}^d} q_{n-1}^b(t-s,x,z)\mathcal{S}_z^b p_0(s,z,y)dzds \quad \text{for } n\geq 1.$$

There is $\varepsilon>0$ so that $\sum_{n=0}^\infty q_n^b(t,x,y)$ converges absolutely on $(0,\varepsilon]\times\mathbb{R}^d\times\mathbb{R}^d$ and $q^b(t,x,y)=\sum_{n=0}^\infty q_n^b(t,x,y)$ on $(0,\varepsilon]\times\mathbb{R}^d\times\mathbb{R}^d$.

(ii) $q^b(t,x,y)=p_0(t,x,y)+\int_0^t\int_{\mathbb{R}^d} q^b(t-s,x,z)\mathcal{S}_z^b p_0(s,z,y)dzds$ *on* $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.

(iii) *For every $A>0$ and $\lambda>0$, there are positive constants $c_k=c_k(d,\alpha,\beta,A), k=1,2,3$ and $c_k=c_k(d,\alpha,\beta,A,\lambda)$, $k=4,\cdots,9$, such that for any bounded function $b(x,z)$ on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying (1.2) and (1.19) with $\|b\|_\infty\leq A$,*

$$c_1e^{-c_2t}\overline{p}_0(t,c_3x,c_3y)\leq q^b(t,x,y)\leq c_4e^{c_5t}\,p_{M_{b+,\lambda}}(t,x,y) \quad \text{on } (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$$

and for any non-negative function $b(x,z)$ on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying (1.2) with $\|b\|_\infty\leq A$,

$$c_6e^{-c_7t}\,p_{m_{b,\lambda}}(t,x,y)\leq q^b(t,x,y)\leq c_8e^{c_9t}\,p_{M_{b,\lambda}}(t,x,y) \quad \text{on } (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d.$$

(iv) *For every* $A > 0$ *and* $M \geq 1$, *there are positive constants* $c_k = c_k(d, \alpha, \beta, A, M)$, $k = 10, \cdots, 13$, *such that for any bounded function* $b(x,z)$ *on* $\mathbb{R}^d \times \mathbb{R}^d$ *satisfying* (1.2) *and* (1.27) *with* $\|b\|_\infty \leq A$,

$$c_{10} e^{-c_{11} t} p_0(t,x,y) \leq q^b(t,x,y) \leq c_{12} e^{c_{13} t} p_0(t,x,y) \quad \text{on } (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d.$$

The rest of the paper is organized as follows. In Section 2, we derive some estimates on $\overline{\Delta}_x^{\beta/2} p_0(t,x,y)$ and $\Delta_x^{\beta/2} p_0(t,x,y)$ that will be used in later. The existence and uniqueness of the fundamental solution $q^b(t,x,y)$ of $\mathcal{L}^b$ are given in Section 3. This is done through a series of lemmas and theorems, which provide more detailed information on $q^b(t,x,y)$ and $q^b_n(t,x,y)$. Theorem 1.1 then follows from these results. We show in Section 4 that the semigroup $\{T^b_t; t > 0\}$ associated with $q^b(t,x,y)$ is a strongly continuous semigroup in $C_\infty(\mathbb{R}^d)$. We then apply Hille-Yosida-Ray theorem and Courrége's first theorem to establish Theorem 1.2. When b satisfies (1.2), (1.18) and (1.19), $q^b(t,x,y)$ determines a conservative Feller process X^b. We first derive a Lévy system of X^b and also prove $(X^b, \mathbb{P}_x)$ is the unique solution to the martingale problem for $(\mathcal{L}^b, \mathcal{S}(\mathbb{R}^d))$ in Section 5. We next establish, for any given $A > 0$, the equi-continuity of $q^b(t,x,y)$ on each $[1/M, M] \times \mathbb{R}^d \times \mathbb{R}^d$ for any b that satisfies (1.2) with $\|b\|_\infty \leq A$. Using this, we can drop the condition (1.18) and establish the Feller process X^b with transition density $q^b(t,x,y)$ for general bounded b that satisfies (1.2) and (1.19) by approximating it with a sequence of $\{k_n(x,z), n \geq 1\}$ that satisfy (1.2), (1.18) and (1.19). The upper bound estimate for $q^b(t,x,y)$ in (1.22) and (1.24) can be obtained from that of $q^{\widehat{b}_\lambda}(t,x,y)$ due to the Meyer's construction of $X^{\widehat{b}_\lambda}$ from X^b, where $\widehat{b}_\lambda(x,z) = b(x,z) 1_{\{|z| \leq \lambda\}}(z) + b^+(x,z) 1_{\{|z|>\lambda\}}(z)$. The lower bound estimates in (1.22) and (1.24) are established by the Lévy system of X^b and some probability estimates. Finally, we use the estimates in (1.24) for b with support in $\{(x,z) \in \mathbb{R}^d \times \mathbb{R}^d : |z| \leq 1\}$ and the non-local Feynman-Kac perturbation results from [10] to obtain Theorem 1.5.

Throughout this paper, we use the capital letters $C_1, C_2, \cdots$ to denote constants in the statement of the results, and their labeling will be fixed. The lowercase constants $c_1, c_2, \cdots$ will denote generic constants used in the proofs, whose exact values are not important and can change from one appearance to another. We will use ":=" to denote a definition. For a differentiable function f on $\mathbb{R}^d$, we use $\partial_i f$ and $\partial^2_{ij} f$ to denote the partial derivatives $\frac{\partial f}{\partial x_i}$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$.

2 Preliminaries

Suppose that Y is a symmetric α-stable process, and Z is a symmetric β-stable process on $\mathbb{R}^d$ that is independent of Z. For any $a \geq 0$, we define Y^a by $Y^a_t := Y_t + a^{1/\beta} Z_t$. We will call the process Y^a the independent sum of the symmetric α-stable process Y and the symmetric β-stable process Z with weight $a^{1/\beta}$. The infinitesimal generator of Y^a is $\Delta^{\alpha/2} + a\Delta^{\beta/2}$. Let $p_a(t,x,y)$ denote the transition density of Y^a (or equivalently the heat kernel of $\Delta^{\alpha/2} + a\Delta^{\beta/2}$) with respect to the Lebesgue measure on $\mathbb{R}^d$. Recently it is proven in [12] that

$$p_1(t,x,y) \asymp \left(t^{-d/\alpha} \wedge t^{-d/\beta} \right) \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{t}{|x-y|^{d+\beta}} \right) \quad \text{on } (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d. \tag{2.1}$$

Unlike the case of the symmetric α-stable process $Y := Y^0$, Y^a does not have the stable scaling for $a > 0$. Instead, the following approximate scaling property holds : for every $\lambda > 0$,

$\{\lambda^{-1}Y^a_{\lambda^\alpha t}, t \geq 0\}$ has the same distribution as $\{Y^{a\lambda^{(\alpha-\beta)}}_t, t \geq 0\}$. Consequently, for any $\lambda > 0$, we have

$$p_{a\lambda^{(\alpha-\beta)}}(t,x,y) = \lambda^d p_a(\lambda^\alpha t, \lambda x, \lambda y) \qquad \text{for } t > 0 \text{ and } x, y \in \mathbb{R}^d. \tag{2.2}$$

In particular, letting $a = 1$, $\lambda = \gamma^{1/(\alpha-\beta)}$, we get

$$p_\gamma(t,x,y) = \gamma^{d/(\alpha-\beta)} p_1(\gamma^{\alpha/(\alpha-\beta)}t, \gamma^{1/(\alpha-\beta)}x, \gamma^{1/(\alpha-\beta)}y) \qquad \text{for } t > 0 \text{ and } x, y \in \mathbb{R}^d.$$

So we deduce from (2.1) that there exists a constant $C > 1$ depending only on d, α and β such that for every $a > 0$ and $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$

$$C^{-1} h_a(t,x,y) \leq p_a(t,x,y) \leq C h_a(t,x,y), \tag{2.3}$$

where

$$h_a(t,x,y) := \left(t^{-d/\alpha} \wedge (at)^{-d/\beta}\right) \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{a\,t}{|x-y|^{d+\beta}}\right).$$

In fact, (2.3) also holds when $a = 0$. Observe (see (1.10)) that for every $A > 0$, there is a constant $c = c(d,\alpha,\beta,A) \geq 1$ so that for every $(t,x,y) \in (0,1] \times \mathbb{R}^d \times \mathbb{R}^d$ and $0 \leq a \leq A$,

$$c^{-1}\, t^{-d/\alpha} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{a\,t}{|x-y|^{d+\beta}}\right) \leq h_a(t,x,y) \leq c\, t^{-d/\alpha} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{a\,t}{|x-y|^{d+\beta}}\right) \tag{2.4}$$

Recall that $p_0(t,x,y) = p_0(t,x-y)$ is the transition density function of the symmetric α-stable process Y^0.

Lemma 2.1. *There exists a constant $C_6 = C_6(d,\alpha) > 0$ such that for every $t > 0$, $x \in \mathbb{R}^d$ and $i, j = 1, \ldots, d$,*

$$\left|\frac{\partial}{\partial x_i} p_0(t,x)\right| \leq C_6 t^{-(d+1)/\alpha} \left(1 \wedge \frac{t^{1/\alpha}}{|x|}\right)^{d+1+\alpha},$$

$$\left|\frac{\partial^2}{\partial x_i \partial x_j} p_0(t,x)\right| \leq C_6 t^{-(d+2)/\alpha} \left(1 \wedge \frac{t^{1/\alpha}}{|x|}\right)^{d+2+\alpha}.$$

Proof. By [6, Lemma 5], there is a positive constant c_1 so that for all $t > 0$ and $x, y \in \mathbb{R}^d$

$$|\nabla_x p_0(t,x)| \leq c_1 |x| \left(t^{-(d+2)/\alpha} \wedge \frac{t}{|x|^{d+2+\alpha}}\right) \leq c_1 \left(t^{-(d+1)/\alpha} \wedge \frac{t}{|x|^{d+1+\alpha}}\right).$$

That is, the first inequality holds. Let $\eta_t(r)$ be the density function of the $\alpha/2$-stable subordinator at time t and $g(t,x) = (4\pi t)^{-d/2} e^{-|x|^2/4t}$ be the Gaussian kernel on $\mathbb{R}^d$. There is a constant c so that $\eta_t(r) \leq c t r^{-1-\alpha/2}$ for all $r, t > 0$, see [6, Lemma 5]. Noting that

$$\left|\frac{\partial^2}{\partial x_i \partial x_j} g(s,x)\right| \leq \left(\frac{|x|^2}{s^2} + \frac{2}{s}\right) g(s,x) = (4\pi)^2 |x|^2 g^{(d+4)}(s,x_1) + 8\pi g^{(d+2)}(s,x_2),$$

where $x_1 \in \mathbb{R}^{d+4}$ and $x_2 \in \mathbb{R}^{d+2}$ with $|x_1| = |x_2| = |x|$, $g^{(d+2)}(s,x_2)$ and $g^{(d+4)}(s,x_1)$ are the Gaussian kernels on $\mathbb{R}^{d+2}$ and $\mathbb{R}^{d+4}$, respectively. Since $p_0(t,x) = \int_0^\infty g(s,x)\eta_t(s)ds$, we have by the dominated convergence theorem that there is a positive constant c_2 so that for all $t > 0$ and $x \in \mathbb{R}^d$

$$
\begin{aligned}
\left|\frac{\partial^2}{\partial x_i \partial x_j} p_0(t,x)\right| &\le \int_0^\infty \left|\frac{\partial^2}{\partial x_i \partial x_j} g(s,x)\right| \eta_t(s)\,ds \\
&\le (4\pi)^2 |x|^2 p_0^{(d+4)}(t,x_1) + 8\pi p_0^{(d+2)}(t,x_2) \\
&\le c_2 \left(t^{-(d+2)/\alpha} \wedge \frac{t}{|x|^{d+2+\alpha}} \right),
\end{aligned}
$$

where $p_0^{(d+2)}(t,x_2)$ and $p_0^{(d+4)}(t,x_1)$ are the transition density functions of the symmetric α-stable processes in $\mathbb{R}^{d+2}$ and $\mathbb{R}^{d+4}$, respectively. This establishes the second inequality in Lemma 2.1. $\square$

Define for $t > 0$ and $x, y \in \mathbb{R}^d$, the function

$$
|\Delta_x^{\beta/2}| p_0(t,x,y) \begin{cases} = \mathcal{A}(d,-\beta)\Big(\int_{|z|\le t^{1/\alpha}} \left|p_0(t,x+z,y) - p_0(t,x,y) - \frac{\partial}{\partial x} p_0(t,x,y)\cdot z\right| \frac{1}{|z|^{d+\beta}} dz \\ \qquad + \int_{|z|>t^{1/\alpha}} |p_0(t,x+z,y) - p_0(t,x,y)| \frac{dz}{|z|^{d+\beta}} \Big) & \text{for } |x-y|^\alpha \le t, \\ = \mathcal{A}(d,-\beta)\Big(\int_{|z|\le |x-y|/2} |p_0(t,x+z,y) - p_0(t,x,y) - \frac{\partial}{\partial x} p_0(t,x,y)\cdot z| \frac{1}{|z|^{d+\beta}} dz \\ \qquad + \int_{|z|>|x-y|/2} |p_0(t,x+z,y) - p_0(t,x,y)| \frac{dz}{|z|^{d+\beta}} \Big) & \text{for } |x-y|^\alpha > t. \end{cases}
$$

Let

$$
f_0(t,x,y) := \left(t^{1/\alpha} \vee |x-y| \right)^{-(d+\beta)} = t^{-(d+\beta)/\alpha} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|} \right)^{d+\beta}. \tag{2.5}
$$

Lemma 2.2. *There exists a constant $C_7 = C_7(d,\alpha,\beta) > 0$ such that*

$$
|\Delta_x^{\beta/2}| p_0(t,x,y) \le C_7 f_0(t,x,y) \qquad \text{on } (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d. \tag{2.6}
$$

Proof. We only need to prove $|\Delta_x^{\beta/2}| p_0(t,x) \le C_7 f_0(t,x,0)$ for all $t > 0$ and $x \in \mathbb{R}^d$.

(i) We first consider the case $|x|^\alpha \le t$. In this case,

$$
\begin{aligned}
|\Delta_x^{\beta/2}| p_0(t,x) &= \mathcal{A}(d,-\beta) \int_{|z|\le t^{1/\alpha}} |p_0(t,x+z) - p_0(t,x) - \frac{\partial}{\partial x} p_0(t,x)\cdot z| \frac{dz}{|z|^{d+\beta}} \\
&\quad + \mathcal{A}(d,-\beta) \int_{|z|\ge t^{1/\alpha}} |p_0(t,x+z) - p_0(t,x)| \frac{dz}{|z|^{d+\beta}} \\
&= I + II.
\end{aligned}
$$

Note that by Lemma 2.1,

$$
\sup_{u\in\mathbb{R}^d} \left| \frac{\partial^2}{\partial u_i \partial u_j} p_0(t,u) \right| \le C_6 t^{-(d+2)/\alpha},
$$

and so by Taylor's formula,

$$
I \le \mathcal{A}(d,-\beta) \sup_{u\in\mathbb{R}^d} \left| \frac{\partial^2}{\partial u_i \partial u_j} p_0(t,u) \right| \int_{|z|\le t^{1/\alpha}} \frac{|z|^2}{|z|^{d+\beta}}\, dz \le c_1 t^{-(d+2)/\alpha} t^{(2-\beta)/\alpha} \le c_1 t^{-(d+\beta)/\alpha}.
$$

On the other hand, by (1.7)

$$II \le \mathcal{A}(d,-\beta)\int_{|z|\ge t^{1/\alpha}} \left(p_0(t,x+z)+p_0(t,x)\right)\frac{dz}{|z|^{d+\beta}} \le c_2 t^{-d/\alpha}\int_{|z|\ge t^{1/\alpha}} \frac{1}{|z|^{d+\beta}}dz \le c_3 t^{-(d+\beta)/\alpha}.$$

(ii) Next, we consider the case $|x|^\alpha \ge t$. In this case,

$$\begin{aligned}|\Delta_x^{\beta/2}|p_0(t,x) &= \mathcal{A}(d,-\beta)\int_{|z|\le|x|/2} |p_0(t,x+z)-p_0(t,x)-\frac{\partial}{\partial x}p_0(t,x)\cdot z|\frac{dz}{|z|^{d+\beta}}\\ &\quad+\mathcal{A}(d,-\beta)\int_{|z|\ge|x|/2} |p_0(t,x+z)-p_0(t,x)|\frac{dz}{|z|^{d+\beta}}\\ &=: I+II.\end{aligned}$$

Note that $|x+z|\ge|x|/2$ for $|z|\le|x|/2$. So by Lemma 2.1,

$$\sup_{|z|\le|x|/2}\Big|\frac{\partial^2}{\partial x_i\partial x_j}p_0(t,x+z)\Big| \le C_6\sup_{|z|\le|x|/2} t|x+z|^{-(d+2+\alpha)} \le 2^{(d+2+\alpha)}C_6 t|x|^{-(d+2+\alpha)}.$$

Hence, by Taylor's formula

$$\begin{aligned}I &\le \mathcal{A}(d,-\beta)\sup_{|z|\le|x|/2}\Big|\frac{\partial^2}{\partial x_i\partial x_j}p_0(t,x+z)\Big|\int_{|z|\le|x|/2}\frac{|z|^2}{|z|^{d+\beta}}\,dz\\ &\le c_4 t|x|^{-(d+2+\alpha)}|x|^{2-\beta} = c_4 t|x|^{-(d+\alpha+\beta)}.\end{aligned} \tag{2.7}$$

Noting that $|x|^\alpha\ge t$, thus $I\le c_4|x|^{-(d+\beta)}$. On the other hand, note that symmetric α-stable process is a subordinate Brownian motion, so $p_0(t,x+z)\le p_0(t,x)$ if $|x+z|\ge|x|$ and $p_0(t,x)\le p_0(t,x+z)$ if $|x+z|\le|x|$. Hence, by (1.7) and the condition that $|x|^\alpha\ge t$, we obtain

$$\begin{aligned}II &\le \mathcal{A}(d,-\beta)\int_{|z|\ge|x|/2,|x+z|\ge|x|} 2p_0(t,x)\frac{dz}{|z|^{d+\beta}}+\mathcal{A}(d,-\beta)\int_{|z|\ge|x|/2,|x+z|\le|x|} 2p_0(t,x+z)\frac{dz}{|z|^{d+\beta}}\\ &\le 2\mathcal{A}(d,-\beta)p_0(t,x)\int_{|z|\ge|x|/2}\frac{dz}{|z|^{d+\beta}}+2^{d+1+\beta}\mathcal{A}(d,-\beta)|x|^{-(d+\beta)}\int_{z\in\mathbb{R}^d}p_0(t,x+z)\,dz\\ &\le c_5 t|x|^{-(d+\alpha)}|x|^{-\beta}+2^{d+1+\beta}\mathcal{A}(d,-\beta)|x|^{-(d+\beta)}\le c_6|x|^{-(d+\beta)}.\end{aligned} \tag{2.8}$$

This establishes the lemma. □

In order to get the upper bound estimates in (1.16) in terms of weight $M_{b,\lambda}$ rather than $\|b\|_\infty$, we define, for $t>0,\lambda>0$ and $x,y\in\mathbb{R}^d$, the function

$$|\Delta_{\lambda,x}^{\beta/2}|\,p_0(t,x,y)\begin{cases} =\mathcal{A}(d,-\beta)\Big(\int_{|z|\le\lambda\wedge t^{1/\alpha}}|p_0(t,x+z,y)-p_0(t,x,y)-\frac{\partial}{\partial x}p_0(t,x,y)\cdot z|\,\frac{1}{|z|^{d+\beta}}dz\\ \quad+\int_{\lambda>|z|>(\lambda\wedge t^{1/\alpha})}|p_0(t,x+z,y)-p_0(t,x,y)|\frac{dz}{|z|^{d+\beta}}\Big) & \text{for } |x-y|^\alpha\le t,\\ =\mathcal{A}(d,-\beta)\Big(\int_{|z|\le\lambda\wedge|x-y|/2}|p_0(t,x+z,y)-p_0(t,x,y)-\frac{\partial}{\partial x}p_0(t,x,y)\cdot z|\frac{1}{|z|^{d+\beta}}dz\\ \quad+\int_{\lambda>|z|>(\lambda\wedge|x-y|/2)}|p_0(t,x+z,y)-p_0(t,x,y)|\frac{dz}{|z|^{d+\beta}}\Big) & \text{for } |x-y|^\alpha>t.\end{cases}$$

Observe that

$$|\Delta_{\lambda,x}^{\beta/2}|\,p_0(t,x,y)|\le|\Delta_x^{\beta/2}|\,p_0(t,x,y).$$

Set

$$f_{0,\lambda}(t,x,y) = \begin{cases} t^{-(d+\beta)/\alpha} & \text{when } |x-y| \le t^{1/\alpha}, \\ |x-y|^{-(d+\beta)} \mathbb{1}_{\{|x-y|\le\lambda\}} + |x-y|^{-(d+\alpha)} \mathbb{1}_{\{|x-y|>\lambda\}} & \text{when } |x-y| > t^{1/\alpha}. \end{cases}$$

Observe that when $\lambda = \infty$, $f_{0,\infty}$ is just the function f_0 defined in (2.5).

Lemma 2.3. *For each $\lambda > 0$ and $T > 0$, there exists a constant $C_8 = C_8(d, \alpha, \beta, \lambda, T) > 0$ such that*

$$|\Delta^{\beta/2}_{\lambda,x}|\, p_0(t,x,y) \le C_8 f_{0,\lambda}(t,x,y) \qquad \text{on } (0,T] \times \mathbb{R}^d \times \mathbb{R}^d. \tag{2.9}$$

Proof. (i) We first consider the case $|x-y|^\alpha \le t$. Note that

$$|\Delta^{\beta/2}_{\lambda,x}|\, p_0(t,x,y) \le |\Delta^{\beta/2}_x| p_0(t,x,y).$$

Hence, by the first part (i) in the proof of Lemma 2.2, there exists a positive constant c_1 so that

$$|\Delta^{\beta/2}_{\lambda,x}|\, p_0(t,x,y) \le c_1 t^{-(d+\beta)/\alpha}.$$

(ii) Next, we consider the case $|x-y|^\alpha > t$. In this case

$$\begin{aligned} |\Delta^{\beta/2}_{\lambda,x}|\, p_0(t,x,y) &\le \mathcal{A}(d,-\beta) \int_{|z|\le|x-y|/2} |p_0(t,x+z,y) - p_0(t,x,y) - \frac{\partial}{\partial x} p_0(t,x,y)\cdot z| \frac{dz}{|z|^{d+\beta}} \\ &\quad + \mathcal{A}(d,-\beta) \int_{\lambda \ge |z| \ge (\lambda\wedge|x-y|/2)} |p_0(t,x+z,y) - p_0(t,x,y)| \frac{dz}{|z|^{d+\beta}} \\ &=: I + II. \end{aligned}$$

By (2.7), there is a positive constant c_2 so that

$$I \le c_2 t |x-y|^{-(d+\alpha+\beta)} \le c_3 \left(|x-y|^{-(d+\beta)} \mathbb{1}_{\{|x-y|\le 2\lambda\}} + |x-y|^{-(d+\alpha)} \mathbb{1}_{\{|x-y|>2\lambda\}} \right).$$

Here the last inequality holds since $t|x-y|^{-(d+\alpha+\beta)} \le T(2\lambda)^{-\beta}|x-y|^{-(d+\alpha)}$ when $|x-y| > 2\lambda$ and $t|x-y|^{-(d+\alpha+\beta)} \le |x-y|^{-(d+\beta)}$ due to $|x-y|^\alpha \ge t$.

It is clear that $II = 0$ if $|x-y| > 2\lambda$. On the other hand, if $|x-y| \le 2\lambda$, then there exists a positive constant c_4 so that $II \le c_4 |x-y|^{-(d+\beta)}$ by (2.8). Finally, noting that $|x-y|^{-(d+\beta)} \asymp |x-y|^{-(d+\alpha)}$ for $\lambda < |x-y| \le 2\lambda$. This establishes the lemma. □

For each $\lambda > 0$ and $a \ge 0$, we extend the definition of $f_{0,\lambda}(t,x,y)$ to define

$$f_{a,\lambda}(t,x,y) := \begin{cases} t^{-(d+\beta)/\alpha} & \text{when } |x-y| \le t^{1/\alpha}, \\ |x-y|^{-(d+\beta)} \mathbb{1}_{\{|x-y|\le\lambda\}} + \left(|x-y|^{-(d+\alpha)} + a\,|x-y|^{-(d+\beta)} \right) \mathbb{1}_{\{|x-y|>\lambda\}} & \\ & \text{when } |x-y| > t^{1/\alpha}. \end{cases} \tag{2.10}$$

Note that $f_{a,\infty}(t,x,y) = f_0(t,x,y)$.

Lemma 2.4. *For each $\lambda > 0$, there is a constant $C_9 = C_9(d,\alpha,\beta,\lambda) > 0$ such that for every $a \in [0,1]$,*

$$\int_0^t \int_{\mathbb{R}^d} f_{a,\lambda}(s,z,y) dz ds \le C_9 \,(t^{1-\beta/\alpha} + t), \qquad t \in (0,\infty),\ y \in \mathbb{R}^d. \tag{2.11}$$

Proof. By the definition of $f_{a,\lambda}$,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} f_{a,\lambda}(s,z,y)\,dz\,ds \\
\le\;& \int_0^t \int_{|y-z|\le s^{1/\alpha}} s^{-(d+\beta)/\alpha}\,dz\,ds + \int_0^t \int_{\lambda\ge|y-z|>s^{1/\alpha}} \frac{1}{|y-z|^{d+\beta}}\,dz\,ds \\
&+ \int_0^t \int_{|y-z|\ge\lambda} (|y-z|^{-(d+\alpha)} + |y-z|^{-(d+\beta)})\,dz\,ds \\
\le\;& c_1 \int_0^t (s^{-\beta/\alpha}+1)\,ds \le c_2(t^{1-\beta/\alpha}+t).
\end{aligned}$$

□

For every $a\ge 0$, define

$$g_a(t,x,y) = \begin{cases} t^{-d/\alpha} & \text{when } |x-y|\le t^{1/\alpha}, \\ \frac{t}{|x-y|^{d+\alpha}} + \frac{at}{|x-y|^{d+\beta}} & \text{when } |x-y|>t^{1/\alpha}. \end{cases} \tag{2.12}$$

Observe that

$$\int_{\mathbb{R}^d} g_a(t,x,y)dy \asymp 1+at^{1-\beta/\alpha} \qquad \text{on } (0,\infty)\times\mathbb{R}^d. \tag{2.13}$$

Recall that $p_a(t,x,y)$ is the heat kernel of the operator $\Delta^{\alpha/2}+a\Delta^{\beta/2}$. Moreover, in view of (1.10),

$$g_a(t,x,y) \asymp p_a(t,x,y) \qquad \text{on } (0,1]\times\mathbb{R}^d\times\mathbb{R}^d. \tag{2.14}$$

Lemma 2.5. *For each $\lambda>0$ and $T>0$, there exists $C_{10}=C_{10}(d,\alpha,\beta,\lambda,T)>0$ such that for every $a\in[0,1]$ and all $t\in(0,T]$, $x,y\in\mathbb{R}^d$,*

$$\int_0^t \int_{\mathbb{R}^d} g_a(t-s,x,z) f_{a,\lambda}(s,z,y)\,dz\,ds \le C_{10} g_a(t,x,y).$$

Proof. Denote by $I=\int_0^t\int_{\mathbb{R}^d} g_a(t-s,x,z)f_{a,\lambda}(s,z,y)\,dz\,ds$.

(i) Suppose that $|x-y|\le t^{1/\alpha}$. Then

$$\begin{aligned}
I \;=\;& \int_0^t \int_{|x-z|\le 2t^{1/\alpha}} g_a(t-s,x,z) f_{a,\lambda}(s,z,y)\,dz\,ds \\
&+ \int_0^t \int_{|x-z|>2t^{1/\alpha}} g_a(t-s,x,z) f_{a,\lambda}(s,z,y)\,dz\,ds \\
=:\;& I_1+I_2.
\end{aligned}$$

We write I_1 as

$$\begin{aligned}
I_1 \;=\;& \int_0^{t/2} \int_{|x-z|\le 2t^{1/\alpha}} g_a(t-s,x,z) f_{a,\lambda}(s,z,y)\,dz\,ds \\
&+ \int_{t/2}^t \int_{|x-z|\le 2t^{1/\alpha}} g_a(t-s,x,z) f_{a,\lambda}(s,z,y)\,dz\,ds \\
=\;& I_{11}+I_{12}.
\end{aligned}$$

If $s \in (0, t/2)$, then $t - s \in (t/2, t)$. In this case, $g_a(t-s, x, z) \le c_1 t^{-d/\alpha}$ when $|x - z| \le 2t^{1/\alpha}$ by (2.12). Hence, by Lemma 2.4,

$$I_{11} \le c_1 t^{-d/\alpha} \int_0^t \int_{\mathbb{R}^d} f_{a,\lambda}(s, z, y)\, dz\, ds \le c_2 (T^{1-\beta/\alpha} + T)\, t^{-d/\alpha}.$$

When $s \in [t/2, t]$, since $|x - y| \le t^{1/\alpha}$ and $|x - z| \le 2t^{1/\alpha}$, $|y - z| \le 3t^{1/\alpha} \le 3(2s)^{1/\alpha}$. Thus $f_{a,\lambda}(s, z, y) \le c_3 s^{-(d+\beta)/\alpha} \le 2^{(d+\beta)/\alpha} c_3 t^{-(d+\beta)/\alpha}$. Hence,

$$I_{12} \le 2^{(d+\beta)/\alpha} c_3 t^{-(d+\beta)/\alpha} \int_0^t \int_{\mathbb{R}^d} g_a(t-s, x, z)\, dz\, ds \le c_4 T^{1-\beta/\alpha}(1 + T^{1-\beta/\alpha})\, t^{-d/\alpha}.$$

Next we consider I_2. Noting that $|x - z| > 2t^{1/\alpha}$, so we have by (2.12) and Lemma 2.4,

$$\begin{aligned}
I_2 &\le c_5 \int_0^t \int_{|x-z|>2t^{1/\alpha}} \left(\frac{t-s}{|x-z|^{d+\alpha}} + \frac{t-s}{|x-z|^{d+\beta}} \right) f_{a,\lambda}(s, z, y)\, dz\, ds \\
&\le c_6 t^{-d/\alpha} \left(1 + t^{1-\beta/\alpha} \right) \int_0^t \int_{\mathbb{R}^d} f_{a,\lambda}(s, z, y)\, dz\, ds \\
&\le c_7 (1 + T^{1-\beta/\alpha})(T^{1-\beta/\alpha} + T)\, t^{-d/\alpha}.
\end{aligned}$$

We thus conclude from the above that there is a $c_8 = c_8(d, \alpha, \beta, \lambda, T) > 0$ such that $I \le c_8\, t^{-d/\alpha}$ for every $t \in (0, T]$ whenever $|x - y| \le t^{1/\alpha}$.

(ii) Next assume that $|x - y| > t^{1/\alpha}$. Then

$$\begin{aligned}
I &= \int_0^t \int_{|x-z| \le |x-y|/2} g_a(t-s, x, z) f_{a,\lambda}(s, z, y)\, dz\, ds \\
&\quad + \int_0^t \int_{|x-z| > |x-y|/2} g_a(t-s, x, z) f_{a,\lambda}(s, z, y)\, dz\, ds \\
&=: I_1 + I_2.
\end{aligned}$$

If $|x - z| \le |x - y|/2$, then $|y - z| \ge |x - y|/2 > t^{1/\alpha}/2$. Hence, there is a constant c_9 so that

$$f_{a,\lambda}(s, z, y) \le c_9 \left(|x-y|^{-(d+\alpha)} + a|x-y|^{-(d+\beta)} \right)$$

for $s \in (0, t)$. Therefore,

$$\begin{aligned}
I_1 &\le c_9 (|x-y|^{-(d+\alpha)} + a|x-y|^{-(d+\beta)}) \cdot \int_0^t \int_{\mathbb{R}^d} g_a(t-s, x, z)\, dz\, ds \\
&\le c_{10} (1 + T^{1-\beta/\alpha}) \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{at}{|x-y|^{d+\beta}} \right).
\end{aligned}$$

If $|x - z| > |x - y|/2$, then $|x - z| > t^{1/\alpha}/2$. Hence $g_a(t-s, x, z) \le c_{11} \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{at}{|x-y|^{d+\beta}} \right)$ by (2.12). Thus by Lemma 2.4, we obtain

$$\begin{aligned}
I_2 &\le c_{11} \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{at}{|x-y|^{d+\beta}} \right) \int_0^t \int_{\mathbb{R}^d} f_{a,\lambda}(s, z, y)\, dz\, ds \\
&\le c_{12} (T^{1-\beta/\alpha} + T) \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{at}{|x-y|^{d+\beta}} \right).
\end{aligned}$$

This completes the proof of the Lemma. □

3 Fundamental solution

Throughout the rest of this paper, $b(x,z)$ is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying condition (1.2). Recall the definition of the non-local operator $\mathcal{S}^b$ from (1.3). Let $|q^b|_0(t,x,y) = p_0(t,x,y)$, and define for each $n \geq 1$,

$$|q^b|_n(t,x,y) = \int_0^t \int_{\mathbb{R}^d} |q^b|_{n-1}(t-s,x,z)|\mathcal{S}^b_z p_0(s,z,y)|\, dzds.$$

For each $\lambda > 0$, define

$$b_\lambda(x,z) = b(x,z)1_{\{|z|>\lambda\}}(z).$$

In view of (1.7), there exists a constant $C_{11} = C_{11}(d,\alpha,\beta) > 0$ such that $p_0(t,x,y) \leq C_{11} g_a(t,x,y)$ for all $t > 0, a \in [0,1]$ and $x,y \in \mathbb{R}^d$, where g_a is the function defined by (2.12). On the other hand, note that

$$\begin{aligned}
|\mathcal{S}^b f(x)| &= \left| \mathcal{A}(d,-\beta) \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - \langle \nabla f(x), z\rangle \mathbb{1}_{\{|z|\leq\lambda\}} \right) \frac{b(x,z)}{|z|^{d+\beta}} dz \right| \\
&\leq \left| \mathcal{A}(d,-\beta) \int_{|z|\leq\lambda} \left(f(x+z) - f(x) - \langle \nabla f(x), z\rangle \right) \frac{b(x,z)}{|z|^{d+\beta}} dz \right| \\
&\quad + \left| \mathcal{A}(d,-\beta) \int_{\mathbb{R}^d} \left(f(x+z) - f(x) \right) \frac{b_\lambda(x,z)}{|z|^{d+\beta}} dz \right| \\
&\leq \|b\|_\infty \cdot |\Delta^{\beta/2}_{\lambda,x}| f(x) + \|b_\lambda\|_\infty \cdot |\Delta^{\beta/2}_x| f(x)
\end{aligned}$$

Then by Lemma 2.2 and Lemma 2.3, for every $A > 0, \lambda > 0$ and $T > 0$ and every bounded function b with $\|b\|_\infty \leq A$,

$$\begin{aligned}
|\mathcal{S}^b_z p_0(t,z,y)| &\leq \|b\|_\infty \cdot |\Delta^{\beta/2}_{\lambda,z}| p_0(t,z,y) + \|b_\lambda\|_\infty \cdot |\Delta^{\beta/2}_z| p_0(t,z,y) \\
&\leq C_8 A\, f_{0,\lambda}(t,z,y) + C_7 M_{b,\lambda}\, f_0(t,z,y) \\
&\leq (C_7 + C_8) A f_{M_{b,\lambda}/A,\lambda}(t,z,y), \qquad t \in (0,T].
\end{aligned} \tag{3.1}$$

Here recall that $M_{b,\lambda} = \operatorname{esssup}_{x,z\in\mathbb{R}^d, |z|>\lambda} |b(x,z)|$, $f_{a,\lambda}$ is the function defined in (2.10). The above estimate is a refinement of Lemma 2.2. The latter corresponds to the case where $\lambda = \infty$.

Lemma 3.1. *For each $\lambda > 0, A > 0$ and $T > 0$ and every bounded function b on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying condition* (1.2) *with $\|b\|_\infty \leq A$,*

$$|q^b|_n(t,x,y) \leq C_{11} \left(A(C_7 + C_8)C_{10} \right)^n g_{M_{b,\lambda}/A}(t,x,y) < \infty, \quad t \in (0,T],\ x,y \in \mathbb{R}^d. \tag{3.2}$$

Proof. We prove this lemma by induction. Since $p_0(t,x,y) \leq C_{11} g_{M_{b,\lambda}/A}(t,x,y)$ and $M_{b,\lambda}/A \leq 1$, in view of Lemma 2.5 and (3.1), (3.2) clearly holds for $n = 1$. Suppose that (3.2) holds for $n = j \geq 1$. Then by Lemma 2.5 and (3.1),

$$\begin{aligned}
& |q^b|_{j+1}(t,x,y) \\
\leq\ & C_{11} \left(A(C_7 + C_8)C_{10} \right)^j \int_0^t \int_{\mathbb{R}^d} g_{M_{b,\lambda}/A}(t-s,x,z) |\mathcal{S}^b_z p_0(s,z,y)|\, dz\, ds \\
\leq\ & C_{11} \left(A(C_7 + C_8)C_{10} \right)^j\ (C_7 + C_8)A \int_0^t \int_{\mathbb{R}^d} g_{M_{b,\lambda}/A}(t-s,x,z) f_{M_{b,\lambda}/A,\lambda}(s,z,y)\, dz\, ds \\
\leq\ & C_{11} \left(A(C_7 + C_8)C_{10} \right)^{j+1} g_{M_{b,\lambda}/A}(t,x,y)
\end{aligned}$$

for $t \in (0, T]$ and $x, y \in \mathbb{R}^d$. This proves that (3.2) holds for $n = j + 1$ and thus for every $n \geq 1$. $\square$

Now we define $q_n^b : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ as follows. For $t > 0$ and $x, y \in \mathbb{R}^d$, let $q_0^b(t, x, y) = p_0(t, x, y)$, and for each $n \geq 1$, define

$$q_n^b(t, x, y) = \int_0^t \int_{\mathbb{R}^d} q_{n-1}^b(t - s, x, z) \mathcal{S}_z^b p_0(s, z, y) \, dz \, ds. \tag{3.3}$$

Clearly by Lemma 3.1, each $q_n^b(t, x, y)$ is well defined.

Lemma 3.2. *For every $n \geq 0$, $q_n^b(t, x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$.*

Proof. We prove it by induction. Clearly $q_0^b(t, x, y)$ is continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. Suppose that $q_n^b(t, x, y)$ is continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. For every $M \geq 2$, it follows from (3.1) , Lemma 3.1 and the dominated convergence theorem that for $\varepsilon < 1/(2M)$,

$$(t, x, y) \mapsto \int_\varepsilon^{t-\varepsilon} \int_{\mathbb{R}^d} q_n^b(t - s, x, z) \mathcal{S}_z^b p_0(s, z, y) dz ds$$

is jointly continuous on $[1/M, M] \times \mathbb{R}^d \times \mathbb{R}^d$. On the other hand, it follows from (3.1) and (2.13) that

$$\begin{aligned}
& \sup_{t \in [1/M, M]} \sup_{x, y} \int_{t-\varepsilon}^t \int_{\mathbb{R}^d} g_{M_{b,\lambda}}(t - s, x, z) |\mathcal{S}_z^b p_0(s, z, y)| \, dz \, ds \\
\leq \;& c_1 A \left(\sup_{t \in [1/M, M]} [(t - \varepsilon)^{-(d+\beta)/\alpha} + (t - \varepsilon)^{-(d+\alpha)/\alpha}] \right) \sup_{x \in \mathbb{R}^d} \int_{t-\varepsilon}^t \int_{\mathbb{R}^d} g_{M_{b,\lambda}}(t - s, x, z) \, dz \, ds \\
\leq \;& c_2 A (2M)^{(d+\alpha)/\alpha} \int_0^\varepsilon (1 + r^{1-\beta/\alpha}) dr \\
\leq \;& c_3 A (2M)^{(d+\alpha)/\alpha} \varepsilon,
\end{aligned}$$

which goes to zero as $\varepsilon \to 0$; while by (3.1) and (2.11),

$$\begin{aligned}
& \sup_{t \in [1/M, M]} \sup_{x, y} \int_0^\varepsilon \int_{\mathbb{R}^d} g_{M_{b,\lambda}}(t - s, x, z) |\mathcal{S}_z^b p_0(s, z, y)| \, dz \, ds \\
\leq \;& c_4 \left(\sup_{t \in [1/M, M]} (t - \varepsilon)^{-d/\alpha} \right) \sup_{y \in \mathbb{R}^d} \int_0^\varepsilon \int_{\mathbb{R}^d} |\mathcal{S}_z^b p_0(s, z, y)| \, dz \, ds \\
\leq \;& c_5 (2M)^{d/\alpha} \, \|b\|_\infty \, \varepsilon^{1-\beta/\alpha} \to 0
\end{aligned} \tag{3.4}$$

as $\varepsilon \to 0$. We conclude from Lemma 3.1 and the above argument that

$$q_{n+1}^b(t, x, y) = \int_0^t \int_{\mathbb{R}^d} q_n^b(t - s, x, z) \mathcal{S}_z^b p_0(s, z, y) \, dz \, ds$$

is jointly continuous in $(t, x, y) \in [1/M, M] \times \mathbb{R}^d \times \mathbb{R}^d$ and so in $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. This completes the proof of the lemma. $\square$

Recall $f_0(t, x, y)$ is the function defined in (2.5) and

$$|\Delta_x^{\beta/2}| p_0(t, x, y) \leq C_7 f_0(t, x, y) \qquad \text{on } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d.$$

Lemma 3.3. *There is a constant $C_{12} = C_{12}(d, \alpha, \beta) > 0$ so that for every $A > 0$ and every bounded function b on $\mathbb{R}^d \times \mathbb{R}^d$ with $\|b\|_\infty \le A$ and for every integer $n \ge 0$ and $\varepsilon > 0$,*

$$\left| \int_{\{z \in \mathbb{R}^d : |z| > \varepsilon\}} \left(q_n^b(t, x+z, y) - q_n^b(t, x, y) \right) \frac{\mathcal{A}(d, -\beta) b(x, z)}{|z|^{d+\beta}} dz \right| \le (C_{12} A)^{n+1} f_0(t, x, y) \qquad (3.5)$$

for $(t, x, z) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$, and $\mathcal{S}_x^b q_n^b(t, x, y)$ exists pointwise for $(t, x, z) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ in the sense of (1.5) with

$$\mathcal{S}_x^b q_{n+1}^b(t, x, y) = \int_0^t \int_{\mathbb{R}^d} \mathcal{S}_x^b q_n^b(t-s, x, z) \mathcal{S}_z^b p_0(s, z, y) dz ds \qquad (3.6)$$

and

$$|\mathcal{S}_x^b q_n^b(t, x, y)| \le (C_{12} A)^{n+1} f_0(t, x, y) \qquad \text{on } (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d. \qquad (3.7)$$

Moreover,

$$q_{n+1}^b(t, x, y) = \int_0^t \int_{\mathbb{R}^d} p_0(t-s, x, z) \mathcal{S}_z^b q_n^b(s, z, y) dz ds \qquad \text{for } (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d. \quad (3.8)$$

Proof. Let $q(t, x, y)$ denote the transition density function of the symmetric β-stable process on $\mathbb{R}^d$. Then by (1.7) but with β in place of α, we have

$$q(t, x, y) \asymp t^{-d/\beta} \left(1 \wedge \frac{t^{1/\beta}}{|x-y|} \right)^{d+\beta} \qquad \text{on } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \qquad (3.9)$$

Observe that (2.5) and (3.9) yield

$$f_0(t, x, y) \asymp t^{-\beta/\alpha} q(t^{\beta/\alpha}, x, y) \qquad \text{on } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \qquad (3.10)$$

Hence on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^d} f_0(t-s, x, z) f_0(s, z, y) ds dz \\
\asymp\ & \int_0^t (t-s)^{-\beta/\alpha} s^{-\beta/\alpha} \left(\int_{\mathbb{R}^d} q((t-s)^{\beta/\alpha}, x, z) q(s^{\beta/\alpha}, z, y) dz \right) ds \\
=\ & \int_0^t (t-s)^{-\beta/\alpha} s^{-\beta/\alpha} q((t-s)^{\beta/\alpha} + s^{\beta/\alpha}, x, y) ds \\
\asymp\ & q(t^{\beta/\alpha}, x, y) \int_0^t (t-s)^{-\beta/\alpha} s^{-\beta/\alpha} ds \\
=\ & q(t^{\beta/\alpha}, x, y)\, t^{1-(2\beta/\alpha)} \int_0^1 (1-u)^{-\beta/\alpha} u^{-\beta/\alpha} du \\
\asymp\ & t^{1-\beta/\alpha} f_0(t, x, y).
\end{aligned}$$

In the second $\asymp$ above, we used the fact that

$$(t/2)^{\beta/\alpha} \le (t-s)^{\beta/\alpha} + s^{\beta/\alpha} \le 2 t^{\beta/\alpha} \qquad \text{for every } s \in (0, t)$$

and the estimate (3.9), while in the last equality, we used a change of variable $s = tu$. So there is a constant $c_1 = c_1(d, \alpha, \beta) > 0$ so that

$$\int_0^t \int_{\mathbb{R}^d} f_0(t-s, x, z) f_0(s, z, y) ds dz \le c_1 \, f_0(t, x, y) \qquad \text{for every } t \in (0, 1] \text{ and } x, y \in \mathbb{R}^d. \tag{3.11}$$

By increasing the value of c_1 if necessary, we may and do assume that c_1 is larger than 1.

We now proceed by induction. Let $C_{12} := c_1 C_7$. Note that

$$|\mathcal{S}_x^b p_0(t, x, y)| \le A |\Delta_x^{\beta/2}| p_0(t, x, y) \le C_7 A f_0(t, x, y). \tag{3.12}$$

When $n = 0$, (3.8) holds by definition. By Lemma 2.2, (3.5) and (3.7) hold for $n = 0$. Suppose that (3.5) and (3.7) hold for $n = j$. Then for every $\varepsilon > 0$, by the definition of q_{j+1}^b, Lemma 3.1, (3.11) and Fubini's theorem,

$$\begin{aligned} & \int_{\{w \in \mathbb{R}^d : |\omega| > \varepsilon\}} \left(q_{j+1}^b(t, x+w, y) - q_{j+1}^b(t, x, y) \right) \frac{\mathcal{A}(d, -\beta) b(x, w)}{|w|^{d+\beta}} dw \\ = & \int_0^t \int_{\mathbb{R}^d} \left(\int_{\{w \in \mathbb{R}^d : |w| > \varepsilon\}} \left(q_j^b(t-s, x+w, z) - q_j^b(t-s, x, z) \right) \frac{\mathcal{A}(d, -\beta) b(x, w)}{|w|^{d+\beta}} dw \right) \\ & \quad \times \mathcal{S}_z^b p_0(s, z, y) \, dz ds \end{aligned} \tag{3.13}$$

and so

$$\begin{aligned} & \left| \int_{\{w \in \mathbb{R}^d : |w| > \varepsilon\}} \left(q_{j+1}^b(t, x+w, y) - q_{j+1}^b(t, x, y) \right) \frac{\mathcal{A}(d, -\beta) b(x, w)}{|w|^{d+\beta}} dw \right| \\ \le & \int_0^t \int_{\mathbb{R}^d} (C_{12} A)^{j+1} f_0(t-s, x, z) \, |\mathcal{S}_z^b p_0(s, z, y)| dz ds \\ \le & \int_0^t \int_{\mathbb{R}^d} (C_{12} A)^{j+1} f_0(t-s, x, z) \, C_7 A f_0(s, z, y) dz ds \\ \le & (C_{12} A)^{j+2} f_0(t, x, y). \end{aligned}$$

By (3.13) and Lebesgue dominated convergence theorem, we conclude that

$$\begin{aligned} & S_x^b q_{j+1}^b(t, x, y) \\ := & \lim_{\varepsilon \to 0} \int_{\{w \in \mathbb{R}^d : |w| > \varepsilon\}} \left(q_{j+1}^b(t, x+w, y) - q_{j+1}^b(t, x, y) \right) \frac{\mathcal{A}(d, -\beta) b(x, w)}{|w|^{d+\beta}} dw \\ = & \int_0^t \int_{\mathbb{R}^d} \left(\lim_{\varepsilon \to 0} \int_{\{w \in \mathbb{R}^d : |w| > \varepsilon\}} \left(q_j^b(t-s, x+w, z) - q_j^b(t-s, x, z) \right) \frac{\mathcal{A}(d, -\beta) b(x, w)}{|w|^{d+\beta}} dw \right) \\ & \quad \times \mathcal{S}_z^b p_0(s, z, y) \, dz ds \\ = & \int_0^t \int_{\mathbb{R}^d} \mathcal{S}_x^b q_j^b(t-s, x, z) \, \mathcal{S}_z^b p_0(s, z, y) \, dz ds \end{aligned}$$

exists and (3.6) as well as (3.7) holds for $n = j + 1$. (The same proof verifies (3.6) when $n = 0$.)

On the other hand, in view of (3.7) and (3.8) for $n = j$, we have by the Fubini theorem,

$$
\begin{aligned}
& q^b_{j+1}(t,x,y) \\
= & \int_0^t \int_{\mathbb{R}^d} q^b_j(s,x,z)\mathcal{S}^b_z p_0(t-s,z,y)dzds \\
= & \int_0^t \int_{\mathbb{R}^d} \left(\int_0^s \int_{\mathbb{R}^d} p_0(r,x,w)\mathcal{S}^b_w q^b_{j-1}(s-r,w,z)drdw \right) \mathcal{S}^b_z p_0(t-s,z,y)dzds \\
= & \int_0^t \int_{\mathbb{R}^d} p_0(r,x,w) \left(\int_r^t \int_{\mathbb{R}^d} \mathcal{S}^b_w q^b_{j-1}(s-r,w,z)\mathcal{S}^b_z p_0(t-s,z,y)dsdz \right) dwdr \\
= & \int_0^t \int_{\mathbb{R}^d} p_0(r,x,w)\mathcal{S}^b_w q^b_j(t-r,w,y)dwdr.
\end{aligned}
$$

This verifies that (3.8) also holds for $n = j+1$. The lemma is now established by induction. $\square$

Recall that $M_{b,\lambda} = \mathrm{esssup}_{x,z\in\mathbb{R}^d,|z|>\lambda}|b(x,z)| = \|b_\lambda(x,z)\|_\infty$.

Lemma 3.4. *For each $\lambda > 0$, there are positive constants $A_0 = A_0(d,\alpha,\beta,\lambda)$ and $C_{13} = C_{13}(d,\alpha,\beta,\lambda)$ so that if $\|b\|_\infty \le A_0$, then for every integer $n \ge 0$,*

$$|q^b_{n+1}(t,x,y)| \le C_{13}2^{-n}\, p_{M_{b,\lambda}}(t,x,y) \quad \text{for } t\in(0,1] \text{ and } x,y\in\mathbb{R}^d, \tag{3.14}$$

(3.5) holds and so $\mathcal{S}^b_x q^b_n(t,x,y)$ exists pointwise in the sense of (1.5) with

$$|\mathcal{S}^b_x q^b_n(t,x,y)| \le 2^{-n}\, f_0(t,x,y) \quad \text{for } t\in(0,1] \text{ and } x,y\in\mathbb{R}^d, \tag{3.15}$$

and

$$\sum_{n=0}^\infty q^b_n(t,x,y) \ge \frac{1}{2}p_0(t,x,y) \quad \text{for } t\in(0,1] \text{ and } |x-y|\le 3t^{1/\alpha}. \tag{3.16}$$

Proof. We take a positive constant A_0 so that $A_0 \le 1\wedge[2(C_7+C_8)C_{10}+2C_{12}]^{-1}$. We have by Lemma 3.1 and Lemma 3.3 that for every b with $\|b\|_\infty \le A_0$,

$$|q^b_{n+1}(t,x,y)| \le C_{11}2^{-n}g_{M_{b,\lambda}/A_0}(t,x,y) \le C_{11}A_0^{-1}2^{-n}g_{M_{b,\lambda}}(t,x,y) \quad \text{and} \quad |\mathcal{S}^b_x q^b_n(t,x,y)| \le 2^{-n}f_0(t,x,y)$$

for every $t\in(0,1]$ and $x,y\in\mathbb{R}^d$. This together with (2.14) establishes (3.14) and (3.15).

On the other hand, by (2.12), there exists $c = c(d,\alpha,\beta) \ge 1$ so that $g_a(t,x,y) \le cp_0(t,x,y)$ for $a\in[0,1]$ and $|x-y|\le 3t^{1/\alpha}$ and $t\in(0,1]$. Take A_0 small enough so that $A_0 \le 1\wedge[2(C_7+C_8)C_{10}+2C_{12}]^{-1}$ and $\sum_{n=1}^\infty (A_0(C_7+C_8)C_{10})^n \le \frac{1}{2cC_{11}}$. Then for every b with $\|b\|_\infty \le A_0$, we have by Lemma 3.1 for $|x-y|\le 3t^{1/\alpha}$ and $t\in(0,1]$ that

$$\sum_{n=1}^\infty |q^b|_n(t,x,y) \le cC_{11}\sum_{n=1}^\infty (A_0(C_7+C_8)C_{10})^n p_0(t,x,y) \le \frac{1}{2}p_0(t,x,y).$$

Consequently, for $|x-y|\le 3t^{1/\alpha}$ and $t\in(0,1]$,

$$\sum_{n=0}^\infty q^b_n(t,x,y) \ge p_0(t,x,y) - \sum_{n=1}^\infty |q^b_n(t,x,y)| \ge \frac{1}{2}p_0(t,x,y).$$

$\square$

We now extend the results in Lemma 3.4 to any bounded b that satisfies condition (1.2). For $\lambda > 0$, define

$$b^{(\lambda)}(x,z) = \lambda^{\beta/\alpha-1} b(\lambda^{-1/\alpha}x, \lambda^{-1/\alpha}z). \tag{3.17}$$

For a function f on $\mathbb{R}^d$, set

$$f^{(\lambda)}(x) := f(\lambda^{-1/\alpha}x).$$

By a change of variable, one has from (1.1) and (1.3) that

$$\Delta^{\alpha/2} f^{(\lambda)}(x) = \lambda^{-1}(\Delta^{\alpha/2} f)(\lambda^{-1/\alpha}x)$$

and

$$\mathcal{S}^{b^{(\lambda)}} f^{(\lambda)}(x) = \lambda^{-1}(\mathcal{S}^b f)(\lambda^{-1/\alpha}x). \tag{3.18}$$

We remark here that condition (1.2) used in establishing (3.18). Note that the transition density function $p_0(t,x,y)$ of the symmetric α-stable process has the following scaling property:

$$p_0(t,x,y) = \lambda^{-d/\alpha} p_0(\lambda^{-1}t, \lambda^{-1/\alpha}x, \lambda^{-1/\alpha}y) \tag{3.19}$$

Recall $q_n^b(t,x,y)$ is the function defined inductively by (3.3) with $q_0^b(t,x,y) := p_0(t,x,y)$.

Lemma 3.5. *Suppose that b is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2). For every $\lambda > 0$ and for every integer $n \geq 0$,*

$$q_n^{b^{(\lambda)}}(t,x,y) = \lambda^{-d/\alpha}\, q_n^b(\lambda^{-1}t, \lambda^{-1/\alpha}x, \lambda^{-1/\alpha}y), \qquad x,y \in \mathbb{R}^d; \tag{3.20}$$

or, equivalently,

$$q_n^b(t,x,y) = \lambda^{d/\alpha}\, q_n^{b^{(\lambda)}}(\lambda t, \lambda^{1/\alpha}x, \lambda^{1/\alpha}y), \qquad x,y \in \mathbb{R}^d. \tag{3.21}$$

Proof. We prove it by induction. Clearly in view of (3.19), (3.20) holds when $n = 0$. Suppose that (3.20) holds for $n = j \geq 0$. Then by the definition (3.3), (3.18) and (3.19),

$$\begin{aligned}
q_{j+1}^{b^{(\lambda)}}(t,x,y) &= \int_0^t \int_{\mathbb{R}^d} q_j^{b^{(\lambda)}}(t-s,x,z)\mathcal{S}_z^{b^{(\lambda)}} p_0(s,z,y)\, dzds \\
&= \int_0^t \int_{\mathbb{R}^d} \lambda^{-d/\alpha} q_j^b(\lambda^{-1}(t-s), \lambda^{-1/\alpha}x, \lambda^{-1/\alpha}z)\lambda^{-d/\alpha-1}\left(\mathcal{S}_z^b p_0(\lambda^{-1}s,\cdot,\lambda^{-1/\alpha}y)\right)(\lambda^{-1/\alpha}z)\, dzds \\
&= \lambda^{-d/\alpha}\int_0^{\lambda^{-1}t} \int_{\mathbb{R}^d} q_j^b(\lambda^{-1}t - r, \lambda^{-1/\alpha}x, w)\left(\mathcal{S}_w^b p_0(r,\cdot,\lambda^{-1/\alpha}y)\right)(w)\, dwdr \\
&= \lambda^{-d/\alpha} q_{j+1}^b(\lambda^{-1}t, \lambda^{-1/\alpha}x, \lambda^{-1/\alpha}y).
\end{aligned}$$

This proves that (3.20) holds for $n = j+1$ and so, by induction, it holds for every $n \geq 0$. $\square$

Recall that A_0 is the positive constant in Lemma 3.4.

Theorem 3.6. *For every $\lambda > 0$ and $A > 0$, there is a positive constant $C_{14} = C_{14}(d,\alpha,\beta,A,\lambda) > 0$ so that for every bounded function b with $\|b\|_\infty \leq A$, that satisfies condition (1.2) and $n \geq 0$,*

$$|q_n^b(t,x,y)| \leq C_{14}2^{-n}\left(t^{-d/\alpha} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{M_{b,\lambda}\, t}{|x-y|^{d+\beta}}\right)\right) \tag{3.22}$$

for every $0 < t \le 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ *and* $x, y \in \mathbb{R}^d$, *and*

$$\sum_{n=0}^\infty q_n^b(t,x,y) \ge \frac{1}{2} p_0(t,x,y) \quad \text{for } 0 < t \le 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)} \text{ and } |x-y| \le 3t^{1/\alpha}. \tag{3.23}$$

Moreover, for every $n \ge 0$, (3.5) *holds and so* $\mathcal{S}_x^b q_n^b(t,x,y)$ *exists pointwise in the sense of* (1.5) *with*

$$|\mathcal{S}_x^b q_n^b(t,x,y)| \le 2^{-n} f_0(t,x,y) \tag{3.24}$$

for every $0 < t \le 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ *and* $x, y \in \mathbb{R}^d$. *Moreover,* (3.6) *and* (3.8) *hold.*

Proof. In view of Lemma 3.4, it suffices to prove the theorem for $A_0 < \|b\|_\infty \le A$. Set $r = (\|b\|_\infty/A_0)^{\alpha/(\alpha-\beta)}$. The function $b^{(r)}$ defined by (3.17) has the property $\|b^{(r)}\|_\infty = A_0$. Thus by Lemma 3.4, there is a constant $C_{14} = C_{14}(d,\alpha,\beta,A,\lambda) := C_{13}(d,\alpha,\beta,r^{1/\alpha}\lambda) > 0$ so that for every integer $n \ge 0$,

$$|q_n^{b^{(r)}}(t,x,y)| \le C_{14}\, 2^{-n}\, p_{M_{b^{(r)},r^{1/\alpha}\lambda}}(t,x,y) \quad \text{for } t \in (0,1] \text{ and } x, y \in \mathbb{R}^d, \tag{3.25}$$

(3.5) holds and so $\mathcal{S}_x^b q_n^{b^{(r)}}(t,x,y)$ exists pointwise in the sense of (1.5) with

$$|\mathcal{S}_x^b q_n^{b^{(r)}}(t,x,y)| \le 2^{-n}\, f_0(t,x,y) \quad \text{for } t \in (0,1] \text{ and } x, y \in \mathbb{R}^d, \tag{3.26}$$

and

$$\sum_{n=0}^\infty q_n^{b^{(r)}}(t,x,y) \ge \frac{1}{2} p_0(t,x,y) \quad \text{for } t \in (0,1] \text{ and } |x-y| \le 3t^{1/\alpha}. \tag{3.27}$$

Noting $r^{1-\beta/\alpha} M_{b^{(r)},r^{1/\alpha}\lambda} = M_{b,\lambda}$, we have by (3.21), (3.25) and (2.3) that for every $0 < t \le 1/r = (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
|q_n^b(t,x,y)| &= r^{d/\alpha}\, |q_n^{b^{(r)}}(rt, r^{1/\alpha}x, r^{1/\alpha}y)| \\
&\le C_{14} 2^{-n} r^{d/\alpha}\, p_{M_{b^{(r)},r^{1/\alpha}\lambda}}(rt, r^{1/\alpha}x, r^{1/\alpha}y) \\
&\le 2C\, C_{14}\, 2^{-n} \left(t^{-d/\alpha} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{r^{1-\beta/\alpha} M_{b^{(r)},r^{1/\alpha}\lambda}\, t}{|x-y|^{d+\beta}} \right) \right) \\
&\le 2C\, C_{14}\, 2^{-n} \left(t^{-d/\alpha} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{M_{b,\lambda}\, t}{|x-y|^{d+\beta}} \right) \right),
\end{aligned}$$

which establishes (3.22). Similarly, (3.23) follows from (3.19), and (3.27), while the conclusion of (3.24) is a direct consequence of (3.18), (3.21) and (3.26). That (3.6) and (3.8) hold follows directly from Lemma 3.3 and Lemma 3.5. □

Recall that $q^b(t,x,y) := \sum_{n=0}^\infty q_n^b(t,x,y)$, whenever it is convergent. The following theorem follows immediately from Lemmas 3.2, 3.4 and Theorem3.6.

Theorem 3.7. *For every* $\lambda > 0$ *and* $A > 0$, *let* $C_{14} = C_{14}(d,\alpha,\beta,A,\lambda)$ *be the constant in Theorem 3.6. Then for every bounded function* b *with* $\|b\|_\infty \le A$ *that satisfies condition* (1.2), $q^b(t,x,y)$ *is well defined and is jointly continuous in* $(0, 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}] \times \mathbb{R}^d \times \mathbb{R}^d$. *Moreover,*

$$|q^b(t,x,y)| \le 2C_{14} \left(t^{-d/\alpha} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{M_{b,\lambda}\, t}{|x-y|^{d+\beta}} \right) \right)$$

and $\mathcal{S}^b_x q^b(t,x,y)$ *exists pointwise in the sense of* (1.5) *with*

$$|\mathcal{S}^b_x q^b(t,x,y)| \le 2 f_0(t,x,y)$$

for every $0<t\le 1\wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ *and* $x,y\in\mathbb{R}^d$, *and*

$$q^b(t,x,y)\ge \frac{1}{2}p_0(t,x,y) \quad \text{for } 0<t\le 1\wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)} \text{ and } |x-y|\le 3t^{1/\alpha}. \tag{3.28}$$

Moreover, for every $0<t\le 1\wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ *and* $x,y\in\mathbb{R}^d$,

$$\begin{aligned} q^b(t,x,y) &= p_0(t,x,y)+\int_0^t\int_{\mathbb{R}^d} q^b(t-s,x,z)\mathcal{S}^b_z p_0(s,z,y)dzds && (3.29)\\ &= p_0(t,x,y)+\int_0^t\int_{\mathbb{R}^d} p_0(t-s,x,z)\mathcal{S}^b_z q^b(s,z,y)dzds. && (3.30)\end{aligned}$$

Theorem 3.8. *Suppose that* b *is a bounded function on* $\mathbb{R}^d\times\mathbb{R}^d$ *satisfying* (1.2). *Let* A_0 *be the constant in Lemma 3.4. Then for every* $t,s>0$ *with* $t+s\le 1\wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ *and* $x,y\in\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} q^b(t,x,z)q^b(s,z,y)dz = q^b(t+s,x,y). \tag{3.31}$$

Proof. In view of Theorem 3.6, we have

$$\int_{\mathbb{R}^d} q^b(t,x,z)q^b(s,z,y)dz=\sum_{j=0}^\infty\sum_{n=0}^j\int_{\mathbb{R}^d} q^b_n(t,x,z)q^b_{j-n}(s,z,y)dz.$$

So it suffices to show that for every $j\ge 0$,

$$\sum_{n=0}^j\int_{\mathbb{R}^d} q^b_n(t,x,z)q^b_{j-n}(s,z,y)dz = q^b_j(t+s,x,y) \tag{3.32}$$

Clearly, (3.32) holds for $j=0$. Suppose that (3.32) holds for $j=l\ge 1$. Then we have by Fubini's theorem and the estimates in (3.1) and Theorem 3.6,

$$\begin{aligned} &\sum_{n=0}^{l+1}\int_{\mathbb{R}^d} q^b_n(t,x,z)q^b_{l+1-n}(s,z,y)dz\\ =\;& \int_{\mathbb{R}^d} q^b_{l+1}(t,x,z)p_0(s,z,y)dz+\sum_{n=0}^{l}\int_{\mathbb{R}^d} q^b_n(t,x,z)q^b_{l+1-n}(s,z,y)dz\\ =\;& \int_{\mathbb{R}^d}\left(\int_0^t\int_{\mathbb{R}^d} q^b_l(t-r,x,w)\mathcal{S}^b_w p_0(r,w,z)dwdr\right)p_0(s,z,y)dz\\ &+\sum_{n=0}^{l}\int_{\mathbb{R}^d} q^b_n(t,x,z)\left(\int_0^s\int_{\mathbb{R}^d} q^b_{l-n}(s-r,z,w)\mathcal{S}^b_w p_0(r,w,y)dwdr\right)dz\\ =\;& \int_0^t\int_{\mathbb{R}^d} q^b_l(t-r,x,w)\mathcal{S}^b_w p_0(r+s,w,y)dwdr\\ &+\int_0^s\int_{\mathbb{R}^d} q^b_l(t+s-r,x,w)\mathcal{S}^b_w p_0(r,w,y)dwdr\\ =\;& q^b_{l+1}(t+s,x,y).\end{aligned}$$

This proves that (3.32) holds for $j = l+1$. So by induction, we conclude that (3.32) holds for every $j \geq 0$. $\square$

For notational simplicity, denote $1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ by δ_0. In view of Theorem 3.8, we can uniquely extend the definition of $q^b(t,x,y)$ to $t > \delta_0$ by using the Chapman-Kolmogorov equation recursively as follows.

Suppose that $q^b(t,x,y)$ has been defined and satisfies the Chapman-Kolmogorov equation (3.31) on $(0, k\delta_0] \times \mathbb{R}^d \times \mathbb{R}^d$. Then for $t \in (k\delta_0, (k+1)\delta_0]$, define

$$q^b(t,x,y) = \int_{\mathbb{R}^d} q^b(s,x,z) q^b(r,z,y)\, dz, \quad x,y \in \mathbb{R}^d \tag{3.33}$$

for any $s, r \in (0, k\delta_0]$ so that $s + r = t$. Such $q^b(t,x,y)$ is well defined on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and satisfies (3.31) for every $s,t > 0$. Moreover, since Chapman-Kolmogorov equation holds for $q^b(t,x,y)$ for all $t,s > 0$, we have by Theorem 3.7 and (2.3)-(2.4) that for every $A \geq A_0$, there are constants $c_i = c_i(d,\alpha,\beta,A)$, $i = 1,2$, so that for every $b(x,z)$ satisfying (1.2) with $\|b\|_\infty \leq A$,

$$|q^b(t,x,y)| \leq c_1\, e^{c_2 t}\, p_{M_{b,\lambda}}(t,x,y) \qquad \text{for every } t > 0 \text{ and } x,y \in \mathbb{R}^d. \tag{3.34}$$

Theorem 3.9. *$q^b(t,x,y)$ satisfies (3.29) and (3.30) for every $t > 0$ and $x,y \in \mathbb{R}^d$.*

Proof. Let $\delta_0 := 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$. It suffices to prove that for every $n \geq 1$, (3.29) and (3.30) hold for all $t \in (0, n\delta_0]$ and $x,y \in \mathbb{R}^d$.

Clearly, (3.29) holds for $t \in (0, n\delta_0]$ with $n = 1$. Suppose that (3.29) holds for $t \in (0, n\delta_0]$ with $n = k$. For $t \in (k\delta_0, (k+1)\delta_0]$, take $l, s \in (0, k\delta_0]$ so that $l+s = t$. Then we have by Fubini's theorem, Chapman-Kolmogorov equation of q^b, Lemma 2.5, (3.1) and (3.34),

$$\begin{aligned}
q^b(l+s,x,y) &= \int_{\mathbb{R}^d} q^b(l,x,z) q^b(s,z,y)\, dz \\
&= \int_{\mathbb{R}^d} q^b(l,x,z) \left(p_0(s,z,y) + \int_0^s \int_{\mathbb{R}^d} q^b(s-r,z,\omega) \mathcal{S}^b_\omega p_0(r,\omega,y)\, d\omega\, dr \right) dz \\
&= \int_{\mathbb{R}^d} p_0(l,x,z) p_0(s,z,y)\, dz \\
&\quad + \int_{\mathbb{R}^d} \left(\int_0^l \int_{\mathbb{R}^d} q^b(l-u,x,\eta) \mathcal{S}^b_\omega p_0(u,\eta,z)\, d\eta\, du \right) p_0(s,z,y)\, dz \\
&\quad + \int_0^s \int_{\mathbb{R}^d} q^b(l+s-r,x,\omega) \mathcal{S}^b_\omega p_0(r,\omega,y)\, d\omega\, dr \\
&= p_0(l+s,x,y) + \int_0^l \int_{\mathbb{R}^d} q^b(l-u,x,\eta) \mathcal{S}^b_\omega p_0(u+s,\eta,y)\, d\eta\, du \\
&\quad + \int_0^s \int_{\mathbb{R}^d} q^b(l+s-r,x,\omega) \mathcal{S}^b_\omega p_0(r,\omega,y)\, d\omega\, dr \\
&= p_0(l+s,x,y) + \int_0^{l+s} \int_{\mathbb{R}^d} q^b(l+s-r,x,z) \mathcal{S}^b_z p_0(r,z,y)\, dz\, dr.
\end{aligned}$$

By the similar procedure as above, we can also prove that (3.30) holds for every $t > 0$ and $x,y \in \mathbb{R}^d$. $\square$

Theorem 3.10. *Suppose that b is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying* (1.2)*. $q^b(t,x,y)$ is the unique continuous kernel that satisfies the Chapman-Kolmogorov equation* (3.31) *on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and that for some $\varepsilon > 0$,*

$$|q^b(t,x,y)| \le c\, p_1(t,x,y) \tag{3.35}$$

and (3.29) *hold for $(t,x,y) \in (0,\varepsilon] \times \mathbb{R}^d \times \mathbb{R}^d$. Moreover,* (3.34) *holds for $q^b(t,x,y)$.*

Proof. Suppose that $\overline{q}$ is any continuous kernel that satisfies, for some $\varepsilon > 0$, (3.29) and (3.35) hold for $(t,x,y) \in (0,\varepsilon] \times \mathbb{R}^d \times \mathbb{R}^d$. Without loss of generality, we may and do assume that $\varepsilon < 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$. Using (3.29) recursively, one gets

$$\overline{q}(t,x,y) = \sum_{j=1}^n q_j^b(t,x,y) + \int_0^t \int_{\mathbb{R}^d} \overline{q}(t-s,x,z)(\mathcal{S}^b p_0)_z^{*,n}(s,z,y)dsdz. \tag{3.36}$$

Here $(\mathcal{S}^b p_0)_z^{*,n}(s,z,y)$ denotes the nth convolution operation of the function $\mathcal{S}_z^b p_0(s,z,y)$; that is, $(\mathcal{S}^b p_0)_z^{*,1}(s,z,y) = \mathcal{S}_z^b p_0(s,z,y)$ and

$$(\mathcal{S}^b p_0)_z^{*,n}(s,z,y) = \int_0^s \int_{\mathbb{R}^d} \mathcal{S}_z^b p_0(r,z,w)\,(\mathcal{S}^b p_0)_w^{*,n-1}(s-r,w,y)dwdr \quad \text{for } n \ge 2. \tag{3.37}$$

It follows from Lemma 2.2, (3.12) and (3.11) that for every $A > 0$ so that $\|b\|_\infty \le A$,

$$|(\mathcal{S}^b p_0)_z^{*,n}(s,z,y)| \le (C_{12}A)^n f_0(t,x,y),$$

where C_{12} is the constant in Lemma 3.3. Noting that the constant A_0 defined in Lemma 3.4 satisfies that $A_0 \le 1/2C_{12}$. So for every bounded function b with $\|b\|_\infty \le A_0$, we have

$$|(\mathcal{S}^b p_0)_z^{*,n}(s,z,y)| \le 2^{-n} f_0(s,z,y), \quad s \in (0,1). \tag{3.38}$$

Then by the scale change formulas (3.18) and (3.19), when $\|b\|_\infty > A_0$,

$$|(\mathcal{S}^b p_0)_z^{*,n}(s,z,y)| \le 2^{-n} f_0(s,z,y), \quad s \in (0,(A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}).$$

By the condition (3.35), there is a constant $c_1 > 0$ so that for every $n \ge 1$,

$$\left| \int_0^t \int_{\mathbb{R}^d} \overline{q}(t-s,x,z)(\mathcal{S}^b p_0)_z^{*,n}(s,z,y)dsdz \right| \le c_1 2^{-n} \int_0^t \int_{\mathbb{R}^d} p_1(t-s,x,z) f_0(s,z,y)dsdz.$$

Noting that $p_1(t,x,y) \asymp g_1(t,x,y)$ on $(0,1] \times \mathbb{R}^d \times \mathbb{R}^d$ and

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^d} f_0(s,z,y)\,dz\,ds &\le \int_0^t \int_{|y-z| \le s^{1/\alpha}} s^{-(d+\beta)/\alpha}\,dz\,ds + \int_0^t \int_{|y-z|>s^{1/\alpha}} \frac{1}{|y-z|^{d+\beta}}\,dz\,ds \\ &= c_2 t^{1-\beta/\alpha}. \end{aligned} \tag{3.39}$$

Then by the similar proof in Lemma 2.5, we can get

$$\int_0^t \int_{\mathbb{R}^d} p_1(t-s,x,z) f_0(s,z,y)dsdz \le c_3 p_1(t,x,y).$$

It follows that

$$\overline{q}(t,x,y) = \sum_{n=0}^\infty q_n^b(t,x,y) = q^b(t,x,y)$$

for every $t \in (0,\varepsilon]$ and $x,y \in \mathbb{R}^d$. Since both $\overline{q}$ and q^b satisfy the Chapman-Kolmogorov equation (3.31), $\overline{q} = q^b$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$. □

Remark 3.11. It follows from the definition of $q^b_n(t,x,y)$ and Lemma 3.3 that $(\mathcal{S}^b p_0)^{*,n+1}(s,z,y) = \mathcal{S}^b_z q^b_n(s,z,y)$. □

In view of Lemma 3.5 and Chapman-Kolmogorov equation, we have

Theorem 3.12. *Suppose that b is a bounded function on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying* (1.2)*. $q^b(t,x,y) = \lambda^{d/\alpha} q^{b^{(\lambda)}}(\lambda t, \lambda^{1/\alpha}x, \lambda^{1/\alpha}y)$ on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$, where $b^{(\lambda)}(x,z) := \lambda^{\beta/\alpha-1} b(\lambda^{-1/\alpha}x, \lambda^{-1/\alpha}z)$.*

For a bounded function f on $\mathbb{R}^d$, $t>0$ and $x\in\mathbb{R}^d$, we define

$$T^b_t f(x) = \int_{\mathbb{R}^d} q^b(t,x,y) f(y)\,dy \quad \text{and} \quad P_t f(x) = \int_{\mathbb{R}^d} p_0(t,x,y)f(y)dy.$$

The following lemma follows immediately from (3.31) and (3.33).

Lemma 3.13. *Suppose that b is a bounded function on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying* (1.2)*. For all $s,t>0$, we have $T^b_{t+s} = T^b_t T^b_s$.*

Theorem 3.14. *Let b be a bounded function on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying* (1.2)*. Then for every $f\in C^2_b(\mathbb{R}^d)$,*

$$T^b_t f(x) - f(x) = \int_0^t T^b_s \mathcal{L}^b f(x) ds \qquad \text{for every } t>0,\ x\in\mathbb{R}^d.$$

Proof. Note that by Theorem 3.9, for $f\in C^2_b(\mathbb{R}^d)$,

$$T^b_t f(x) = P_t f(x) + \int_0^t T^b_{t-s}\mathcal{S}^b P_s f(x) ds = P_t f(x) + \int_0^t T^b_s \mathcal{S}^b P_{t-s} f(x) ds. \tag{3.40}$$

Hence

$$\begin{aligned}
& T^b_t f(x) - f(x) \\
= \;& P_t f(x) - f(x) + \int_0^t T^b_s\mathcal{S}^b f(x)ds + \int_0^t T^b_s \mathcal{S}^b(P_{t-s}f - f)(x)ds \\
= \;& \int_0^t P_s \Delta^{\alpha/2} f(x) ds + \int_0^t T^b_s\mathcal{S}^b f(x)ds + \int_0^t T^b_s \mathcal{S}^b(P_{t-s}f - f)(x)ds \\
= \;& \int_0^t T^b_s \Delta^{\alpha/2} f(x) ds - \int_0^t \left(\int_0^s T^b_r \mathcal{S}^b P_{s-r}(\Delta^{\alpha/2} f)(x) dr \right) ds \\
& + \int_0^t T^b_s\mathcal{S}^b f(x)ds + \int_0^t T^b_s \mathcal{S}^b(P_{t-s}f - f)(x)ds \\
= \;& \int_0^t T^b_s \left(\Delta^{\alpha/2} + \mathcal{S}^b\right) f(x) ds - \int_0^t \left(\int_r^t T^b_r \mathcal{S}^b P_{s-r}(\Delta^{\alpha/2} f)(x) ds \right) dr \\
& + \int_0^t T^b_s \mathcal{S}^b(P_{t-s}f - f)(x)ds \\
= \;& \int_0^t T^b_s \mathcal{L}^b f(x) ds - \int_0^t T^b_r \mathcal{S}^b(P_{t-r}f - f)(x)dr + \int_0^t T^b_s \mathcal{S}^b(P_{t-s}f - f)(x)ds \\
= \;& \int_0^t T^b_s \mathcal{L}^b f(x) ds.
\end{aligned}$$

Here in the third inequality, we used (3.40); while in the fifth inequality we used Lemma 2.2 and (3.34), which allow the interchange of the integral sign $\int_r^t$ with $T_r^b \mathcal{S}^b$, and the fact that

$$\int_r^t P_{s-r}(\Delta^{\alpha/2} f)(x) ds = \int_r^t \left(\frac{d}{ds} P_{s-r} f(x) \right) ds = P_{t-r} f(x) - f(x).$$

□

Theorem 3.15. *Let b be a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying* (1.2)*. Then $q^b(t,x,y)$ is jointly continuous in $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and $\int_{\mathbb{R}^d} q^b(t,x,y)\, dy = 1$ for every $x \in \mathbb{R}^d$ and $t > 0$.*

Proof. By Lemma 3.13, we have

$$q^b(t+s,x,y) = \int_{\mathbb{R}^d} q^b(t,x,z) q^b(s,z,y)\, dz, \quad x,y \in \mathbb{R}^d, s,t > 0. \tag{3.41}$$

Continuity of $q^b(t,x,y)$ in $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ follows from Theorem 3.7, (3.41) and the dominated convergence theorem. For $n \geq 1$ and $t \in (0,T]$, it follows from (3.1), Lemma 2.5, Theorem 3.6 and Fubini's Theorem that for every $t \in (0, 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}]$,

$$\begin{aligned} \int_{\mathbb{R}^d} q_n^b(t,x,y)\, dy &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^t q_{n-1}^b(t-s,x,z) \mathcal{S}_z^b p_0(s,z,y)\, ds\, dz\, dy \\ &= \int_{\mathbb{R}^d} \int_0^t q_{n-1}^b(t-s,x,z) \mathcal{S}_z^b \left(\int_{\mathbb{R}^d} p_0(s,z,y)\, dy \right) ds\, dz = 0. \end{aligned}$$

Hence we have by Lemma 3.4,

$$\int_{\mathbb{R}^d} q^b(t,x,y)\, dy = \int_{\mathbb{R}^d} p_0(t,x,y)\, dy = 1$$

for $t \in (0, 1 \wedge (A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}]$. This conservativeness property extends to all $t > 0$ by (3.41). □

Theorem 1.1 now follows from (2.3)-(2.4), Theorems 3.7, 3.9, 3.10, 3.14 and 3.15.

4 C_∞-Semigroups and Positivity

Recall that A_0 is the positive constant in Lemma 3.4.

Lemma 4.1. *Suppose that b is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying condition* (1.2)*. Then $\{T_t^b, t > 0\}$ is a strongly continuous semigroup in $C_\infty(\mathbb{R}^d)$.*

Proof. The following proof is a minor modification of that for [9, Proposition 2.3]. For reader's convenience, we spell out the details. Since $q^b(t,x,y)$ is continuous by Theorem 3.15, it follows that T_t^b maps bounded continuous functions to continuous function for every $t > 0$. Moreover, by (3.34) and the semigroup of $q^b(t,x,y)$, there are constants c_1 and c_2 so that

$$|q^b(t,x,y)| \leq c_1 e^{c_2 t} p_{M_{b,\lambda}}(t,x,y) \qquad \text{for every } t > 0 \text{ and } x,y \in \mathbb{R}^d. \tag{4.1}$$

Thus, for every $f \in C_\infty(\mathbb{R}^d)$ and $t > 0$,

$$\lim_{x\to\infty} |T_t^b f(x)| \le \lim_{x\to\infty} c_1 e^{c_2 t} \int_{\mathbb{R}^d} p_{M_{b,\lambda}}(t,x,y)|f(y)|\,dy = 0$$

and so $T_t^b f \in C_\infty(\mathbb{R}^d)$. On the other hand, given $f \in C_\infty(\mathbb{R}^d)$, for every $\varepsilon > 0$, there is $\delta > 0$ so that $|f(x) - f(y)| \le \varepsilon$ whenever $|x - y| \le \delta$. Since

$$\lim_{t_0\to 0} \sup_{t\le t_0} \sup_{x\in\mathbb{R}^d} \int_{|y-x|\ge\delta} |q^b(t,x,y)|\,dy \le \lim_{t_0\to 0} c e^{c_2 t_0} \sup_{t\le t_0} \sup_{x\in\mathbb{R}^d} \int_{|y-x|\ge\delta} p_{M_{b,\lambda}}(t,x,y)\,dy = 0,$$

we have

$$\begin{aligned}
& \lim_{t\to 0} \sup_{x\in\mathbb{R}^d} |T_t^b f(x) - f(x)| \\
= \; & \lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \left| \int_{\mathbb{R}^d} q^b(t,x,y)(f(y)-f(x))\,dy \right| \\
\le \; & \lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \int_{|y-x|<\delta} |q^b(t,x,y)|\,|f(y)-f(x)|\,dy + 2\|f\|_\infty \lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \int_{|y-x|\ge\delta} |q^b(t,x,y)|\,dy \\
\le \; & \varepsilon \lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} c_1 e^{c_2 t} p_1(t,x,y)dy = c_1\varepsilon.
\end{aligned}$$

Thus $\lim_{t\to 0} \sup_{x\in\mathbb{R}^d} |T_t^b f(x) - f(x)| = 0$. This proves that T_t^b is a strongly continuous semigroup in $C_\infty(\mathbb{R}^d)$. □

Lemma 4.2. *Let b be a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.18). For each $f \in C_\infty^2(\mathbb{R}^d)$, $\mathcal{L}^b f(x)$ exists pointwise and is in $C_\infty(\mathbb{R}^d)$.*

Proof. Suppose that $\gamma \in (0,2)$ and $f \in C_\infty^2(\mathbb{R}^d)$. Denote $\sum_{i,j=1}^d |\partial_{ij}^2 f(x)|$ by $|D^2 f(x)|$. Let $R > 1$ to be chosen later. Then for each $x \in \mathbb{R}^d$, we have by Taylor expansion,

$$\begin{aligned}
\Phi_f(x) \; := \; & \int_{\mathbb{R}^d} \left| f(x+z) - f(x) - \nabla f(x)\cdot z \mathbb{1}_{\{|z|\le 1\}} \right| \frac{1}{|z|^{d+\gamma}} dz \\
\le \; & \int_{|z|\le 1} \left| f(x+z) - f(x) - \nabla f(x)\cdot z \mathbb{1}_{\{|z|\le 1\}} \right| \frac{1}{|z|^{d+\gamma}} dz \\
& + \int_{1<|z|\le R} |f(x+z) - f(x)| \frac{1}{|z|^{d+\gamma}} dz + \int_{|z|>R} |f(x+z) - f(x)| \frac{1}{|z|^{d+\gamma}} dz \\
\le \; & \sup_{|y|\le 1} |D^2 f(x+y)| \int_{|z|\le 1} |z|^{2-d-\gamma} dz + \int_{1<|z|\le R} |f(x+z) - f(x)| \frac{1}{|z|^{d+\gamma}} dz \\
& + 2\|f\|_\infty \int_{|z|>R} |z|^{-d-\gamma} dz \\
= \; & c R^{2-\gamma} \sup_{|y|\le 1} |D^2 f(x+y)| + \int_{1<|z|\le R} |f(x+z) - f(x)| \frac{1}{|z|^{d+\gamma}} dz + c R^{-\gamma}\|f\|_\infty.
\end{aligned}$$

For any given $\varepsilon > 0$, we can take R large so that $cR^{-\gamma}\|f\|_\infty < \varepsilon/2$ to conclude that

$$\lim_{|x|\to\infty} \int_{\mathbb{R}^d} \left| f(x+z) - f(x) - \nabla f(x)\cdot z \mathbb{1}_{\{|z|\le 1\}} \right| \frac{1}{|z|^{d+\gamma}} dz = 0. \tag{4.2}$$

By the same reason, applying the above argument to function $x \mapsto f(x+y) - f(x)$ in place of f yields that for every $\varepsilon > 0$ and $x_0 \in \mathbb{R}^d$, there is $\delta > 0$ so that

$$\Phi_{f(\cdot+y)-f}(x_0) < \varepsilon \quad \text{for every } |y| < \delta. \tag{4.3}$$

It follows from the last two displays, the definition of $\mathcal{L}^b$ and (1.4) that $\mathcal{L}^b f(x)$ exists for every $x \in \mathbb{R}^d$ and $\mathcal{L}^b f \in C_\infty(\mathbb{R}^d)$. $\square$

Proof of Theorem 1.2. Since b satisfies condition (1.18), it is easy to verify that $\mathcal{L}^b f \in C_\infty(\mathbb{R}^d)$ for every $f \in C_c^2(\mathbb{R}^d)$. Let $\widehat{\mathcal{L}}^b$ denote the infinitesimal generator of the strongly continuous semigroup $\{T_t^b; t \geq 0\}$ in $C_\infty(\mathbb{R}^d)$, which is a closed linear operator. It follows from Theorem 3.14, Lemmas 4.1 and 4.2 that for every $f \in C_\infty^2(\mathbb{R}^d)$, $(T_t^b f(x) - f(x))/t$ converges uniformly to $\mathcal{L}^b f(x)$ as $t \to 0$. So

$$C_\infty^2(\mathbb{R}^d) \subset D(\widehat{\mathcal{L}}^b) \quad \text{ and } \quad \widehat{\mathcal{L}}^b f = \mathcal{L}^b f \quad \text{for } f \in C_\infty^2(\mathbb{R}^d). \tag{4.4}$$

In view of Theorem 3.7, there are constants $c_1, c_2 > 0$ so that (4.1) holds. This implies that

$$\sup_{x \in \mathbb{R}^d} \int_0^\infty e^{-\lambda t} |T_t^b f|(x) dt \leq c_\lambda \|f\|_\infty, \quad f \in C_\infty(\mathbb{R}^d),$$

for every $\lambda > c_2$. Observe that $e^{-c_2 t} T_t^b$ is a strongly continuous semigroup in $C_\infty(\mathbb{R}^d)$ whose infinitesimal generator is $\widehat{\mathcal{L}}^b - c_2$. The above display implies that $(0, \infty)$ is contained in the residual set $\rho(\widehat{\mathcal{L}}^b - c_2)$ of $\widehat{\mathcal{L}}^b - c_2$. Therefore by Theorem 3.15 and the Hille-Yosida-Ray theorem [16, p165], $\{e^{-c_2 t} T_t^b; t \geq 0\}$ is a positive preserving semigroup on $C_\infty(\mathbb{R}^d)$ if and only if $\widehat{\mathcal{L}}^b - c_2$ satisfies the positive maximum principle. On the other hand, Courrége's first theorem (see [1, p158]) tells us that $\widehat{\mathcal{L}}^b - c_2$ satisfies the positive maximum principle if and only if for each $x \in \mathbb{R}^d$,

$$\frac{\mathcal{A}(d, -\alpha)}{|z|^{d+\alpha}} + \frac{\mathcal{A}(d, -\beta) b(x, z)}{|z|^{d+\beta}} \geq 0 \quad \text{for a.e. } z \in \mathbb{R}^d.$$

Since $e^{-c_2 t} T_t^b$ has a continuous integral kernel $e^{-c_2 t} q^b(t, x, y)$, it follows that $q^b(t, x, y) \geq 0$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ if and only if for each $x \in \mathbb{R}^d$, (1.19) holds. If $b(x, z) = b(x)$ is a function of x only, then by taking $|z| \to \infty$, one concludes that (1.19) holds if and only if $b(x) \geq 0$ on $\mathbb{R}^d$. $\square$

5 Feller process and heat kernel estimates

Throughout this section, b is a bounded function satisfying condition (1.2) and (1.19). We will show that $q^b(t, x, y) > 0$ and so it generates a Feller process X^b that has strong Feller property. We further derive the upper and lower bound estimates on $q^b(t, x, y)$. We will first establish the Feller process X^b and its connection to the martingale problem for $(\mathcal{L}^b, \mathcal{S}(\mathbb{R}^d))$ under an additional assumption (1.18). We will then remove this additional assumption using an approximation method and the uniqueness result on $q^b(t, x, y)$ from Theorem 3.10.

Suppose that b is a bounded function satisfying conditions (1.2), (1.18) and (1.19). Then it follows from Theorem 1.2, Theorem 3.15, Lemma 4.1 and Theorem 3.8, T^b is a Feller semigroup. So it uniquely determines a conservative Feller process $X^b = \{X_t^b, t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^d\}$ having $q^b(t, x, y)$ as its transition density function. Since, by Theorem 3.10, $q^b(t, x, y)$ is continuous and $q^b(t, x, y) \leq c_1 e^{c_2 t} p_{M_{b,\lambda}}(t, x, y)$ for some positive constants c_1 and c_2, X^b enjoys the strong Feller property.

Proposition 5.1. *Suppose that b is a bounded function satisfying conditions* (1.2), (1.18) *and* (1.19). *For each $x \in \mathbb{R}^d$ and $f \in C_b^2(\mathbb{R}^d)$,*

$$M_t^f := f(X_t^b) - f(X_0^b) - \int_0^t \mathcal{L}^b f(X_s^b)\, ds$$

is a martingale under $\mathbb{P}_x$. So in particular, the Feller process $(X^b, \mathbb{P}_x, x \in \mathbb{R}^d)$ solves the martingale problem for $(\mathcal{L}^b, C_\infty^2(\mathbb{R}^d))$.

Proof. This follows immediately from Theorem 3.14 and the Markov property of X^b. □

We next determine the Lévy system of X^b. Recall that

$$J^b(x,y) = \frac{\mathcal{A}(d,-\alpha)}{|x-y|^{d+\alpha}} + \frac{\mathcal{A}(d,-\beta)\, b(x,y-x)}{|x-y|^{d+\beta}}. \tag{5.1}$$

Proposition 5.2. *Suppose that b is a bounded function satisfying conditions* (1.2), (1.18) *and* (1.19). *Assume that A and B are disjoint compact sets in $\mathbb{R}^d$. Then*

$$\sum_{s\le t} \mathbb{1}_{\{X_{s-}^b \in A, X_s^b \in B\}} - \int_0^t \mathbb{1}_A(X_s^b) \int_B J^b(X_s^b, y) dy\, ds$$

is a $\mathbb{P}_x$-martingale for each $x \in \mathbb{R}^d$.

Proof. The proof is similar to that for [9, Theorem 2.6]. For reader's convenience, we give the details here. Let $f \in C^\infty(\mathbb{R}^d)$ with $f = 0$ in an open neighborhood of A and $f = 1$ in an open neighborhood of B. Define

$$M_t^f := f(X_t^b) - f(X_0^b) - \int_0^t \mathcal{L}^b f(X_s^b)\, ds.$$

Then M_t^f is a martingale under $\mathbb{P}_x$ by Proposition 5.1, and so is $N_t^f := \int_0^t \mathbb{1}_A(X_{s-}^b) dM_s^f$. Proposition 5.1 in particular implies that $X_t^b = (X_t^{b,1}, \dots, X_t^{b,d})$ is a semi-martingale. So by Ito's formula, we have that,

$$f(X_t^b) - f(X_0^b) = \sum_{i=1}^d \int_0^t \partial_i f(X_{s-}^b) dX_s^{b,i} + \sum_{s\le t} \eta_s(f) + \frac{1}{2} A_t(f), \tag{5.2}$$

where

$$\eta_s(f) = f(X_s^b) - f(X_{s-}^b) - \sum_{i=1}^d \partial_i f(X_{s-}^b)(X_s^{b,i} - X_{s-}^{b,i}) \tag{5.3}$$

and

$$A_t(f) = \sum_{i,j=1}^d \int_0^t \partial_i \partial_j f(X_{s-}^b) d\langle (X^{b,i})^c, (X^{b,j})^c \rangle_s. \tag{5.4}$$

Since f vanishes in an open neighborhood of A, we have by (5.2)–(5.4), (1.1) and (1.3) that

$$\begin{aligned} N_t^f &= \sum_{s\le t} \mathbf{1}_A(X_{s-}^b) f(X_s^b) - \int_0^t \mathbf{1}_A(X_s^b) \left(\mathcal{L}^b f(X_s^b) \right) ds \\ &= \sum_{s\le t} \mathbf{1}_A(X_{s-}^b) f(X_s^b) - \int_0^t \mathbf{1}_A(X_s^b) \int_{\mathbb{R}^d} f(y) J^b(X_s^b, y) dy ds. \end{aligned}$$

By taking a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^d)$ with $f_n = 0$ on A, $f_n = 1$ on B and $f_n \downarrow \mathbf{1}_B$, we get that, for any $x \in \mathbb{R}^d$,

$$\sum_{s \le t} \mathbf{1}_A(X^b_{s-}) \mathbf{1}_B(X^b_s) - \int_0^t \mathbf{1}_A(X^b_s) \int_B J^b(X^b_s, y) dy ds$$

is a $\mathbb{P}_x$-martingale for every $x \in \mathbb{R}^d$. □

Proposition 5.2 implies that

$$\mathbb{E}_x \left[\sum_{s \le t} \mathbf{1}_A(X^b_{s-}) \mathbf{1}_B(X^b_s) \right] = \mathbb{E}_x \left[\int_0^t \int_{\mathbb{R}^d} \mathbf{1}_A(X^b_s) \mathbf{1}_B(y) J^b(X^b_s, y) dy ds \right].$$

Using this and a routine measure theoretic argument, we get

$$\mathbb{E}_x \left[\sum_{s \le t} f(s, X^b_{s-}, X^b_s) \right] = \mathbb{E}_x \left[\int_0^t \int_{\mathbb{R}^d} f(s, X^b_s, y) J^b(X^b_s, y) dy ds \right]$$

for any non-negative measurable function f on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ vanishing on $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\}$. Finally, following the same arguments as in [11, Lemma 4.7] and [12, Appendix A], we get

Proposition 5.3. *Suppose that b is a bounded function satisfying conditions* (1.2), (1.18) *and* (1.19). *Let f be a nonnegative function on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal. Then for stopping time T with respect to the minimal admissible filtration generated by X^b,*

$$\mathbb{E}_x \left[\sum_{s \le T} f(s, X^b_{s-}, X^b_s) \right] = \mathbb{E}_x \left[\int_0^T \int_{\mathbb{R}^d} f(s, X^b_s, u) J^b(X^b_s, u)\, du\, ds \right].$$

To remove the assumption (1.18) on b, we approximate a general measurable function $b(x, z)$ by continuous $k_n(x, z)$. To show that $q^{k_n}(t, x, y)$ converges to $q^b(t, x, y)$, we establish equi-continuity of $q^b(t, x, y)$ and apply the uniqueness result, Theorem 3.10.

Proposition 5.4. *For each $0 < t_0 < T < \infty$ and $A > 0$, the function $q^b(t, x, y)$ is uniform continuous in $(t, x) \in (t_0, T) \times \mathbb{R}^d$ for every b with $\|b\|_\infty \le A$ that satisfies* (1.2) *and for all $y \in \mathbb{R}^d$.*

Proof. In view of Theorem 3.12, it suffices to prove the theorem for $A = A_0$, where A_0 is the constant in Lemma 3.4 (or in Theorem 1.1). Using the Chapman-Kolmogorov equation for $q^b(t, x, y)$ (see Lemma 3.13) and (3.34), it suffices to prove the Proposition for $T = 1$.

Noting that q^b_n can also be rewritten in the following form:

$$q^b_n(t, x, y) = \int_0^t \int_{\mathbb{R}^d} p_0(t - r, x, z)(\mathcal{S}^b p_0)^{*,n}_z(r, z, y)\, dz\, dr.$$

Here $(\mathcal{S}^b p_0)^{*,n}_z(r,z,y)$ is defined in (3.37). Hence, for $T > t > s > t_0, x_1, x_2 \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$, we have

$$
\begin{aligned}
&|q^b_n(s,x_1,y) - q^b_n(t,x_2,y)| \\
\le& \int_0^s \int_{\mathbb{R}^d} |p_0(s-r,x_1,z) - p_0(t-r,x_2,z)||(\mathcal{S}^b p_0)^{*,n}_z(r,z,y)|\, dz\, dr \\
&+ \int_s^t \int_{\mathbb{R}^d} p_0(t-r,x_2,z)|(\mathcal{S}^b p_0)^{*,n}_z(r,z,y)|\, dz\, dr \\
=:& I + II.
\end{aligned}
$$

It is known (see [11]) that there are positive constants c_1 and θ so that for any $t, s \in [t_0, T]$ and $x_i \in \mathbb{R}^d$ with $i = 1, 2$,

$$
|p_0(s,x_1,y) - p_0(t,x_2,y)| \le c_1\, t_0^{-(d+\theta)/\alpha} \left(|t-s|^{1/\alpha} + |x_1 - x_2|\right)^\theta, \quad y \in \mathbb{R}^d,
$$

we have by (2.5), (3.38) and (3.39), for $\rho \in (0, s/2)$,

$$
\begin{aligned}
I =& \int_0^{s-\rho} \int_{\mathbb{R}^d} |p_0(s-r,x_1,z) - p_0(t-r,x_2,z)||(\mathcal{S}^b p_0)^{*,n}_z(r,z,y)|\, dz\, dr \\
&+ \int_{s-\rho}^s \int_{\mathbb{R}^d} |p_0(s-r,x_1,z) - p_0(t-r,x_2,z)||(\mathcal{S}^b p_0)^{*,n}_z(r,z,y)|\, dz\, dr \\
\le& c_2 2^{-(n-1)} \rho^{-(d+\theta)/\alpha} \left(|t-s|^{1/\alpha} + |x_1 - x_2|\right)^\theta \int_0^{s-\rho} \int_{\mathbb{R}^d} f_0(r,z,y)\, dz\, dr \\
&+ c_2 2^{-(n-1)} \int_{s-\rho}^s \int_{\mathbb{R}^d} (p_0(s-r,x_1,z) + p_0(t-r,x_2,z)) f_0(r,z,y)\, dz\, dr \\
\le& c_3 2^{-(n-1)} \rho^{-(d+\theta)/\alpha} \left(|t-s|^{1/\alpha} + |x_1 - x_2|\right)^\theta s^{1-\beta/\alpha} + c_3 2^{-(n-1)} (s-\rho)^{-(d+\beta)/\alpha} \rho.
\end{aligned}
\tag{5.5}
$$

Moreover, by (2.5) and (3.38),

$$
II \le 2^{-(n-1)} \int_s^t \int_{\mathbb{R}^d} p_0(t-r,x_2,z) f_0(r,z,y)\, dz\, dr \le 2^{-(n-1)} s^{-(d+\beta)/\alpha} |t-s|. \tag{5.6}
$$

Therefore, noting that

$$
|q^b(s,x_1,y) - q^b(t,x_2,y)| \le |p_0(s,x_1,y) - p_0(t,x_2,y)| + \sum_{n=1}^\infty |q^b_n(s,x_1,y) - q^b_n(t,x_2,y)|,
$$

then first taking $|t-s|$ and $|x_1 - x_2|$ small, and then making ρ small in (5.5) and (5.6) yields the conclusion of this Proposition. □

Proposition 5.5. *For each $0 < t_0 < T < \infty$ and $A > 0$, the function $q^b(t,x,y)$ is uniform continuous in y for every b with $\|b\|_\infty \le A$ that satisfies (1.2) and for all $(t,x) \in (t_0,T) \times \mathbb{R}^d$.*

Proof. In view of Theorem 3.12, it suffices to prove the theorem for $A = A_0$, where A_0 is the constant in Lemma 3.4 (or in Theorem 1.1). Using the Chapman-Kolmogorov equation for $q^b(t,x,y)$ (see Lemma 3.13) and (3.34), it suffices to prove the Proposition for $T = 1$.

Define $P(s,x,y) = p_0(s,x) - p_0(s,y)$. For $s>0$, we have

$$
\begin{aligned}
&|\mathcal{S}^b p_0(s,y_1) - \mathcal{S}^b p_0(s,y_2)|\\
\le& c_1 \int_{\mathbb{R}^d} |P(s,y_1+h,y_2+h) - P(s,y_1,y_2) - \langle \nabla_{(y_1,y_2)} P(s,y_1,y_2), h 1\!\!1_{|h|\le 1}\rangle| \frac{dh}{|h|^{d+\beta}}\\
\le& c_1 \int_{|h|\le 1} |h|^2 \sup_{\theta\in(0,1)} |\frac{\partial^2}{\partial y_1^2} p_0(s,y_1+\theta h) - \frac{\partial^2}{\partial y_2^2} p_0(s,y_2+\theta h)| \frac{dh}{|h|^{d+\beta}}\\
&+ c_1 \int_{|h|>1} |p_0(s,y_1+h) - p_0(s,y_2+h) - p_0(s,y_1) + p_0(s,y_2)| \frac{dh}{|h|^{d+\beta}}\\
\le& c_2 \sup_y |\frac{\partial^3}{\partial y^3} p_0(s,y)||y_1-y_2| \int_{|h|\le 1} |h|^2 \frac{dh}{|h|^{d+\beta}} + c_2 \sup_y |\frac{\partial}{\partial y} p_0(s,y)||y_1-y_2| \int_{|h|>1} \frac{dh}{|h|^{d+\beta}}\\
\le& c_3 |y_1-y_2| [s^{-(d+3)/\alpha} + s^{-(d+1)/\alpha}],
\end{aligned}
\tag{5.7}
$$

where in the fourth inequality, $|\frac{\partial^3}{\partial y^3} p_0(s,y)| \le c_3 s^{-(d+3)/\alpha}$ can be proved similarly by the argument in Lemma 2.1. Take $\rho \in (0, t_0/2)$. Then for each $n\ge 1$, we have by (1.8), (3.39), Lemma 2.4, Lemma 3.4 and (5.7), that for $(t,x,y) \in (t_0,1)\times\mathbb{R}^d\times\mathbb{R}^d$,

$$
\begin{aligned}
&|q^b_n(t,x,y_1) - q^b_n(t,x,y_2)|\\
\le\ & \int_0^\rho \int_{\mathbb{R}^d} q^b_{n-1}(t-s,x,z) |\mathcal{S}^b_z p_0(s,z,y_1) - \mathcal{S}^b_z p_0(s,z,y_2)|\, dz\, ds\\
&+ \int_\rho^t \int_{\mathbb{R}^d} q^b_{n-1}(t-s,x,z) |\mathcal{S}^b_z p_0(s,z,y_1) - \mathcal{S}^b_z p_0(s,z,y_2)|\, dz\, ds\\
\le\ & c_4 2^{-(n-1)} \int_0^\rho \int_{\mathbb{R}^d} p_1(t-s,x,z) |\mathcal{S}^b_z p_0(s,z,y_1) - \mathcal{S}^b_z p_0(s,z,y_2)|\, dz\, ds\\
&+ c_4 2^{-(n-1)} \int_\rho^t \int_{\mathbb{R}^d} p_1(t-s,x,z) \left|\mathcal{S}^b_z p_0(s,z-y_1) - \mathcal{S}^b_z p_0(s,z-y_2)\right| dz\, ds\\
\le\ & c_5 2^{-(n-1)} t_0^{-d/\alpha} \int_0^\rho \int_{\mathbb{R}^d} \Big(|\mathcal{S}^b_z p_0(s,z,y_1)| + |\mathcal{S}^b_z p_0(s,z,y_2)|\Big)\, dz\, ds\\
&+ c_5 2^{-(n-1)} \rho^{-(d+3)/\alpha} |y_1-y_2| \int_\rho^t \int_{\mathbb{R}^d} p_1(t-s,x,z)\, dz\, ds\\
\le\ & c_6\, 2^{-(n-1)}\, t_0^{-d/\alpha} \rho^{1-\beta/\alpha} + c_6 2^{-(n-1)} \rho^{-(d+3)/\alpha} |y_1-y_2|.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&|q^b(t,x,y_1) - q^b(t,x,y_2)|\\
\le& |p_0(t,x,y_1) - p_0(t,x,y_2)| + \sum_{n=1}^\infty c_6 2^{-(n-1)}\, t_0^{-d/\alpha} \rho^{1-\beta/\alpha} + \sum_{n=1}^\infty c_6 2^{-(n-1)} \rho^{-(d+3)/\alpha} |y_1-y_2|.
\end{aligned}
$$

By first taking $|y_1-y_2|$ small and then making ρ small yields the desired uniform continuity of $q^b(t,x,y)$. $\square$

Theorem 5.6. *Suppose b is a bounded function on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying* (1.2) *and* (1.19). *The kernel $q^b(t,x,y)$ uniquely determines a Feller process $X^b = (X^b_t, t\ge 0, \mathbb{P}_x, x\in\mathbb{R}^d)$ on the canonical Skorokhod space $\mathbb{D}([0,\infty),\mathbb{R}^d)$ such that*

$$
\mathbb{E}_x \left[f(X^b_t) \right] = \int_{\mathbb{R}^d} q^b(t,x,y) f(y) dy
$$

for every bounded continuous function f on $\mathbb{R}^d$. The Feller process X^b is conservative and has a Lévy system $(J^b(x,y)dy,t)$, where

$$J^b(x,y)=\frac{\mathcal{A}(d,-\alpha)}{|x-y|^{d+\alpha}}+\frac{\mathcal{A}(d,-\beta)\,b(x,y-x)}{|x-y|^{d+\beta}}.$$

Moreover, for each $x\in\mathbb{R}^d$, $(X^b,\mathbb{P}_x)$ is the unique solution to the martingale problem $(\mathcal{L}^b,\mathcal{S}(\mathbb{R}^d))$ with initial value x. Here $\mathcal{S}(\mathbb{R}^d)$ denotes the space of tempered functions on $\mathbb{R}^d$.

Proof. When b is a bounded function satisfying (1.2), (1.18) and (1.19), the theorem has already been established via Propositions 5.1-5.3. We now remove the assumption (1.18). Suppose that $b(x,z)$ is a bounded function that satisfies (1.2) and (1.19). Let φ be a non-negative smooth function with compact support in $\mathbb{R}^d$ so that $\int_{\mathbb{R}^d}\varphi(x)dx=1$. For each $n\geq 1$, define $\varphi_n(x)=n^d\varphi(nx)$ and

$$k_n(x,z):=\int_{\mathbb{R}^d}\varphi_n(x-y)b(y,z)dy.$$

Then k_n is a function that satisfies (1.2), (1.18) and (1.19) with $\|k_n\|_\infty\leq\|b\|_\infty$. By Theorem 1.1, Proposition 5.4 and Proposition 5.5, $q^{k_n}(t,x,y)$ is uniformly bounded and equi-continuous on $[1/M,M]\times\mathbb{R}^d\times\mathbb{R}^d$ for each $M\geq 1$, then there is a subsequence $\{n_j\}$ of $\{n\}$ so that $q^{k_{n_j}}(t,x,y)$ converges boundedly and uniformly on compacts of $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$, to some continuous function $\overline{q}(t,x,y)$, which again satisfies (1.16). Obviously, $\overline{q}(t,x,y)$ also satisfies the Chapman-Kolmogorov equation and $\int_{\mathbb{R}^d}\overline{q}(t,x,y)\,dy=1$. By (3.29) and Theorem 3.7,

$$q^{k_{n_j}}(t,x,y)=p_0(t,x,y)+\int_0^t\int_{\mathbb{R}^d}q^{k_{n_j}}(t-s,x,z)\mathcal{S}_z^{k_{n_j}}p_0(s,z,y)dzds$$

and

$$q^{k_{n_j}}(t,x,y)\leq c\,g_{M_{b,\lambda}}(t,x,y)$$

for every $0<t\leq 1\wedge(A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ and $x,y\in\mathbb{R}^d$, where c is a positive constant that depends only on d, α,β and $\|b\|_\infty$. Letting $j\to\infty$, we have by (3.1), Lemma 2.5 and the dominated convergence theorem that

$$\overline{q}(t,x,y)=p_0(t,x,y)+\int_0^t\int_{\mathbb{R}^d}\overline{q}(t-s,x,z)\mathcal{S}_z^b p_0(s,z,y)dyds$$

and $\overline{q}(t,x,y)\leq c\,g_{M_{b,\lambda}}(t,x,y)$ for every $0<t\leq 1\wedge(A_0/\|b\|_\infty)^{\alpha/(\alpha-\beta)}$ and $x,y\in\mathbb{R}^d$. Hence we conclude from Theorem 3.10 that $\overline{q}(t,x,y)=q^b(t,x,y)$. This in particular implies that $q^b(t,x,y)\geq 0$. So there is a Feller process X^b having $q^b(t,x,y)$ as its transition density function. The proof of Propositions 5.1-5.3 only uses the condition (1.18) through its implication that $q^b(t,x,y)\geq 0$. So in view of what we just established, Propositions 5.1-5.3 continue to hold for X^b under the current setting without the additional assumption (1.18). The non-local operator $\mathcal{L}^b$ satisfies the assumptions $[A_1]$ and $[A_2]$ of [19]. So by [19, Theorem 3], solution to the martingale problem $(\mathcal{L}^b,\mathcal{S}(\mathbb{R}^d))$ is unique. Since $\mathcal{S}(\mathbb{R}^d)\subset C^2_\infty(\mathbb{R}^d)$, the proof of the theorem is now complete. $\square$

For each $\lambda>0$, define

$$\widehat{b}_\lambda(x,z)=b(x,z)1_{\{|z|\leq\lambda\}}(z)+b^+(x,z)1_{\{|z|>\lambda\}}(z). \tag{5.8}$$

In the following, we use a method of Meyer [20] to construct from X^b, by adding suitable jumps, a strong Markov process Y corresponding to the jumping kernel $J^{\widehat{b}_\lambda}$ defined by (1.25) but with $\widehat{b}_\lambda$ in place of b. Define

$$\mathcal{J}(x) = \int_{\mathbb{R}^d} (J^{\widehat{b}_\lambda}(x,y) - J^b(x,y))\, dy.$$

Then there exists a positive constant c_1 so that $0 \le \mathcal{J}(x) \le c_1$ for all $x \in \mathbb{R}^d$. Let

$$q(x,y) = \frac{J^{\widehat{b}_\lambda}(x,y) - J^b(x,y)}{\mathcal{J}(x)}.$$

Let S_1 be an exponential random variable of parameter 1 independent of X^b. Set

$$C_t = \int_0^t \mathcal{J}(X^b_s)\, ds, \quad U_1 = \inf\{t \ge 0 : C_t \ge S_1\}. \tag{5.9}$$

We let $Y_t = X^b_t$ for $0 \le t < U_1$ and define Y_{U_1} with law $q(Y_{U_1-}, \cdot) = q(X^b_{U_1-}, \cdot)$, and then repeat using an independent exponential random variable S_2 to define U_2, etc. So the construction proceeds now in the same way from the new starting point (U_1, Y_{U_1}). Since $\mathcal{J}(x)$ is bounded, only finitely many new jumps are introduced in any bounded time interval. In [20], it is proved that the resulting process Y is a strong Markov process. By slightly abusing the notation, we still use $\mathbb{P}_x$ and $\mathbb{E}_x$ to denote the above constructed probability law and expectation induced on such enlarged probability space under which $Y_0 = x$.

Lemma 5.7. *For each $x \in \mathbb{R}^d$ and $f \in C^2_b(\mathbb{R}^d)$,*

$$\mathbb{E}_x\left[f(Y_t); t < U_1\right] = f(x) + \mathbb{E}_x\left[\int_0^t \left(\mathcal{L}^b - \mathcal{J}(Y_s)\right) f(Y_s) \mathbb{1}_{\{s<U_1\}} ds\right],$$

Proof. By the definition of U_1 and Ito's formula, for each function $f \in C^2_b(\mathbb{R}^d)$,

$$\begin{aligned}
\mathbb{E}_x\left[f(Y_t); t < U_1\right] &= \mathbb{E}_x\left[f(X^b_t)\mathbb{1}_{\{U_1>t\}}\right] = \mathbb{E}_x\left[f(X^b_t)e^{-C_t}\right] \\
&= f(x) + \mathbb{E}_x\left[\int_0^t (\mathcal{L}^b - \mathcal{J}(X^b_s))f(X^b_s)e^{-C_s}\, ds\right] \\
&= f(x) + \mathbb{E}_x\left[\int_0^t \left(\mathcal{L}^b - \mathcal{J}(Y_s)\right) f(Y_s)\mathbb{1}_{\{s<U_1\}} ds\right].
\end{aligned}$$

□

Proposition 5.8. *For each $x \in \mathbb{R}^d$ and $f \in C^2_b(\mathbb{R}^d)$,*

$$M^f_t := f(Y_t) - f(Y_0) - \int_0^t \mathcal{L}^{\widehat{b}_\lambda} f(Y_s)\, ds$$

is a martingale under $\mathbb{P}_x$. So in particular, the strongly Markov process $(Y, \mathbb{P}_x, x \in \mathbb{R}^d)$ solves the martingale problem for $(\mathcal{L}^{\widehat{b}_\lambda}, C^2_\infty(\mathbb{R}^d))$.

Proof. Note that M_t^f is an additive function of Y. So by the Markov property of Y, it suffices to show that $\mathbb{E}_x\left[M_t^f\right]=0$ for every $x\in\mathbb{R}^d$ and $t>0$.

Recall that U_1 is defined in (5.9), and denote by $\{U_n, n\geq 2\}$ the subsequent jump adding times inductively defined according to the construction of Meyer [20]. For every $\alpha>0$, set $u_\alpha(x)=\mathbb{E}_x\left[\int_0^{U_1}e^{-\alpha t}f(Y_t)dt\right]$. Since by Lemma 5.7,

$$\mathbb{E}_x\left[f(Y_t);t<U_1\right]=f(x)+\mathbb{E}_x\left[\int_0^t\left(\mathcal{L}^b-\mathcal{J}(Y_s)\right)f(Y_s)\mathbb{1}_{\{s<U_1\}}ds\right],$$

we have by Fubini theorem that

$$u_\alpha(x)=\frac{f(x)}{\alpha}+\frac{1}{\alpha}\mathbb{E}_x\left[\int_0^{U_1}e^{-\alpha s}(\mathcal{L}^b-\mathcal{J}(Y_s))f(Y_s)ds\right].$$

Observe that in view of [24, p.286] (see, for example, the proof of [14, Proposition 2.2]), for any non-negative function φ on $\mathbb{R}^d$ and $x\in\mathbb{R}^d$,

$$\mathbb{E}_x\left[e^{-\alpha U_1}\varphi(Y_{U_1-})\right]=\mathbb{E}_x\left[\int_0^{U_1}e^{-\alpha s}\mathcal{J}(Y_s)\varphi(Y_s)ds\right].$$

Set $U_0=0$ and let θ_t to denote the time shift operator for the Markov process Y. Then we have from above and the strong Markov property of Y that

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^\infty e^{-\alpha t}f(Y_t)dt\right]=\sum_{j=0}^\infty\mathbb{E}_x\left[\int_{U_j}^{U_{j+1}}e^{-\alpha t}f(Y_t)dt\right]=\sum_{j=0}^\infty\mathbb{E}_x\left[e^{-\alpha U_j}u_\alpha(Y_{U_j})\right]\\
=&\ \frac{f(x)}{\alpha}+\frac{1}{\alpha}\sum_{j=1}^\infty\mathbb{E}_x\left[e^{-\alpha U_j}f(Y_{U_j})\right]+\frac{1}{\alpha}\sum_{j=0}^\infty\mathbb{E}_x\left[\int_{U_j}^{U_{j+1}}e^{-\alpha s}(\mathcal{L}^b-\mathcal{J}(Y_s))f(Y_s)ds\right]\\
=&\ \frac{f(x)}{\alpha}+\frac{1}{\alpha}\sum_{j=1}^\infty\mathbb{E}_x\left[e^{-\alpha U_j}\int_{\mathbb{R}^d}f(y)q(Y_{U_j-},y)dy\right]+\frac{1}{\alpha}\mathbb{E}_x\left[\int_0^\infty e^{-\alpha s}(\mathcal{L}^b-\mathcal{J}(Y_s))f(Y_s)ds\right]\\
=&\ \frac{f(x)}{\alpha}+\frac{1}{\alpha}\sum_{j=1}^\infty\mathbb{E}_x\left[e^{-\alpha U_{j-1}}\int_{\mathbb{R}^d}f(y)\left(e^{-\alpha U_1}q(Y_{U_1-},y)\right)\circ\theta_{U_{j-1}}dy\right]\\
&+\frac{1}{\alpha}\mathbb{E}_x\left[\int_0^\infty e^{-\alpha s}(\mathcal{L}^b-\mathcal{J}(Y_s))f(Y_s)ds\right]\\
=&\ \frac{f(x)}{\alpha}+\frac{1}{\alpha}\sum_{j=1}^\infty\mathbb{E}_x\left[e^{-\alpha U_{j-1}}\int_{\mathbb{R}^d}f(y)\left(\int_0^{U_1}e^{-\alpha s}\mathcal{J}(Y_s)q(Y_s,y)ds\right)\circ\theta_{U_{j-1}}dy\right]\\
&+\frac{1}{\alpha}\mathbb{E}_x\left[\int_0^\infty e^{-\alpha s}(\mathcal{L}^b-\mathcal{J}(Y_s))f(Y_s)ds\right]\\
=&\ \frac{f(x)}{\alpha}+\frac{1}{\alpha}\mathbb{E}_x\left[\int_{\mathbb{R}^d}f(y)\left(\int_0^\infty e^{-\alpha s}\mathcal{J}(Y_s)q(Y_s,y)ds\right)dy\right]\\
&+\frac{1}{\alpha}\mathbb{E}_x\left[\int_0^\infty e^{-\alpha s}(\mathcal{L}^b-\mathcal{J}(Y_s))f(Y_s)ds\right]\\
=&\ \frac{f(x)}{\alpha}+\frac{1}{\alpha}\mathbb{E}_x\left[\int_0^\infty e^{-\alpha s}\left(\mathcal{L}^bf(Y_s)+\int_{\mathbb{R}^d}\mathcal{J}(Y_s)q(Y_s,y)(f(y)-f(Y_s))dy\right)ds\right]\\
=&\ \frac{f(x)}{\alpha}+\frac{1}{\alpha}\mathbb{E}_x\left[\int_0^\infty e^{-\alpha s}\mathcal{L}^{\widehat{b}_\lambda}f(Y_s)ds\right].
\end{aligned}$$

By the uniqueness of the Laplace transform, we conclude from above that $\mathbb{E}_x\left[M_t^f\right]=0$ for all $t\geq 0$ and $x\in\mathbb{R}^d$. $\square$

Note that $\widehat{b}_\lambda$ defined by (5.8) is a bounded function on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying (1.2) and (1.19). By Theorem 5.6, the kernel $q^{\widehat{b}_\lambda}(t,x,y)$ uniquely determines a Feller process $X^{\widehat{b}_\lambda}=(X_t^{\widehat{b}_\lambda},t\geq 0,\mathbb{P}_x,x\in\mathbb{R}^d)$ on the canonical Skorokhod space $\mathbb{D}([0,\infty),\mathbb{R}^d)$, and $(X^{\widehat{b}_\lambda},\mathbb{P}_x)$ is the unique solution to the martingale problem for $(\mathcal{L}^{\widehat{b}_\lambda},\mathcal{S}(\mathbb{R}^d))$ with initial value x. This, together with Proposition 5.8 implies that the process Y coincides with $X^{\widehat{b}_\lambda}$ in the sense of distribution.

Theorem 5.9. *For every $\lambda>0$ and $A>0$, there is a positive constant $C_{15}=C_{15}(d,\alpha,\beta,A,\lambda)$ such that for any bounded b satisfying* (1.2) *and* (1.19) *with $\|b\|_\infty\leq A$,*

$$q^b(t,x,y)\leq C_{15}p_{M_{b^+,\lambda}}(t,x,y)\quad \text{for } t\in(0,1] \text{ and } x,y\in\mathbb{R}^d.$$

Proof. Noting that $\widehat{b}_\lambda$ is a bounded function on $\mathbb{R}^d\times\mathbb{R}^d$ with $\|\widehat{b}_\lambda\|_\infty\leq\|b\|_\infty$ satisfying (1.2) and (1.19), then by Theorem 1.1, there is a positive constant $C=C(d,\alpha,\beta,A,\lambda)$ so that

$$q^{\widehat{b}_\lambda}(t,x,y)\leq Cp_{M_{b^+,\lambda}}(t,x,y)\quad \text{for } t\in(0,1] \text{ and } x,y\in\mathbb{R}^d. \tag{5.10}$$

Let $\{\mathcal{M}_t\}_{t\geq 0}$ be the filtration generated by X^b. Note that $X^{\widehat{b}_\lambda}$ has the same distribution as Y. Then by Lemma 3.6 in [2], for any $A\in\mathcal{M}_t$,

$$\mathbb{P}^x(X_t^{\widehat{b}_\lambda}\in A)=\mathbb{P}^x(Y_t\in A)\geq\mathbb{P}^x(\{Y_s=X_s^b \text{ for all } 0\leq s\leq t\}\cap A)\geq e^{-t\|\mathcal{J}\|_\infty}\mathbb{P}^x(X_t^b\in A).$$

Hence, by (5.10)

$$q^b(t,x,y)\leq e^{\|\mathcal{J}\|_\infty}q^{\widehat{b}_\lambda}(t,x,y)\leq Ce^{\|\mathcal{J}\|_\infty}p_{M_{b^+,\lambda}}(t,x,y)\quad \text{for } t\in(0,1] \text{ and } x,y\in\mathbb{R}^d.$$

$\square$

For a Borel set $B\subset\mathbb{R}^d$, we define $\tau_B^b=\inf\{t>0:X_t^b\notin B\}$ and $\sigma_B^b:=\inf\{t\geq 0:X_t^b\in B\}$.

Proposition 5.10. *For each $A>0$ and $R_0>0$, there exists a positive constant*

$$\kappa=\kappa(d,\alpha,\beta,A,R_0)<2^\alpha\left(1-(1/3)^\alpha\right)$$

so that for every b satisfying (1.2) *and* (1.19) *with $\|b\|_\infty\leq A$, $r\in(0,R_0]$ and $x\in\mathbb{R}^d$,*

$$\mathbb{P}_x\left(\tau^b_{B(x,r)}\leq\kappa r^\alpha\right)\leq\frac{1}{2}.$$

Proof. Let f be a C^2 function taking values in $[0,1]$ such that $f(0)=0$ and $f(u)=1$ if $|u|\geq 1$. Set $f_{x,r}(y)=f(\frac{y-x}{r})$. Note that $f_{x,r}$ is a C^2 function taking values in $[0,1]$ such that $f_{x,r}(x)=0$ and $f_{x,r}(y)=1$ if $y\notin B(x,r)$. Moreover,

$$\sup_{y\in\mathbb{R}^d}\left|\frac{\partial^2 f_{x,r}(y)}{\partial y_i\partial y_j}\right|\leq r^{-2}\sup_{y\in\mathbb{R}^d}\left|\frac{\partial^2 f(y)}{\partial y_i\partial y_j}\right|.$$

Denote $\sum_{i,j=1}^{d} |\partial_{ij}^2 f(x)|$ by $|D^2 f(x)|$. By Taylor's formula, it follows that

$$
\begin{aligned}
|\mathcal{L}^b f_{x,r}(u)| &\le c_1 \int \left| f_{x,r}(u+h) - f_{x,r}(u) - \langle \nabla f_{x,r}(u), h\rangle \mathbb{1}_{\{|h|\le r\}} \right| \left(\frac{1}{|h|^{d+\alpha}} + \frac{1}{|h|^{d+\beta}} \right) dh \\
&= c_1 \int_{\{|h|\le r\}} |f_{x,r}(u+h) - f_{x,r}(u) - \langle \nabla f_{x,r}(u), h\rangle| \left(\frac{1}{|h|^{d+\alpha}} + \frac{1}{|h|^{d+\beta}} \right) dh \\
&\quad + c_1 \int_{\{|h|>r\}} |f_{x,r}(u+h) - f_{x,r}(u)| \left(\frac{1}{|h|^{d+\alpha}} + \frac{1}{|h|^{d+\beta}} \right) dh \\
&\le c_2 \|D^2 f\|_\infty r^{-2} \int_{|h|\le r} |h|^2 \left(\frac{1}{|h|^{d+\alpha}} + \frac{1}{|h|^{d+\beta}} \right) dh \\
&\quad + c_2 \|f\|_\infty \int_{\{|h|>r\}} \left(\frac{1}{|h|^{d+\alpha}} + \frac{1}{|h|^{d+\beta}} \right) dh \\
&\le c_3 (r^{-\alpha} + r^{-\beta}) \le c_3 (1 + R_0^{\alpha-\beta}) r^{-\alpha},
\end{aligned}
\tag{5.11}
$$

where $c_i = c_i(d,\alpha,\beta,A), i = 1,2,3$ are positive constants. Therefore, for each $t > 0$,

$$
\begin{aligned}
\mathbb{P}_x(\tau^b_{B(x,r)} \le t) &\le \mathbb{E}_x \left[f_{x,r}(X^b_{\tau^b_{B(x,r)} \wedge t}) \right] - f_{x,r}(x) \\
&= \mathbb{E}_x \left[\int_0^{\tau^b_{B(x,r)} \wedge t} \mathcal{L}^b f_{x,r}(X^b_s)\, ds \right] \le c_3 \big(1 + R_0^{\alpha-\beta}\big) \frac{t}{r^\alpha}.
\end{aligned}
$$

Set $\kappa = (2^\alpha [1 - (1/3)^\alpha]) \wedge (2c_3(1 + R_0^{\alpha-\beta}))^{-1}$, then

$$
\mathbb{P}_x(\tau^b_{B(x,r)} \le \kappa r^\alpha) \le \frac{1}{2}.
$$

$\square$

Recall that $m_{b,\lambda} = \operatorname{essinf}_{x,z\in\mathbb{R}^d, |z|>\lambda} b(x,z)$.

Proposition 5.11. *For every $A > 0, \lambda > 0, 0 < \varepsilon < 1$ and $R_0 > 0$, there exists a constant $C_{16} = C_{16}(d,\alpha,\beta,A,\lambda,\varepsilon,R_0) > 0$ so that for every b satisfying (1.2) and (1.23) with $\|b\|_\infty \le A$, $r \in (0, R_0]$ and $x, y \in \mathbb{R}^d$ with $|x-y| \ge 3r$,*

$$
\mathbb{P}_x\big(\sigma^b_{B(y,r)} < \kappa r^\alpha\big) \ge C_{16}\, r^{d+\alpha} \left(\frac{1}{|x-y|^{d+\alpha}} + \frac{m_{b^+,\lambda}}{|x-y|^{d+\beta}} \right).
$$

Proof. By Proposition 5.10 ,

$$
\mathbb{E}_x \left[\kappa r^\alpha \wedge \tau^b_{B(x,r)} \right] \ge \kappa r^\alpha\, \mathbb{P}_x\big(\tau^b_{B(x,r)} \ge \kappa r^\alpha\big) \ge \frac{1}{2} \kappa r^\alpha.
$$

Since $J^b(x,y) \ge m_{b^+,\lambda}\, \mathcal{A}(d,-\beta) |x-y|^{-d-\beta} \mathbb{1}_{\{|x-y|>\lambda\}}$, (1.23) implies that

$$
J^b(x,y) \ge \frac{1}{2} \left(\varepsilon |x-y|^{-(d+\alpha)} + m_{b^+,\lambda}\, \mathcal{A}(d,-\beta) |x-y|^{-(d+\beta)} \mathbb{1}_{\{|x-y|>\lambda\}} \right)
$$

Thus by Proposition 5.3, there are positive constants $c_1 = c_1(d, \alpha, \beta)$ and $c_2 = c_2(d, \alpha, \beta, A, \lambda, \varepsilon, R_0)$ so that

$$\begin{aligned}
\mathbb{P}_x(\sigma^b_{B(y,r)} < \kappa r^\alpha) &\geq \mathbb{P}_x(X^b_{\kappa r^\alpha \wedge \tau^b_{B(x,r)}} \in B(y,r)) \\
&= \mathbb{E}_x \int_0^{\kappa r^\alpha \wedge \tau^b_{B(x,r)}} \int_{B(y,r)} J^b(X^b_s, u)\, du\, ds \\
&\geq c_1 \mathbb{E}_x\left[\kappa r^\alpha \wedge \tau^b_{B(x,r)}\right] \int_{B(y,r)} \left(\frac{\varepsilon}{|x-y|^{d+\alpha}} + \frac{m_{b^+,\lambda}}{|x-y|^{d+\beta}} \mathbb{1}_{\{|x-y|>\lambda\}} \right) du \\
&\geq c_2 \varepsilon \kappa r^{d+\alpha} \left(\frac{1}{|x-y|^{d+\alpha}} + \frac{m_{b^+,\lambda}}{|x-y|^{d+\beta}} \right).
\end{aligned}$$

Here in the last inequality, we used the fact that $|x-y|^{-(d+\alpha)} \geq (1+\lambda^{\alpha-\beta}A)^{-1}[|x-y|^{-(d+\alpha)} + m_{b,\lambda} \cdot |x-y|^{-(d+\beta)}]$ for $|x-y| \leq \lambda$. $\square$

Proposition 5.12. *For every $A > 0$, there exists a constant $C_{17} = C_{17}(d, \alpha, \beta, A) > 0$ so that for every bounded b that satisfies* (1.2) *and* (1.19) *with $\|b\|_\infty \leq A$, and $3r \leq |x-y| \leq R_* := \frac{1}{3}\left(2A\frac{\mathcal{A}(d,-\beta)}{\mathcal{A}(d,-\alpha)}\right)^{1/(\beta-\alpha)}$,*

$$\mathbb{P}_x\big(\sigma^b_{B(y,r)} < \kappa r^\alpha\big) \geq C_{17} \frac{r^{d+\alpha}}{|x-y|^{d+\alpha}}.$$

Proof. Note that when $|x-u| \leq 3R_*$, $\frac{1}{2}\frac{\mathcal{A}(d,-\alpha)}{\mathcal{A}(d,-\beta)}|x-u|^{\beta-\alpha} \geq A$ and so

$$\begin{aligned}
J^b(x,u) &= \frac{\mathcal{A}(d,-\alpha)}{|x-u|^{d+\alpha}} + \frac{\mathcal{A}(d,-\beta)\, b(x, u-x)}{|x-u|^{d+\beta}} \\
&\geq \frac{\mathcal{A}(d,-\alpha)}{|x-u|^{d+\alpha}} - A\frac{\mathcal{A}(d,-\beta)}{|x-u|^{d+\beta}} \geq \frac{1}{2}\frac{\mathcal{A}(d,-\alpha)}{|x-u|^{d+\alpha}}.
\end{aligned} \tag{5.12}$$

By Propositions 5.3 and 5.10, we have

$$\begin{aligned}
\mathbb{P}_x\left(\sigma^b_{B(y,r)} < \kappa r^\alpha\right) &\geq \mathbb{P}_x\left(X^b_{\kappa r^\alpha \wedge \tau^b_{B(x,r)}} \in B(y,r)\right) \\
&= \mathbb{E}_x \int_0^{\kappa r^\alpha \wedge \tau^b_{B(x,r)}} \int_{B(y,r)} J^b(X^b_s, u)\, du\, ds \\
&\geq c_1 \mathbb{E}_x\left[\kappa r^\alpha \wedge \tau^b_{B(x,r)}\right] \int_{B(y,r)} \frac{1}{|x-y|^{d+\alpha}} du \\
&\geq c_2 \kappa r^{d+\alpha} \frac{1}{|x-y|^{d+\alpha}},
\end{aligned}$$

where the second inequality holds due to $|X^b_s - u| \leq 3|x-y| \leq 3R_*$ and (5.12). $\square$

Theorem 5.13. *For every $\lambda > 0, \varepsilon \in (0,1)$ and $A > 0$, there are positive constants $C_{18} = C_{18}(d, \alpha, \beta, A, \lambda, \varepsilon)$ and $C_{19} = C_{19}(d, \alpha, \beta, A, \lambda)$ such that for any b with $\|b\|_\infty \leq A$ that satisfies* (1.2) *and* (1.23),

$$C_{18}\, p_{m_{b^+,\lambda}}(t,x,y) \leq q^b(t,x,y) \leq C_{19}\, p_{M_{b^+,\lambda}}(t,x,y), \quad t \in (0,1],\ x,y \in \mathbb{R}^d. \tag{5.13}$$

Proof. Noting that the condition (1.23) in particular implies (1.19), so the upper bound estimate follows immediately from Theorem 5.9. We only need to prove the lower bound. Let $\delta_0 := 1 \wedge (A_0/A)^{\alpha/(\alpha-\beta)}$. (3.28) together with (1.7) also yields that for any $\|b\|_\infty \le A$,

$$q^b(t,x,y) \ge c_0 t^{-d/\alpha} \quad \text{for } t \in (0,\delta_0] \text{ and } |x-y| \le 3t^{1/\alpha}. \tag{5.14}$$

Here $c_0 = c_0(d,\alpha,\beta)$ is a positive constant. For every $t \in (0,\delta_0]$, by Proposition 5.10 and Proposition 5.11 with $R_0 = 1$, $r = t^{1/\alpha}/2$ and the strong Markov property of the process X^b, we get for $|x-y| > 3t^{1/\alpha}$,

$$\begin{aligned}
&\mathbb{P}_x(X^b_{2^{-\alpha}\kappa t} \in B(y,t^{1/\alpha})) \\
\ge\ & \mathbb{P}_x\left(X^b \text{ hits} B(y,t^{1/\alpha}/2) \text{ before } 2^{-\alpha}\kappa t \text{ and stays there for at least } 2^{-\alpha}\kappa t \text{ units of time}\right) \\
\ge\ & \mathbb{P}_x\left(\sigma^b_{B(y,t^{1/\alpha}/2)} < 2^{-\alpha}\kappa t\right) \inf_{z\in B(y,t^{1/\alpha}/2)} \mathbb{P}_z\left(\tau^b_{B(y,t^{1/\alpha})} \ge 2^{-\alpha}\kappa t\right) \\
\ge\ & \mathbb{P}_x\left(\sigma^b_{B(y,t^{1/\alpha}/2)} < 2^{-\alpha}\kappa t\right) \inf_{z\in B(y,t^{1/\alpha}/2)} \mathbb{P}_z\left(\tau^b_{B(z,t^{1/\alpha}/2)} \ge 2^{-\alpha}\kappa t\right) \\
\ge\ & c_1\, t^{(d+\alpha)/\alpha}\left(\frac{1}{|x-y|^{d+\alpha}} + \frac{m_{b^+,\lambda}}{|x-y|^{d+\beta}}\right).
\end{aligned} \tag{5.15}$$

Here $c_1 = c_1(d,\alpha,\beta,A,\lambda,\varepsilon)$ is a positive constant. Hence, by (5.14) and (5.15), for $|x-y| > 3t^{1/\alpha}$ and $t \in (0,\delta_0]$,

$$\begin{aligned}
q^b(t,x,y) &\ge \int_{B(y,t^{1/\alpha})} q^b(2^{-\alpha}\kappa t,x,z)q^b((1-2^{-\alpha}\kappa)t,z,y)\,dz \\
&\ge \inf_{z\in B(y,t^{1/\alpha})} q^b((1-2^{-\alpha}\kappa)t,z,y)\mathbb{P}_x(X^b_{2^{-\alpha}\kappa t} \in B(y,t^{1/\alpha})) \\
&\ge c_2 t^{-d/\alpha}\, t^{(d+\alpha)/\alpha}\left(\frac{1}{|x-y|^{d+\alpha}} + \frac{m_{b^+,\lambda}}{|x-y|^{d+\beta}}\right) \\
&\ge c_2\left(\frac{t}{|x-y|^{d+\alpha}} + \frac{t\,m_{b^+,\lambda}}{|x-y|^{d+\beta}}\right),
\end{aligned} \tag{5.16}$$

where $c_2 = c_2(d,\alpha,\beta,A,\lambda,\varepsilon) > 0$, the third inequality holds due to $|z-y| \le t^{1/\alpha} \le 3((1-2^{-\alpha}\kappa)t)^{1/\alpha}$ when $\kappa \le 2^\alpha(1-3^{-\alpha})$ and (5.14)-(5.15). Finally, (5.14), (5.16) together with (1.10) and the Chapman-Kolmogorov equation yields the desired lower bound estimate. □

Theorem 5.14. *For every $\lambda > 0$ and $A > 0$, there are positive constants $C_k = C_k(d,\alpha,\beta,A), k = 20,21$ and $C_{22} = C_{22}(d,\alpha,\beta,A,\lambda)$ such that for any bounded b satisfying* (1.2) *and* (1.19) *with $\|b\|_\infty \le A$,*

$$C_{20}\overline{p}_0(t,C_{21}x,C_{21}y) \le q^b(t,x,y) \le C_{22}p_{M_{b^+,\lambda}}(t,x,y) \quad \text{for } t\in(0,1] \text{ and } x,y\in\mathbb{R}^d. \tag{5.17}$$

Proof. By Theorem 5.9, it suffices to prove the lower bound of q^b. Let $\delta_0 := 1 \wedge (A_0/A)^{\alpha/(\alpha-\beta)}$. By Chapman-Kolmogorov equation, we only need to consider (5.17) for $t \in (0,\delta_0]$. By (1.20), (1.21) and (3.28), it suffices to prove (5.17) when $|x-y| > 3t^{1/\alpha}$ and $t \in (0,\delta_0]$. Let R_* be the constant defined in Proposition 5.12.

(i) First, we consider the case $R_* \ge |x-y| > 3t^{1/\alpha}$. For every $t \in (0,\delta_0]$, by Proposition 5.10 and Proposition 5.12 with $r = t^{1/\alpha}/2$ and the strong Markov property of the process X^b, we get, by the similar procedure in (5.15), for $R_* \ge |x-y| > 3t^{1/\alpha}$,

$$\mathbb{P}_x\left(X^b_{2^{-\alpha}\kappa t} \in B(y,t^{1/\alpha})\right) \ge c_1\, t^{(d+\alpha)/\alpha}\frac{1}{|x-y|^{d+\alpha}}. \tag{5.18}$$

Here $c_1 = c_1(d, \alpha, \beta, A)$ is a positive constant. Hence, for $R_* \geq |x-y| > 3t^{1/\alpha}$, by (5.14) and (5.18), we have

$$\begin{aligned} q^b(t,x,y) &\geq \int_{B(y,t^{1/\alpha})} q^b(2^{-\alpha}\kappa t, x, z) q^b((1-2^{-\alpha}\kappa)t, z, y)\, dz \\ &\geq \inf_{z\in B(y,t^{1/\alpha})} q^b((1-2^{-\alpha}\kappa)t, z, y) \mathbb{P}_x \left(X^b_{2^{-\alpha}\kappa t} \in B(y, t^{1/\alpha}) \right) \\ &\geq c_2 t^{-d/\alpha}\, t^{(d+\alpha)/\alpha} \frac{1}{|x-y|^{d+\alpha}} \\ &\geq c_2 \frac{t}{|x-y|^{d+\alpha}} \end{aligned} \tag{5.19}$$

where $c_2 = c_2(d, \alpha, \beta, A) > 0$.

(ii) Next, we consider the case $|x-y| > R_* > 3t^{1/\alpha}$. Take $C_* = R_*^{-1}$. Then $|x-y| > R_* = C_*^{-1} \geq t/C_*$ for $t \in (0, \delta_0]$. The following proof is similar to [8, Theorem 3.6]. For the reader's convenience, we spell out the details here.

Let $R := |x-y|$ and $c_+ = R_*^{-1} \vee 1$. Let $l \geq 2$ be a positive integer such that $c_+ R \leq l \leq c_+ R + 1$ and let $x = x_0, x_1, \cdots, x_l = y$ be such that $|x_i - x_{i-1}| \asymp R/l \asymp 1/c_+$ for $i = 1, \cdots, l-1$. Since $t/l \leq C_* R/l \leq C_*/c_+ \leq 1$ and $R/l \leq 1/c_+ \leq R_*$, we have by (5.14) and (5.19),

$$q^b(t/l, x_i, x_{i+1}) \geq c_2 \left((t/l)^{-d/\alpha} \wedge \frac{t/l}{(R/l)^{d+\alpha}} \right) \geq c_2 \left((t/l)^{-d/\alpha} \wedge (t/l) \right) \geq c_3 t/l. \tag{5.20}$$

Let $B_i = B(x_i, R_*)$, by (5.20),

$$\begin{aligned} q^b(t,x,y) &\geq \int_{B_1} \cdots \int_{B_{l-1}} q^b(t/l, x, x_1) \cdots q^b(t/l, x_{l-1}, y)\, dx_1 \cdots dx_{l-1} \\ &\geq (c_4 t/l)^l \geq (c_5 t/R)^{c_+ R+1} \geq c_6 (t/R)^{c_7 R} \\ &\geq c_6 \left(\frac{t}{|x-y|} \right)^{c_7 |x-y|}. \end{aligned} \tag{5.21}$$

By (5.19), (5.21) and together with the estimates of $\overline{p}_0$ in (1.20)-(1.21), we get the desired conclusion. □

Proof of Theorem 1.3. Theorem 1.3 now follows from Theorems 5.6, 5.13 and 5.14. □

To prove theorem 1.5, we use the main result in [10] of the heat kernel estimates for non-local operators under the non-local Feynman-Kac perturbation. For each Borel function $q(x)$ on $\mathbb{R}^d$ and Borel function $F(x,y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ that vanishes along the diagonal, we define a non-local Feynman-Kac transform for the process X^b as follows:

$$T_t^{b,F} f(x) = \mathbb{E}_x \left[\exp\left(\int_0^t q(X_s^b)\, ds + \sum_{s\leq t} F(X^b_{s-}, X^b_s) \right) f(X^b_t) \right]. \tag{5.22}$$

Proposition 5.15. *Suppose b is a bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (1.2) and (1.19), q is a bounded function on $\mathbb{R}^d$ and $|F(x,y)| \leq c(|x-y|^2 \wedge 1)$ for some constant c. Then for each f in $C^2_b(\mathbb{R}^d)$,*

$$T_t^{b,F} f(x) = f(x) + \int_0^t T_s^{b,F} \mathcal{L}^{b,F} f(x)\, ds,$$

where

$$\mathcal{L}^{b,F} f(x) = \mathcal{L}^b f(x) + \int_{\mathbb{R}^d} (e^{F(x,y)} - 1) f(y) J^b(x,y)\, dy + q(x) f(x).$$

Proof. First note that since X^b is a semimartingale and $|F(x,y)| \le c(|x-y|^2 \wedge 1)$,

$$\sum_{s\le t} |F(X^b_{s-}, X^b_s)| \le c \sum_{s \le t} |X^b_s - X^b_{s-}|^2 = c\big[X^b, X^b\big]_t < \infty.$$

Let $F_1 = e^F - 1$ and define

$$K_t = \int_0^t q(X^b_s)\, ds + \sum_{s\le t} F_1(X^b_{s-}, X^b_s).$$

Then by [24, A4.17], the Stieljes exponential

$$A_t := \mathrm{Exp}(K)_t = e^{K^c_t} \prod_{0<s\le t} (1 + K_s - K_{s-}) = \exp\left(\int_0^t q(X^b_s)\, ds + \sum_{s \le t} F(X^b_{s-}, X^b_s) \right) \qquad (5.23)$$

is the unique solution to

$$A_t = 1 + \int_0^t A_{s-}\, dK_s. \qquad (5.24)$$

For each function f in $C^2_b(\mathbb{R}^d)$, by Ito's formula, Proposition 5.1 and (5.24), we have

$$\begin{aligned} A_t f(X^b_t) &= f(X^b_0) + \int_0^t f(X^b_{s-}) A_{s-}\, dK_s + \int_0^t A_{s-}\, df(X^b_s) + \sum_{s\le t} (A_s - A_{s-})(f(X^b_s) - f(X^b_{s-})) \\ &= f(X^b_0) + \int_0^t A_s f(X^b_s) q(X^b_s)\, ds + \sum_{s \le t} f(X^b_{s-}) A_{s-} F_1(X^b_{s-}, X^b_s) \qquad (5.25) \\ &\quad + \int_0^t A_s \mathcal{L}^b f(X^b_s)\, ds + \int_0^t A_{s-}\, dM^f_s + \sum_{s\le t} A_{s-} F_1(X^b_{s-}, X^b_s)(f(X^b_s) - f(X^b_{s-})). \end{aligned}$$

By taking expectation on both sides and using the Lévy system formula in Proposition 5.3, we get

$$\begin{aligned} T^{b,F}_t f(x) &= \mathbb{E}_x\big[A_t f(X^b_t)\big] \\ &= f(X^b_0) + \mathbb{E}_x \left[\int_0^t A_s f(X^b_s) q(X^b_s)\, ds + \int_0^t A_s \mathcal{L}^b f(X^b_s)\, ds \right] \\ &\quad + \mathbb{E}_x \left[\int_0^t \int_{\mathbb{R}^d} A_s \Big(f(X^b_s) + \big(f(y) - f(X^b_s)\big)\Big) F_1(X^b_s, y) J^b(X^b_s, y)\, dy\, ds \right] \\ &= f(X^b_0) + \mathbb{E}_x \left[\int_0^t A_s \mathcal{L}^{b,F} f(X^b_s) ds \right] \\ &= f(x) + \int_0^t T^{b,F}_s \mathcal{L}^{b,F} f(x) ds. \end{aligned}$$

That completes the proof. $\square$

Proof of Theorem 1.5. Let $b_0(x,z)=b(x,z)\mathbb{1}_{|z|\le 1}(z)$, which is a bounded function on $\mathbb{R}^d\times\mathbb{R}^d$ satisfying (1.2) and (1.19). By Theorem 1.3, $q^{b_0}(t,x,y)$ is continuous on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ and

$$C_4 p_0(t,x,y)\le q^{b_0}(t,x,y)\le C_3 p_0(t,x,y) \tag{5.26}$$

for all $(t,x,y)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$. In addition, by Proposition 5.3 and (1.25), for each non-negative function f on $\mathbb{R}^d\times\mathbb{R}^d$ that vanishes along the diagonal,

$$\mathbb{E}_x\Big[\sum_{s\le T} f(X^{b_0}_{s-},X^{b_0}_s)\Big]=\mathbb{E}_x\left[\int_0^T\int_{\mathbb{R}^d} f(X^{b_0}_s,u)J^{b_0}(X^{b_0}_s,u)\,du\,ds\right]. \tag{5.27}$$

and there exist two positive constants c_1 and c_2 so that

$$c_1|x-y|^{-(d+\alpha)}\le J^{b_0}(x,y)\le c_2|x-y|^{-(d+\alpha)}. \tag{5.28}$$

Set $F(x,y)=\ln\frac{J^b(x,y)}{J^{b_0}(x,y)}$ and $q(x)=\int_{\mathbb{R}^d}(J^{b_0}(x,y)-J^b(x,y))\,dy$. It is easy to see that q is a bounded function on $\mathbb{R}^d$ and $J^b(x,y)=J^{b_0}(x,y)$ for $|x-y|\le 1$. Moreover, by the (1.27) and (5.28), there exist two positive constants c_3 and c_4 so that $c_3\le\frac{J^b(x,y)}{J^{b_0}(x,y)}\le c_4$ for all $|x-y|>1$ and any bounded b with $\|b\|_\infty\le A$. Hence, there is a positive constant c_5 so that $|F(x,y)|\le c_5(|x-y|^2\wedge 1)$. Let $T^{b_0,F}_t$ be the semigroup $T^{b,F}_t$ defined by (5.22) but with b_0 in place of b. By (5.26)-(5.28) above and [10, Theorem 1.3], the non-local Feynman-Kac semigroup $(T^{b_0,F}_t, t\ge 0)$ has a continuous density $\widetilde{q}(t,x,y)$ and there is a positive constant c_6 so that for all $(t,x,y)\in(0,1]\times\mathbb{R}^d\times\mathbb{R}^d$,

$$c_6^{-1}p_0(t,x,y)\le\widetilde{q}(t,x,y)\le c_6 p_0(t,x,y). \tag{5.29}$$

On the other hand, for each f in $C^2_b(\mathbb{R}^d)$,

$$\begin{aligned}\mathcal{L}^{b_0,F}f(x)&=\mathcal{L}^{b_0}f(x)+\int_{\mathbb{R}^d}(e^{F(x,y)}-1)f(y)J^{b_0}(x,y)\,dy+q(x)f(x)\\&=\mathcal{L}^{b_0}f(x)+\int_{\mathbb{R}^d}\Big(J^b(x,y)-J^{b_0}(x,y)\Big)(f(y)-f(x))\,dy\\&=\mathcal{L}^b f(x).\end{aligned}$$

By taking $f=1$ in Proposition 5.15, we get $T^{b_0,F}_t 1=1$. Hence $\widetilde{q}(t,x,y)$ uniquely determines a conservative Feller process $\widetilde{Y}$ with $\{T^{b_0,F}_t; t\ge 0\}$ as its transition semigroup. Proposition 5.15 implies that the distribution of $\widetilde{Y}$ on the canonical Skorokhod space $\mathbb{D}([0,\infty),\mathbb{R}^d)$ is a solution to the martingale problem $(\mathcal{L}^b, C^2_b(\mathbb{R}^d))$ and in particular to the martingale problem $(\mathcal{L}^b,\mathcal{S}(\mathbb{R}^d))$. However by Theorem 1.3, martingale solution to the operator $(\mathcal{L}^b,\mathcal{S}(\mathbb{R}^d))$ is unique. This yields that $\widetilde{q}=q^b$ and so we get the desired conclusion from (5.29). □

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