Dirichlet Heat Kernel Estimates for Subordinate Brownian Motions with Gaussian Components

Zhen-Qing Chen[*], **Panki Kim**[†] and **Renming Song**[‡]

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Abstract

In this paper, we derive explicit sharp two-sided estimates for the Dirichlet heat kernels, in $C^{1,1}$ open sets D in $\mathbb{R}^d$, of a large class of subordinate Brownian motions with Gaussian components. When D is bounded, our sharp two-sided Dirichlet heat kernel estimates hold for all $t > 0$. Integrating the heat kernel estimates with respect to the time variable t, we obtain sharp two-sided estimates for the Green functions, in bounded $C^{1,1}$ open sets, of such subordinate Brownian motions with Gaussian components.

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1 Introduction

It is well-known that, for a second order elliptic differential operator $\mathcal{L}$ on $\mathbb{R}^d$ satisfying some natural conditions, there is a diffusion process X on $\mathbb{R}^d$ with $\mathcal{L}$ as its infinitesimal generator. The fundamental solution $p(t, x, y)$ of $\partial_t u = \mathcal{L}u$ (also called the heat kernel of $\mathcal{L}$) is the transition density of X. Such relationship is also true for a large class of Markov processes with discontinuous sample paths, which constitute an important family of stochastic processes in probability theory that have been widely used in various applications. Thus obtaining sharp two-sided estimates for $p(t, x, y)$ is a fundamental problem in both analysis and probability theory.

Two-sided heat kernel estimates for diffusions on $\mathbb{R}^d$ have a long history and many beautiful results have been established. See [20, 22] and the references therein. But, due to the complication near the boundary, two-sided estimates for the transition density (equivalently, the Dirichlet heat kernels) of killed Brownian motion in a connected open set D have been established only recently.

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See [21, 22, 23] for upper bound estimates and [31] for the lower bound estimates of the Dirichlet heat kernels in bounded $C^{1,1}$ connected open sets. For discontinuous processes (or, non-local operators), the study of their global heat kernel estimates started quite recently. See [5, 6, 14, 15, 16] and the references therein. See also [3] for a recent survey on this topic. The study of sharp two-sided Dirichlet heat kernel estimates for discontinuous processes is even more recent. In [7], we obtained sharp two-sided estimates for the Dirichlet heat kernel of the fractional Laplacian $\Delta^{\alpha/2}$ in any $C^{1,1}$ open set D with zero exterior condition on D^c (or equivalently, the transition density function of the killed α-stable process in D).

In the last few years, the approach developed in [7] has served as a road map for establishing sharp two-sided Dirichlet heat kernel estimates for other purely discontinuous processes in open subsets of $\mathbb{R}^d$. In [8, 9, 11], the ideas of [7] were adapted and further developed to establish sharp two-sided Dirichlet heat kernel estimates of censored stable-like processes, mixed stable processes and relativistic stable processes in $C^{1,1}$ open subsets of $\mathbb{R}^d$. In [2], a Varopoulos type factorization estimate in terms of surviving probabilities was obtained for the transition densities of symmetric stable processes in κ-fat open sets. Very recently, we obtained, in [12], a Varopoulos type factorization estimate for the Dirichlet heat kernels in non-smooth open sets for a large class of purely discontinuous subordinate Brownian motions. We have also obtained in [12] explicit sharp two-sided Dirichlet heat kernel estimates for a large class of subordinate Brownian motions in $C^{1,1}$ open sets.

Things become more complicated when one deals with Lévy processes having both Gaussian and jump components. In [10], sharped two-sided heat kernel estimates in $C^{1,1}$ open sets are established for Lévy processes that can be written as the independent sum of a Brownian motion and a symmetric α-stable process. A key ingredient is the boundary Harnack principle for $\Delta + \Delta^{\alpha/2}$ in $C^{1,1}$ open sets with explicit boundary decay rates, obtained in [13].

The purpose of this paper is to establish sharp two-sided Dirichlet heat kernel estimates, in $C^{1,1}$ open sets, for a large class of subordinate Brownian motions with Gaussian components. Throughout this paper, we will always assume that $S = (S_t : t \geq 0)$ is a complete subordinator with a positive drift and, without loss of generality, we shall assume that the drift of S is equal to 1. That is, the Laplace exponent of S is a complete Bernstein function which can be written as

$$\phi(\lambda) := \lambda + \psi(\lambda) \qquad \text{with} \quad \psi(\lambda) := \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt), \tag{1.1}$$

where μ is a measure on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge t)\mu(dt) < \infty$. μ is called the Lévy measure of the subordinator S (or of ϕ). We will exclude the trivial case of $S_t = t$, that is, the case of $\psi \equiv 0$. By the definition of complete Bernstein functions, the Lévy measure μ has a complete monotone density. By a slight abuse of notation we will denote the density by $\mu(t)$. For basic facts on complete Bernstein functions, we refer the reader to [30]. In this paper, we will assume the following growth condition on $\mu(t)$ near zero: For any $K > 0$, there exists $c = c(K) > 1$ such that

$$\mu(r) \leq c\mu(2r), \quad r \in (0, K). \tag{1.2}$$

Suppose that $B = (B_t : t \geq 0)$ is a Brownian motion in $\mathbb{R}^d$ with infinitesimal generator Δ and independent of S. Then the process $X = (X_t : t \geq 0)$ defined by $X_t = B_{S_t}$ is called a subordinate Brownian motion. X can be written as the independent sum of a Brownian motion and a purely

discontinuous subordinate Brownian motion. The infinitesimal generator of X is

$$\mathcal{L}^X := -\phi(-\Delta) = \Delta - \psi(-\Delta),$$

and $\psi(-\Delta)$ is a non-local operator. The Lévy density J of X is given by

$$J(x) = j(|x|) = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/4t} \mu(t) dt. \qquad (1.3)$$

The function $J(x)$ determines a Lévy system for X, which describes the jumps of the process X: for any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, y, y) = 0$ for all $y \in \mathbb{R}^d$, any stopping time T (with respect to the filtration of X) and any $x \in \mathbb{R}^d$,

$$\mathbb{E}_x \left[\sum_{s \le T} f(s, X_{s-}, X_s) \right] = \mathbb{E}_x \left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, X_s, y) J(X_s - y) dy \right) ds \right] \qquad (1.4)$$

(see, for example, [14, Proof of Lemma 4.7] and [15, Appendix A]).

The function j is obviously a decreasing function on $(0, \infty)$. Using this and the fact that J is a Lévy density (and so $\int_{\mathbb{R}^d} (1 \wedge |x|^2) j(|x|) dx < \infty$), we can easily get that, for any $K > 0$, there exists $c = c(K) > 0$ such that

$$j(r) \le cr^{-d-2} \qquad \text{for } r \in (0, K]. \qquad (1.5)$$

In fact, we have for $s \in (0, K]$,

$$\frac{1}{d+2} j(s) s^{d+2} = \int_0^s r^{d+1} j(s) dr \le \int_0^s r^{d+1} j(r) dr \le \int_0^K r^{d+1} j(r) dr =: c(K) < \infty,$$

from which (1.5) follows immediately.

The subordinate Brownian motion X has a transition density $p(t, x, y)$ with respect to the Lebesgue measure. Observe that $p(t, x, y)$ is given by $p(t, x, y) = p(t, |x - y|)$ where

$$p(t, r) = \int_0^\infty (4\pi s)^{-d/2} e^{-\frac{r^2}{4s}} \mathbb{P}(S_t \in ds) \quad \forall t > 0, r \ge 0. \qquad (1.6)$$

Clearly $r \to p(t, r)$ is monotonically deceasing. For any open set $D \subset \mathbb{R}^d$, we will use X^D to denote the part process of X killed upon leaving D. The process X^D has a transition density $p_D(t, x, y)$ with respect to the Lebesgue measure on D. The density $p_D(t, x, y)$ is the fundamental solution of $\mathcal{L}^X$ in D with zero exterior condition, which is also called the Dirichlet heat kernel of $\mathcal{L}^X$ in D.

The goal of this paper is to establish explicit sharp two-sided estimates for $p_D(t, x, y)$ in $C^{1,1}$ open sets D under the above assumptions. Throughout the remainder of this paper, we assume that $d \ge 1$. The Euclidean distance between x and y will be denoted as $|x - y|$. We will use $B(x, r)$ to denote the open ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. We also define $J(x, y) = J(y - x)$. For any Borel set $A \subset \mathbb{R}^d$, we will use $\text{diam}(A)$ to denote its diameter and $|A|$ to denote its Lebesgue measure. For any two positive functions f and g, $f \asymp g$ means that there is a positive constant $c \ge 1$ so that $c^{-1} g \le f \le cg$ on their common domain of definition.

We will show in this paper (see Remark 2.7 below) that under the above assumptions, for every $T > 0$, there exists a constant $c_0 = c_0(T, d, \psi) > 0$ so that

$$p(t, r) \geq c_0 \left(t^{-d/2} e^{-r^2/4t} + t^{-d/2} \wedge (tj(r)) \right)$$

for all $(t, r) \in (0, T] \times [0, \infty)$. To get an explicit Dirichlet heat kernel upper bound estimates, we will need to assume the following upper bound condition on $p(t, r)$ for $r \leq \text{diam}(D)$: For any $T > 0$, there exist $C_j \geq 1$, $j = 1, 2, 3$, such that for all $(t, r) \in (0, T] \times [0, \text{diam}(D)]$,

$$p(t, r) \leq C_1 \left(t^{-d/2} e^{-r^2/C_2 t} + t^{-d/2} \wedge (tj(r/C_3)) \right). \tag{1.7}$$

In [16], a DeGiorgi-Nash-Moser-Aronson type theory has been established for a large class of symmetric Markov processes on $\mathbb{R}^d$ with infinitesimal generators of the form

$$\mathcal{L}u(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u(x)}{\partial x_j} \right) + \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d : |y-x| > \varepsilon\}} (u(y) - u(x)) \frac{c(x, y)}{|x - y|^d \Phi(|x - y|)} dy, \tag{1.8}$$

where $(a_{ij}(x))_{1 \leq i,j \leq d}$ is a measurable $d \times d$ matrix-valued measurable function on $\mathbb{R}^d$ that is uniformly elliptic and bounded, $c(x, y)$ is a measurable symmetric kernel that is bounded between two positive constants, and $\Phi(r)$ is a positive increasing function in $r \in (0, \infty)$. If Φ satisfies suitable growth conditions near zero and infinity, sharp two-sided estimates on the transition density of this class of Markov processes have been obtained in [16]. In this case, the transition density $p(t, x, y)$ of such a process admits the following estimates: for any $T > 0$, there exist $c_j \geq 1$ such that for all $(t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$p(t, x, y) \geq c_1^{-1} \left(t^{-d/2} e^{-c_2|x-y|^2/t} + t^{-d/2} \wedge (tj(c_3|x - y|)) \right)$$

$$p(t, x, y) \leq c_1 \left(t^{-d/2} e^{-|x-y|^2/c_2 t} + t^{-d/2} \wedge (tj(|x - y|/c_3)) \right)$$

with $j(r) = r^{-d} (\Phi(r))^{-1}$. These estimates can be regarded as the counterpart of Aronson's estimates for non-local operators. When (a_{ij}) is a constant matrix and $c(x, y) = c(|x-y|)$ is a function of $|x - y|$, the Markov process X with generator $\mathcal{L}$ of (1.8) is a rotationally symmetric Lévy process on $\mathbb{R}^d$ with Lévy measure $j(|z|)dz = \frac{c(|z|)}{|z|^d \Phi(|z|)} dz$. In this case, the Lévy exponent of X is

$$\Psi(\xi) = \sum_{i,j=1}^{d} a_{ij} \xi_i \xi_j + \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot z)) \frac{c(|z|)}{|z|^d \Phi(|z|)} dz, \qquad \xi \in \mathbb{R}^d. \tag{1.9}$$

For the subordinate Brownian motion X considered in this paper, its Lévy exponent is $\phi(|\xi|^2)$, where ϕ is defined in (1.1), which admits an expression of the form

$$\phi(|\xi|^2) = |\xi|^2 + \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot z)) J(z)dz = |\xi|^2 + \psi(|\xi|^2), \tag{1.10}$$

where J is defined in (1.3). When the complete Bernstein function ψ satisfies the following condition near infinity: there exist constants $\delta_k \in (0, 1)$, $a_k > 0$, $k = 1, 2$, and $R_1 > 0$ such that

$$a_1 \lambda^{\delta_1} \psi(r) \leq \psi(\lambda r) \leq a_2 \lambda^{\delta_2} \psi(r) \qquad \text{for } \lambda \geq 1 \text{ and } r \geq R_1, \tag{1.11}$$

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then (see [28, Lemma 3.2])

$$J(x) \asymp \frac{1}{|x|^d \Phi(|x|)} \qquad \text{for } |x| \le 1,$$

with a positive increasing function Φ satisfying the growth conditions in [16] for $r \le 1$. Then one can use the heat kernel estimates in [16] and an argument similar to the proof of [5, Theorem 2.4] to show that the estimate (1.7) holds for X for any bounded open set D with $C_3 = 1$.

If, in addition to (1.11), ψ also satisfies the following condition near zero: there exist constants $\delta_k \in (0,1)$, $a_k > 0$, $k = 3, 4$, and $R_2 > 0$ such that

$$a_3 \lambda^{\delta_3} \psi(r) \le \psi(\lambda r) \le a_4 \lambda^{\delta_4} \psi(r) \qquad \text{for } \lambda \le 1 \text{ and } r \le R_2, \tag{1.12}$$

then (see [28, Theorem 3.4])

$$J(x) \asymp \frac{1}{|x|^d \Phi(|x|)} \qquad \text{for } x \ne 0$$

with a positive increasing function Φ satisfying the conditions in [16] for all $r > 0$. So it follows from the heat kernel estimates in [16] that the estimate (1.7) holds for X with $D = \mathbb{R}^d$ and $C_3 = 1$.

To state the main result of this paper, we first recall that an open set D in $\mathbb{R}^d$ (when $d \ge 2$) is said to be a (uniform) $C^{1,1}$ open set if there exist a localization radius $R_0 > 0$ and a constant $\Lambda_0 > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$-function $\varphi = \varphi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\varphi(0) = 0$, $\nabla \varphi(0) = (0, \ldots, 0)$, $\|\nabla \varphi\|_\infty \le \Lambda_0$, $|\nabla \varphi(x) - \nabla \varphi(z)| \le \Lambda_0 |x - z|$, and an orthonormal coordinate system CS_z with its origin at z such that

$$B(z, R_0) \cap D = \{y = (\widetilde{y}, y_d) \text{ in } CS_z : |y| < R_0, y_d > \varphi(\widetilde{y})\}.$$

The pair (R_0, Λ_0) is called the characteristics of the $C^{1,1}$ open set D. Note that a $C^{1,1}$ open set D with characteristics (R_0, Λ_0) can be unbounded and disconnected; the distance between two distinct components of D is at least R_0. Let $\delta_{\partial D}(x)$ be the Euclidean distance between x and ∂D. It is well-known that any $C^{1,1}$ open set D satisfies both the *uniform interior ball condition* and the *uniform exterior ball condition*: there exists $r_0 = r_0(R_0, \Lambda_0) \in (0, R_0]$ such that for any $x \in D$ with $\delta_{\partial D}(x) < r_0$ and $y \in \mathbb{R}^d \setminus \overline{D}$ with $\delta_{\partial D}(y) < r_0$, there are $z_x, z_y \in \partial D$ so that $|x - z_x| = \delta_{\partial D}(x)$, $|y - z_y| = \delta_{\partial D}(y)$ and that $B(x_0, r_0) \subset D$ and $B(y_0, r_0) \subset \mathbb{R}^d \setminus \overline{D}$ for $x_0 = z_x + r_0(x - z_x)/|x - z_x|$ and $y_0 = z_y + r_0(y - z_y)/|y - z_y|$. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be written as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive.

For an open set $D \subset \mathbb{R}^d$ and $x \in D$, we will use $\delta_D(x)$ to denote the Euclidean distance between x and D^c. For an open set $D \subset \mathbb{R}^d$ and $\lambda_0 \in [1, \infty)$, we say *the path distance in each connected component of D is comparable to the Euclidean distance with characteristic λ_0* if for every x, y in the same component of D there is a rectifiable curve l in D connecting x to y such that the length of l is no larger than $\lambda_0 |x - y|$. Clearly, such a property holds for all bounded $C^{1,1}$ open sets, $C^{1,1}$ open sets with compact complements and connected open sets above graphs of $C^{1,1}$ functions.

For any open set $D \subset \mathbb{R}^d$ and positive constants c_1 and c_2, we define

$$h_{D,c_1,c_2}(t, x, y) := \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right) \left(t^{-d/2} e^{-c_1|x-y|^2/t} + t^{-d/2} \wedge (tJ(c_2 x, c_2 y))\right). \tag{1.13}$$

The following is the main result of this paper.

Theorem 1.1 *Suppose that X is a subordinate Brownian motion with Lévy exponent $\phi(|\xi|^2)$ with ϕ being a complete Bernstein function satisfying (1.1) and (1.2). Suppose that D is a $C^{1,1}$ open subset of $\mathbb{R}^d$ with characteristics (R_0, Λ_0).*

(i) *If the path distance in each connected component of D is comparable to the Euclidean distance with characteristic λ_0, then for every $T > 0$, there exist $c_1 = c_1(R_0, \Lambda_0, \lambda_0, T, \psi, d) > 0$ and $c_2 = c_2(R_0, \Lambda_0, \lambda_0, d) > 0$ such that for all $(t, x, y) \in (0, T] \times D \times D$,*

$$p_D(t, x, y) \geq c_1 \, h_{D, c_2, 1}(t, x, y).$$

(ii) *If D satisfies (1.7), then for every $T > 0$, there exists $c_3 = c_3(R_0, \Lambda_0, T, d, \psi, C_1, C_3, d) > 1$ such that for all $(t, x, y) \in (0, T] \times D \times D$,*

$$p_D(t, x, y) \leq c_3 \, h_{D, C_4, C_5}(t, x, y),$$

where $C_4 = (16C_2)^{-1}$ and $C_5 = (8 \vee 4C_3)^{-1}$.

(iii) *If D is bounded, then for every $T > 0$, there exists $c_4 = c_4(\mathrm{diam}(D), R_0, \Lambda_0, T, \psi, d) > 0$ such that for all $(t, x, y) \in [T, \infty) \times D \times D$,*

$$p_D(t, x, y) \geq c_4 \, e^{-\lambda_1 t} \delta_D(x) \delta_D(y),$$

where $-\lambda_1 < 0$ is the largest eigenvalue of the generator of X^D.

(iv) *If D is bounded and satisfies (1.7), then for every $T > 0$, there exists $c_5 = c_5(\mathrm{diam}(D), R_0, \Lambda_0, T, \psi, d, C_1, C_2, C_3) > 0$ such that for all $(t, x, y) \in [T, \infty) \times D \times D$,*

$$p_D(t, x, y) \leq c_5 \, e^{-\lambda_1 t} \delta_D(x) \delta_D(y).$$

When $D = B(x_0, r)$, it follows as a special case of [17, Theorem 4.5(ii)] that $\phi(\lambda_1^D)/2 \leq \lambda_1 \leq \phi(\lambda_1^D)$, where λ_1^D is the smallest eigenvalue of $-\Delta$ in D. It follows from the scaling property of Brownian motion (or Laplacian) that $\lambda_1^{B(x_0,r)} = cr^{-2}$, where $c = c(d)$ is a positive constant that depends only on the dimension d. When D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with $C^{1,1}$-characteristics (R_0, Λ_0), D contains a ball of radius r_0 and is contained in a ball of radius $\mathrm{diam}(D)$, where $r_0 = r_0(R_0, \Lambda_0)$ is such that D satisfies the uniform interior ball condition with radius r_0. By the domain monotonicity of the first eigenvalue λ_1, one concludes from above that λ_1 is bounded between two positive constants that depend only on $R_0, \Lambda_0, \psi, \mathrm{diam}(D)$ and d.

Note that the explicit upper bound estimates in Theorem 1.1 are established under the assumption that the upper bound (1.7) for $p(t, x, y)$ holds. If, instead of (1.7), we assume that there exist constants $\delta \in (0, 1)$ and $C_6 > 0$ such that the function ψ in (1.1) has the property

$$\psi(\lambda r) \leq C_6 \lambda^\delta \psi(r) \qquad \text{for } \lambda \geq 1 \text{ and } r \geq 1, \tag{1.14}$$

we can establish the following upper bound on the Dirichlet heat kernel in terms of $p(t, x, y)$ and the boundary decay terms.

Theorem 1.2 *Suppose that X is a subordinate Brownian motion with Lévy exponent $\phi(|\xi|^2)$ with ϕ being a complete Bernstein function satisfying (1.1), (1.2) and (1.14). Suppose that D is a $C^{1,1}$ open subset of $\mathbb{R}^d$ with characteristics (R_0, Λ_0).*

(i) *For every $T > 0$, there exists $c_1 = c_1(R_0, \Lambda_0, T, \psi, d) > 0$ such that for all $t \in (0, T]$ and all $x, y \in D$,*

$$p_D(t, x, y) \leq c_1 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right) p(t, x/4, y/4).$$

(ii) *If D is bounded, then for every $T > 0$, there exists $c_2 = c_2(\mathrm{diam}(D), R_0, \Lambda_0, d, T, \psi) \geq 1$ such that for all $(t, x, y) \in [T, \infty) \times D \times D$,*

$$c_2^{-1} e^{-\lambda_1 t} \delta_D(x)\delta_D(y) \leq p_D(t, x, y) \leq c_2 e^{-\lambda_1 t} \delta_D(x)\delta_D(y), \tag{1.15}$$

where $-\lambda_1 < 0$ is the largest eigenvalue of the generator of X^D.

By integrating the two-sided heat kernel estimates in Theorem 1.1 with respect to t, we can easily obtain sharp two-sided estimates on the Green function $G_D(x, y) := \int_0^\infty p_D(t, x, y)dt$. For this, let

$$g_D(x, y) := \begin{cases} \frac{1}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) & \text{when } d \geq 3, \\ \log\left(1 + \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right) & \text{when } d = 2, \\ \left(\delta_D(x)\delta_D(y)\right)^{1/2} \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|} & \text{when } d = 1. \end{cases} \tag{1.16}$$

Corollary 1.3 *Suppose that X is a subordinate Brownian motion with Lévy exponent $\phi(|\xi|^2)$ with ϕ being a complete Bernstein function satisfying (1.1) and (1.2). Suppose that D is a bounded $C^{1,1}$ open subset of $\mathbb{R}^d$ with characteristics (R_0, Λ_0).*

(i) *There exists $c_1 = c_1(\mathrm{diam}(D), R_0, \Lambda_0, \psi, d) > 0$ such that*

$$G_D(x, y) \geq c_1 \, g_D(x, y), \qquad x, y \in D.$$

(ii) *If D satisfies (1.7), then there exists $c_2 = c_2(\mathrm{diam}(D), R_0, \Lambda_0, \psi, C_1, C_2, C_3, d) > 0$ such that*

$$G_D(x, y) \leq c_2 \, g_D(x, y), \qquad x, y \in D.$$

We remark that even though D may be disconnected, in contrast with the Brownian motion case, the process X^D is always irreducible because X^D can jump from one component of D to another. Denote by $G_D^0(x, y)$ the Green function of Brownian motion in D. It is known (see [19]) that $G_D^0(x, y) \asymp g_D(x, y)$ when x and y are in the same component of D, and $G_D^0(x, y) = 0$ otherwise. Thus when D is a bounded $C^{1,1}$ connected open subset of $\mathbb{R}^d$, the estimates in Corollary 1.3 are exactly the same as those for Brownian motion, while our heat kernel estimates (Theorem 1.1(i)–(ii)) detect a short-time and short-distance region, precisely $t \leq |x-y|^2 \leq 1$ and $\delta_D(x) \wedge \delta_D(y) \geq \sqrt{t}$, where the jump part is the dominant term. When $\phi(\lambda) = \lambda + \lambda^{\alpha/2}$, Theorem 1.1 and Corollary 1.3 in particular recover the main results of [10].

Throughout this paper the constants r_0, R_0, λ_0, Λ_0, and C_i, $i = 1, \ldots, 6$, will be fixed. We use $c_1, c_2, \cdots$ to denote generic constants, whose exact values are not important and can change from one appearance to another. The labeling of the constants $c_1, c_2, \cdots$ starts anew in the statement of each result. We use $c(\alpha, \beta, ...)$ to indicate a positive constant that depends on the parameters α, β, ... Dependence on dimension d will not be explicitly mentioned. We will use dx to denote the Lebesgue measure in $\mathbb{R}^d$.

2 Lower bound estimate

In this section we derive the lower bound estimate on $p_D(t,x,y)$ when D is a $C^{1,1}$ open set such that the path distance in each connected component of D is comparable to the Euclidean distance. As a consequence, we also get the lower bound estimate on $p(t,x,y)$ in $\mathbb{R}^d$. We will use some relation between killed subordinate Brownian motions and subordinate killed Brownian motions. In this paper we always assume that X is a subordinate Brownian motion with Lévy exponent $\phi(|\xi|^2)$ with ϕ being a complete Bernstein function satisfying (1.1) and (1.2).

Let $\widetilde{S}_t$ be a subordinator whose Laplace exponent ψ is given by (1.1). Then $t + \widetilde{S}_t$ is a subordinator which has the same law as S_t. Assume that $\widetilde{S}_t$ is independent of the Brownian motion B in $\mathbb{R}^d$. Suppose that U is an open subset of $\mathbb{R}^d$. We denote by B^U the part process of B killed upon leaving U. The process $\{Z_t^U : t \geq 0\}$ defined by $Z_t^U = B_{t+\widetilde{S}_t}^U$ is called a subordinate killed Brownian motion in U. Let $q_U(t,x,y)$ be the transition density of Z^U. Denote by $\zeta^{Z,U}$ the lifetime of Z^U. Clearly, $Z_t^U = B_{t+\widetilde{S}_t}$ for every $t \in [0, \zeta^{Z,U})$. Therefore we have

$$p_U(t,z,w) \geq q_U(t,z,w) \quad \text{for } (t,z,w) \in (0,\infty) \times U \times U. \tag{2.1}$$

By [1, Proposition III.8], for every $b > 0$, there exist constants $T_0 > 0$ and $c > 0$ so that

$$\mathbb{P}(\widetilde{S}_t \leq b\,t) > c \quad \text{for } t \leq T_0.$$

Using the Markov property of $\widetilde{S}_t$, we can easily deduce that for every $b > 0$ and $T > 0$, there exists $c = c(b,T,\psi) > 0$ such that

$$\mathbb{P}(\widetilde{S}_t \leq b\,t) > c \quad \text{for } t \leq T. \tag{2.2}$$

These facts (with $b = 1$) will be used in the proof of the following lemma.

Lemma 2.1 *Suppose that D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ_0) such that the path distance in each connected component of D is comparable to the Euclidean distance with characteristic λ_0. For every constant $T > 0$, there exist positive constants $c_1 = c_1(R_0, \Lambda_0, \lambda_0, MT, \psi)$ and $c_2 = c_2(R_0, \Lambda_0, \lambda_0)$ such that for all $\lambda \in (0, M]$, $t \in (0, T]$ and x, y in the same connected component of $\sqrt{\lambda}D$,*

$$p_{\sqrt{\lambda}D}(\lambda t, x, y) \geq c_1 (\lambda t)^{-d/2} \left(1 \wedge \frac{\delta_{\sqrt{\lambda}D}(x)}{\sqrt{\lambda t}}\right) \left(1 \wedge \frac{\delta_{\sqrt{\lambda}D}(y)}{\sqrt{\lambda t}}\right) e^{-c_2 |x-y|^2/(\lambda t)}. \tag{2.3}$$

Proof. Suppose that $\lambda^{-1/2}x$ and $\lambda^{-1/2}y$ are in the same component, say U, of D. Let $\widetilde{p}_U(t,z,w)$ be the transition density of B^U. By [18, Theorem 3.3] (see also [31, Theorem 1.2]), there exist positive constants $c_1 = c_1(R_0, \Lambda_0, \lambda_0, T)$ and $c_2 = c_2(R_0, \Lambda_0, \lambda_0)$ such that for any $(s,z,w) \in (0, 2T] \times U \times U$,

$$\widetilde{p}_U(s,z,w) \geq c_1 \left(1 \wedge \frac{\delta_U(z)}{\sqrt{s}}\right) \left(1 \wedge \frac{\delta_U(w)}{\sqrt{s}}\right) s^{-d/2} e^{-c_2 |z-w|^2/s}. \tag{2.4}$$

(Although not explicitly mentioned in [18], a careful examination of the proofs in [18] reveals that the constants c_1 and c_2 in the above lower bound estimate can be chosen to depend only on $(R_0, \Lambda_0, \lambda_0, T)$ and $(R_0, \Lambda_0, \lambda_0)$, respectively.) By using the scaling property of Brownian motion, we get that, for every $\lambda > 0$, $t \in (0, T]$ and x, y in $\sqrt{\lambda}U$,

$$\widetilde{p}_{\sqrt{\lambda}U}(\lambda t, x, y) = \lambda^{-d/2} \widetilde{p}_U(t, \lambda^{-1/2}x, \lambda^{-1/2}y).$$

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Thus by (2.4),

$$
\begin{aligned}
\widetilde{p}_{\sqrt{\lambda}U}(\lambda t, x, y) \;&\geq\; c_1(\lambda t)^{-d/2}\left(1 \wedge \frac{\delta_U(\lambda^{-1/2}x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_U(\lambda^{-1/2}y)}{\sqrt{t}}\right)e^{-c_2|x-y|^2/(\lambda t)}\\
&=\; c_1(\lambda t)^{-d/2}\left(1 \wedge \frac{\delta_{\sqrt{\lambda}U}(x)}{\sqrt{\lambda t}}\right)\left(1 \wedge \frac{\delta_{\sqrt{\lambda}U}(y)}{\sqrt{\lambda t}}\right)e^{-c_2|x-y|^2/(\lambda t)}. \qquad (2.5)
\end{aligned}
$$

Now we assume $\lambda \in (0, M]$. Recall that $\widetilde{S}_t$ is independent of B and that $q_{\sqrt{\lambda}U}(t, x, y)$ is the transition density of $Z_t^{\sqrt{\lambda}U} = B_{t+\widetilde{S}_t}^{\sqrt{\lambda}U}$. Note that for every $0 < t \leq T$ and x, y in $\sqrt{\lambda}U$,

$$
q_{\sqrt{\lambda}U}(\lambda t, x, y) = \int_{\lambda t}^{\infty} \widetilde{p}_{\sqrt{\lambda}U}(s, x, y)\mathbb{P}(\lambda t + \widetilde{S}_{\lambda t} \in ds).
$$

So by (2.1), (2.2) and (2.5), for every $0 < t \leq T$, $\lambda \in (0, M]$ and x, y in $\sqrt{\lambda}U$,

$$
\begin{aligned}
p_{\sqrt{\lambda}D}(\lambda t, x, y) \;&\geq\; p_{\sqrt{\lambda}U}(\lambda t, x, y) \geq q_{\sqrt{\lambda}U}(\lambda t, x, y)\\
&\geq\; \int_{\lambda t}^{2\lambda t} \widetilde{p}_{\sqrt{\lambda}U}(s, x, y)\mathbb{P}(\lambda t + \widetilde{S}_{\lambda t} \in ds)\\
&=\; \int_0^{\lambda t} \widetilde{p}_{\sqrt{\lambda}U}(\lambda t + s, x, y)\mathbb{P}(\widetilde{S}_{\lambda t} \in ds)\\
&\geq\; c_3\left(1 \wedge \frac{\delta_{\sqrt{\lambda}U}(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_{\sqrt{\lambda}U}(y)}{\sqrt{t}}\right)(\lambda t)^{-d/2}e^{-c_2|x-y|^2/(\lambda t)}\mathbb{P}(\widetilde{S}_{\lambda t} \leq \lambda t)\\
&\geq\; c_4\left(1 \wedge \frac{\delta_{\sqrt{\lambda}U}(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_{\sqrt{\lambda}U}(y)}{\sqrt{t}}\right)(\lambda t)^{-d/2}e^{-c_2|x-y|^2/(\lambda t)}\\
&=\; c_4\left(1 \wedge \frac{\delta_{\sqrt{\lambda}D}(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_{\sqrt{\lambda}D}(y)}{\sqrt{t}}\right)(\lambda t)^{-d/2}e^{-c_2|x-y|^2/(\lambda t)}.
\end{aligned}
$$

$\square$

Remark 2.2 Note that the Brownian motion B in $\mathbb{R}^d$ with infinitesimal generator Δ has transition density

$$
\widetilde{p}(t, x, y) = (4\pi t)^{-d/2}e^{-\frac{|x-y|^2}{4t}}, \quad x, y \in \mathbb{R}^d, \ t > 0.
$$

Using this instead of (2.4) and an argument similar to (but easier than) the proof of Lemma 2.1 with $\lambda = 1$ we can get that, for any $T > 0$, there exists a positive constant $c = c(T, \psi)$ such that for all $t \in (0, T]$ and x, y in $\mathbb{R}^d$,

$$
p(t, x, y) \geq c_1 t^{-d/2}e^{-\frac{|x-y|^2}{4t}}.
$$

Lemma 2.3 *For any positive constants R and a, there exists $c = c(R, a, \psi) > 0$ such that for all $z \in \mathbb{R}^d$ and $r \in (0, R]$,*

$$
\inf_{y \in B(z,r)} \mathbb{P}_y\left(\tau_{B(z,2r)} > ar^2\right) \geq c.
$$

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Proof. By Lemma 2.1, we have

$$\inf_{y \in B(z,r)} \mathbb{P}_y \left(\tau_{B(z,2r)} > ar^2 \right) \geq \mathbb{P}_0 \left(\tau_{B(0,r)} > ar^2 \right) = \int_{B(0,r)} p_{B(0,r)}(ar^2, 0, y) dy$$

$$\geq \int_{B(0,r/2)} p_{B(0,r)}(ar^2, 0, y) dy$$

$$\geq \int_{B(0,r/2)} c_1 \left(ar^2 \right)^{-d/2} \left(1 \wedge \frac{\delta_{B(0,r)}(0)}{\sqrt{ar^2}} \right) \left(1 \wedge \frac{\delta_{B(0,r)}(y)}{\sqrt{ar^2}} \right) e^{-c_2 |y|^2/(ar^2)} dy$$

$$= c_2 \int_{B(0,1/2)} a^{-d/2} \left(1 \wedge \frac{\delta_{B(0,1)}(0)}{\sqrt{a}} \right) \left(1 \wedge \frac{\delta_{B(0,1)}(z)}{\sqrt{a}} \right) e^{-c_2 |z|^2/a} dz$$

$$= c_3(R, a, \psi) > 0.$$

$\square$

Recall that we assume that (1.2) holds. On the other hand, since ϕ is a complete Bernstein function, it follows from [26, Lemma 2.1] that there exists $c_1 > 1$ such that $\mu(t) \leq c_1 \mu(t+1)$ for every $t > 1$. Thus by [25, Proposition 13.3.5] and its proof, we have that for any $K > 0$, there exists $c_2 = c_2(K) > 1$ such that

$$j(r) \leq c_2 \, j(2r), \qquad \forall r \in (0, K], \tag{2.6}$$

and, there exists $c_3 > 1$ such that

$$j(r) \leq c_3 \, j(r+1), \qquad \forall r \geq 1. \tag{2.7}$$

Lemma 2.4 *Suppose that $R > 0$ and $b > 1$ are constants. Then there exists $c = c(R, b, \psi) > 0$ such that for all $r \in (0, R]$, $t \in [r^2/b, br^2]$ and $u, v \in \mathbb{R}^d$,*

$$p_{B(u,r) \cup B(v,r)}(t, u, v) \geq c \left(t^{-d/2} \wedge (tJ(u,v)) \right).$$

Proof. Let $r \in (0, R]$, $t \in [r^2/b, br^2]$ and $E = B(u,r) \cup B(v,r)$. If $|u - v| \leq r/2$, by Lemma 2.1 (with $T = b$, $\sqrt{\lambda} = r$ and $D = B(0,1)$) and (1.5),

$$p_E(t, u, v) \geq \inf_{|z| < r/2} p_{B(0,r)}(t, 0, z) = \inf_{|z| < r/2} p_{B(0,r)}(r^2(t/r^2), 0, z)$$

$$\geq c_1 t^{-d/2} \left(1 \wedge \frac{r}{\sqrt{t}} \right) \left(1 \wedge \frac{r}{2\sqrt{t}} \right) e^{-c_2 r^2/t} \geq c_3 t^{-d/2} \geq c_4(tJ(u,v) \wedge t^{-d/2}).$$

If $|u - v| \geq r/2$, we have by the strong Markov property and the Lévy system of X in (1.4) that

$$p_E(t, u, v) \geq \mathbb{E}_u \left[p_E(t - \tau_{B(u,r/8)}, X_{\tau_{B(u,r/8)}}, v) : \tau_{B(u,r/8)} < t, X_{\tau_{B(u,r/8)}} \in B(v, r/8) \right]$$

$$= \int_0^t \left(\int_{B(u,r/8)} p_{B(u,r/8)}(s, u, w) \left(\int_{B(v,r/8)} J(w, z) p_E(t - s, z, v) dz \right) dw \right) ds$$

$$\geq \left(\inf_{w \in B(u,r/8), \, z \in B(v,r/8)} J(w, z) \right) \int_0^t \mathbb{P}_u \left(\tau_{B(u,r/8)} > s \right) \left(\int_{B(v,r/8)} p_E(t - s, z, v) dz \right) ds$$

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$$\geq \mathbb{P}_u(\tau_{B(u,r/8)} > t) \left(\inf_{w \in B(u,r/8),\, z \in B(v,r/8)} J(w,z) \right) \int_0^t \int_{B(v,r/8)} p_{B(v,r/8)}(t-s,z,v)dzds$$

$$= \mathbb{P}_0(\tau_{B(0,r/8)} > t) \left(\inf_{w \in B(u,r/8),\, z \in B(v,r/8)} j(|w-z|) \right) \int_0^t \mathbb{P}_0(\tau_{B(0,r/8)} > s)ds$$

$$\geq t \left(\mathbb{P}_0(\tau_{B(0,r/8)} > br^2) \right)^2 \left(\inf_{w \in B(u,r/8),\, z \in B(v,r/8)} j(|w-z|) \right)$$

$$\geq c_5 t \left(\inf_{w \in B(u,r/8),\, z \in B(v,r/8)} j(|w-z|) \right).$$

In the last inequality we have used Lemma 2.3. Note that, if $w \in B(u,r/8)$ and $z \in B(v,r/8)$, then

$$|w-z| \leq |u-w| + |u-v| + |v-z| \leq |u-v| + \frac{r}{4} \leq (2|u-v|) \wedge (|u-v| + \frac{R}{4}).$$

Thus using both (2.6) and (2.7) we have

$$p_E(t,u,v) \geq c_6 t j(|u-v|) \geq c_6(tJ(u,v) \wedge t^{-d/2}).$$

The proof is now complete. $\qquad\square$

The next lemma in particular implies that if x and y are in different components of D, the jumping kernel component of the heat kernel dominates the Gaussian component.

Lemma 2.5 *For any given positive constants c_1, c_2, R and T, there is a positive constant $c_3 = c_3(R, T, c_1, c_2, \psi)$ so that*

$$t^{-d/2}e^{-r^2/(c_1 t)} \leq c_3(t^{-d/2} \wedge (tj(c_2 r))) \qquad \textit{for every } r \geq R \textit{ and } t \in (0,T]. \tag{2.8}$$

Proof. Observe that (2.7) implies that there exist $c_4 > 0$ and $c_5 > 0$ such that

$$j(c_2 r) \geq c_4 e^{-c_5 r} \quad \text{for every } r > 1/c_2. \tag{2.9}$$

For $r > (1/c_2) \vee (2c_1 c_5 T)$ and $t \in (0,T]$, we have $r^2/(2c_1 t) > c_5 r$ and

$$\begin{aligned}
t^{-d/2-1}e^{-r^2/(2c_1 t)} &\leq t^{-d/2-1}e^{-((1/c_2)\vee(2c_1 c_5 T))^2/(2c_1 t)} \\
&\leq \sup_{0<s\leq T} s^{-d/2-1}e^{-((1/c_2)\vee(2c_1 c_5 T))^2/(2c_1 s)} =: c_6 < \infty.
\end{aligned}$$

So by (2.9), when $r > (1/c_2) \vee (2c_1 c_5 T)$ and $t \in (0,T]$, we have

$$t^{-d/2}e^{-r^2/(c_1 t)} \leq c_6 t e^{-r^2/(2c_1 t)} \leq c_6 t e^{-c_5 r} \leq (c_6/c_4)\, t\, j(c_2 r). \tag{2.10}$$

When $R \leq r \leq (1/c_2) \vee (2c_1 c_5 T)$ and $t \in (0,T]$, clearly

$$t^{-d/2}e^{-r^2/(c_1 t)} \leq t \left(\sup_{s\leq T} s^{-d/2-1}e^{-R^2/(c_1 s)} \right) \leq c_7 t j(c_2 r). \tag{2.11}$$

The desired inequality (2.8) now follows from (2.10) and (2.11). $\qquad\square$

Recall that the function $h_{D,c_1,c_2}(t,x,y)$ is defined in (1.13).

Theorem 2.6 *Suppose that D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ_0) such that the path distance in each connected component of D is comparable to the Euclidean distance with characteristic λ_0. For every $T > 0$, there exist $c_1 = c_1(R_0, \Lambda_0, \lambda_0, T, \psi) > 0$ and $c_2 = c_2(R_0, \Lambda_0, \lambda_0) > 0$ such that for all $(t, x, y) \in (0, T] \times D \times D$,*

$$p_D(t, x, y) \geq c_1 h_{D, c_2, 1}(t, x, y). \tag{2.12}$$

Proof. First note that the distance between two distinct connected components of D is at least R_0. Hence in view of Lemmas 2.1 and 2.5, we only need to show that there exists $c = c(R_0, \Lambda_0, \lambda_0, T, \psi) > 0$ such that for all $(t, x, y) \in (0, T] \times D \times D$,

$$p_D(t, x, y) \geq c \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_D(y)}{\sqrt{t}} \wedge 1 \right) ((tJ(x, y)) \wedge t^{-d/2}). \tag{2.13}$$

Since D is a $C^{1,1}$ open set, as mentioned earlier, it satisfies the uniform interior and uniform exterior ball conditions with radius $r_0 = r_0(R_0, \Lambda_0) \in (0, R_0]$. Set $T_0 = (r_0/4)^2$. Consequently, there exists $L = L(r_0) > 1$ such that, for all $t \in (0, T_0]$ and $x, y \in D$, we can choose $\xi_x^t \in D \cap B(x, L\sqrt{t})$ and $\xi_y^t \in D \cap B(y, L\sqrt{t})$ so that $B(\xi_x^t, 2\sqrt{t})$ and $B(\xi_y^t, 2\sqrt{t})$ are subsets of the connected components of D that contains x and y, respectively.

We first consider the case $t \in (0, T_0]$. Note that for $u \in B(\xi_x^t, \sqrt{t})$, we have

$$\delta_D(u) \geq \sqrt{t} \quad \text{and} \quad |x - u| \leq |x - \xi_x^t| + |\xi_x^t - u| \leq L\sqrt{t} + \sqrt{t} = (L + 1)\sqrt{t}.$$

Thus by (2.3) (with $\lambda = 1$), for $t \in (0, T_0]$,

$$\begin{aligned}
\int_{B(\xi_x^t, \sqrt{t})} p_D(t/3, x, u) du &\geq c_1 \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) \int_{B(\xi_x^t, \sqrt{t})} \left(\frac{\delta_D(u)}{\sqrt{t}} \wedge 1 \right) t^{-d/2} e^{-c_2 |x - u|^2 / t} du \\
&\geq c_1 \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) t^{-d/2} e^{-c_2(L+1)^2} |B(\xi_x^t, \sqrt{t})| \geq c_3 \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right).
\end{aligned} \tag{2.14}$$

Similarly, for $t \in (0, T_0]$,

$$\int_{B(\xi_y^t, \sqrt{t})} p_D(t/3, y, u) du \geq c_3 \left(\frac{\delta_D(y)}{\sqrt{t}} \wedge 1 \right). \tag{2.15}$$

By the semigroup property, for $t \in (0, T_0]$,

$$p_D(t, x, y) \geq \int_{B(\xi_x^t, \sqrt{t})} \int_{B(\xi_y^t, \sqrt{t})} p_D(t/3, x, u) p_D(t/3, u, v) p_D(t/3, v, y) du dv. \tag{2.16}$$

We consider the cases $|x - y| \geq \sqrt{t}/8$ and $|x - y| < \sqrt{t}/8$ separately.

Case 1: Suppose $|x - y| \geq \sqrt{t}/8$ and $t \in (0, T_0]$. Note that by (2.16), Lemma 2.4, symmetry and (2.14)–(2.15),

$$\begin{aligned}
&p_D(t, x, y) \\
&\geq \int_{B(\xi_y^t, \sqrt{t})} \int_{B(\xi_x^t, \sqrt{t})} p_D(t/3, x, u) p_{B(u, \sqrt{t}/2) \cup B(v, \sqrt{t}/2)}(t/3, u, v) p_D(t/3, v, y) du dv
\end{aligned}$$

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$$\geq c_4 \int_{B(\xi_y^t, \sqrt{t})} \int_{B(\xi_x^t, \sqrt{t})} p_D(t/3, x, u)(t^{-d/2} \wedge (tJ(u,v))) p_D(t/3, v, y) du dv$$

$$\geq c_4 \left(\inf_{(u,v) \in B(\xi_x^t, \sqrt{t}) \times B(\xi_y^t, \sqrt{t})} (t^{-d/2} \wedge (tJ(u,v))) \right) \int_{B(\xi_y^t, \sqrt{t})} \int_{B(\xi_x^t, \sqrt{t})} p_D(t/3, x, u) p_D(t/3, v, y) du dv$$

$$\geq c_4 c_3^2 \left(\inf_{(u,v) \in B(\xi_x^t, \sqrt{t}) \times B(\xi_y^t, \sqrt{t})} (t^{-d/2} \wedge (tJ(u,v))) \right) \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_D(y)}{\sqrt{t}} \wedge 1 \right). \tag{2.17}$$

Since $|x - y| \geq \sqrt{t}/8$, we have that for $(u, v) \in B(\xi_x^t, \sqrt{t}) \times B(\xi_y^t, \sqrt{t})$,

$$|u - v| \leq |u - \xi_x^t| + |\xi_x^t - x| + |x - y| + |y - \xi_y^t| + |\xi_y^t - v|$$
$$\leq 2(1 + L)\sqrt{t} + |x - y| \leq \left(16(1 + L)|x - y| \right) \wedge \left(2(1 + L)\sqrt{T_0} + |x - y| \right),$$

thus using (2.6) and (2.7) we have

$$\inf_{(u,v) \in B(\xi_x^t, \sqrt{t}) \times B(\xi_y^t, \sqrt{t})} (t^{-d/2} \wedge (tJ(u,v))) \geq c_5 (t^{-d/2} \wedge (tJ(x,y))). \tag{2.18}$$

Thus combining with (2.17) and (2.18), we conclude that, for $|x - y| \geq \sqrt{t}/8$,

$$p_D(t, x, y) \geq c_6 \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_D(y)}{\sqrt{t}} \wedge 1 \right) ((tJ(x,y)) \wedge t^{-d/2}). \tag{2.19}$$

Case 2: Suppose $|x - y| < \sqrt{t}/8$ and $t \in (0, T_0]$. Then for $(u, v) \in B(\xi_x^t, \sqrt{t}) \times B(\xi_y^t, \sqrt{t})$,

$$|u - v| \leq 2(1 + L)\sqrt{t} + |x - y| \leq (2(1 + L) + 8^{-1})\sqrt{t}.$$

Thus by (2.3), we have for every $(u, v) \in B(\xi_x^t, \sqrt{t}) \times B(\xi_y^t, \sqrt{t})$,

$$p_D(t/3, u, v) \geq c_7 \left(\frac{\delta_D(u)}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_D(v)}{\sqrt{t}} \wedge 1 \right) t^{-d/2} e^{-c_8 |u-v|^2/t} \geq c_9 t^{-d/2}.$$

Therefore by (2.14)–(2.16), for $t \leq T_0$,

$$\begin{aligned} p_D(t, x, y) &\geq c_9 c_3^2 \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_D(y)}{\sqrt{t}} \wedge 1 \right) t^{-d/2} \\ &\geq c_9 c_3^2 \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_D(y)}{\sqrt{t}} \wedge 1 \right) ((tJ(x,y)) \wedge t^{-d/2}). \end{aligned} \tag{2.20}$$

Combining (2.19) and (2.20) we get (2.13) for $t \in (0, T_0]$. When $T > T_0$ and $t \in (T_0, T]$, observe that $T_0/3 \leq t - 2T_0/3 \leq T - 2T_0/3 \leq (T/T_0 - 2/3)T_0$, that is, $t - 2T_0/3$ is comparable to $T_0/3$ with some universal constants that depend only on T and T_0. Using the inequality

$$p_D(t, x, y) \geq \int_{B(\xi_x^{T_0}, \sqrt{T_0})} \int_{B(\xi_y^{T_0}, \sqrt{T_0})} p_D(T_0/3, x, u) p_D(t - 2T_0/3, u, v) p_D(T_0/3, v, y) du dv \tag{2.21}$$

instead of (2.16) and by considering the cases $|x - y| \geq \sqrt{T_0}/8$ and $|x - y| < \sqrt{T_0}/8$ separately, we deduce by the same argument as above that (2.13) holds for $t \in [T_0, T]$ and hence for $t \in (0, T]$. □

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Remark 2.7 By Lemma 2.4, we have that for every $T > 0$ there is a positive constant $c_1 = c_1(\psi, T)$ such that for all $(t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$p(t, x, y) \geq p_{B(x, \sqrt{t}) \cup B(y, \sqrt{t})}(t, x, y) \geq c_1 \left(t^{-d/2} \wedge (tJ(x, y)) \right). \tag{2.22}$$

Together with Remark 2.2, (2.22) yields the following global lower bound on $p(t, x, y)$: For every $T > 0$, there is a positive constant $c_2 = c_2(\psi, T)$ such that for all $(t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$p(t, x, y) \geq c_2 \left(t^{-d/2} e^{-|x-y|^2/(4t)} + t^{-d/2} \wedge (tJ(x, y)) \right). \tag{2.23}$$

3 Upper bound estimate

In this section, we derive the upper bound estimate on $p_D(t, x, y)$ for $C^{1,1}$ open sets satisfying the assumption (1.7). We first record a lemma, Lemma 3.1, which serves as the starting point for the upper bound estimate. Applying it and using (1.7) for $p_D(t, x, y)$ on the right hand side of (3.2), we can get an intermediate upper bound estimate for $p_D(t, x, y)$ that has one boundary decay factor. This is done in Proposition 3.2. Applying Lemma 3.1 again but now using the intermediate upper bound estimate for $p_D(t, x, y)$ obtained in Proposition 3.2 on the right hand side of (3.1), we can get the desired short time sharp upper bound estimate for $p_D(t, x, y)$. This is carried out in the proof of Theorem 1.1(ii). Recall that X is a subordinate Brownian motion with Lévy exponent $\phi(|\xi|^2)$ with ϕ being a complete Bernstein function satisfying (1.1) and (1.2).

Lemma 3.1 *Suppose that U_1, U_3, E are open subsets of $\mathbb{R}^d$ with $U_1, U_3 \subset E$ and $\mathrm{dist}(U_1, U_3) > 0$. Let $U_2 := E \setminus (U_1 \cup U_3)$. If $x \in U_1$ and $y \in U_3$, then for every $t > 0$,*

$$p_E(t, x, y) \leq \mathbb{P}_x \left(X_{\tau_{U_1}} \in U_2 \right) \left(\sup_{s < t,\, z \in U_2} p_E(s, z, y) \right)$$

$$+ \int_0^t \mathbb{P}_x(\tau_{U_1} > s) \mathbb{P}_y(\tau_E > t - s) ds \left(\sup_{u \in U_1,\, z \in U_3} J(u, z) \right) \tag{3.1}$$

$$\leq \mathbb{P}_x \left(X_{\tau_{U_1}} \in U_2 \right) \left(\sup_{s < t,\, z \in U_2} p(s, z, y) \right) + (t \wedge \mathbb{E}_x [\tau_{U_1}]) \left(\sup_{u \in U_1,\, z \in U_3} J(u, z) \right). \tag{3.2}$$

Proof. The proof is similar to that of [10, Lemma 3.4]. For the reader's convenience, we spell out the details here. Using the strong Markov property of X, we have

$$
\begin{aligned}
p_E(t, x, y) &= \mathbb{E}_x \left[p_E \left(t - \tau_{U_1}, X_{\tau_{U_1}}, y \right) : \tau_{U_1} < t \right] \\
&= \mathbb{E}_x \left[p_E \left(t - \tau_{U_1}, X_{\tau_{U_1}}, y \right) : \tau_{U_1} < t, X_{\tau_{U_1}} \in U_2 \right] \\
&\quad + \mathbb{E}_x \left[p_E \left(t - \tau_{U_1}, X_{\tau_{U_1}}, y \right) : \tau_{U_1} < t, X_{\tau_{U_1}} \in U_3 \right] =: I + II.
\end{aligned}
$$

Clearly

$$I \leq \mathbb{P}_x \left(X_{\tau_{U_1}} \in U_2 \right) \left(\sup_{s < t,\, z \in U_2} p_E(s, z, y) \right) \leq \mathbb{P}_x \left(X_{\tau_{U_1}} \in U_2 \right) \left(\sup_{s < t,\, z \in U_2} p(s, z, y) \right).$$

On the other hand, by (1.4) and the symmetry,

$$
\begin{aligned}
II &= \int_0^t \left(\int_{U_1} p_{U_1}(s,x,u) \left(\int_{U_3} J(u,z) p_E(t-s,z,y)dz \right) du \right) ds \\
&\leq \left(\sup_{u \in U_1,\, z \in U_3} J(u,z) \right) \int_0^t \mathbb{P}_x(\tau_{U_1} > s) \left(\int_{U_3} p_E(t-s,z,y)dz \right) ds \\
&\leq \left(\sup_{u \in U_1,\, z \in U_3} J(u,z) \right) \int_0^t \mathbb{P}_x(\tau_{U_1} > s) \mathbb{P}_y(\tau_E > t-s) ds.
\end{aligned}
$$

Finally

$$
\int_0^t \mathbb{P}_x(\tau_{U_1} > s) \mathbb{P}_y(\tau_E > t-s) ds \leq \int_0^t \mathbb{P}_x(\tau_{U_1} > s) ds \leq t \wedge \mathbb{E}_x[\tau_{U_1}].
$$

This completes the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Recall that C_1, C_2 and C_3 are the constants in (1.7).

Proposition 3.2 *Suppose that D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). Assume that (1.7) holds. For every $T > 0$, there exists $c = c(C_1, C_3, R_0, \Lambda_0, T, \psi) > 0$ such that for all $t \in (0,T]$ and all $x, y \in D$,*

$$
p_D(t,x,y) \leq c \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left(t^{-d/2} e^{-|x-y|^2/(4C_2 t)} + (t^{-d/2} \wedge t j(|x-y|/(4 \vee 2C_3))) \right).
$$

Proof. There exists $r_0 = r_0(R_0, \Lambda_0) \in (0, R_0]$ such that D satisfies the uniform interior and uniform exterior ball conditions with radius r_0. Fix $T > 0$ and $t \in (0, T]$. Let $x, y \in D$. In view of (1.7), we only need to show the theorem for $\delta_D(x) < r_0\sqrt{t}/(16\sqrt{T}) \leq r_0/(16)$, which we will assume throughout the remainder of this proof. Choose $x_0 \in \partial D$ such that $\delta_D(x) = |x - x_0|$. Let

$$
U_1 := B(x_0, r_0\sqrt{t}/(8\sqrt{T})) \cap D. \tag{3.3}
$$

Let $\mathbf{n}(x_0)$ be the unit inward normal of D at the boundary point x_0. Put

$$
x_1 = x_0 + \frac{r_0\sqrt{t}}{16\sqrt{T}} \mathbf{n}(x_0). \tag{3.4}
$$

Note that $\delta_D(x_1) = \frac{r_0\sqrt{t}}{16\sqrt{T}}$. Applying the boundary Harnack principle in [27] we get

$$
\mathbb{P}_x(X_{\tau_{U_1}} \in D \setminus U_1) \leq c_1 \mathbb{P}_{x_1}(X_{\tau_{U_1}} \in D \setminus U_1) \frac{\delta_D(x)}{\delta_D(x_1)} \leq c_1 \frac{16\sqrt{T}\delta_D(x)}{r_0\sqrt{t}}.
$$

Hence

$$
\mathbb{P}_x(X_{\tau_{U_1}} \in D \setminus U_1) \leq c_2 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right). \tag{3.5}
$$

By [27, Lemma 4.3],

$$
\mathbb{E}_x[\tau_{U_1}] \leq c_3 \sqrt{t}\, \delta_D(x). \tag{3.6}
$$

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Thus we have by (3.5) and (3.6),

$$\mathbb{P}_x\left(\tau_D > t/2\right) \leq \mathbb{P}_x\left(\tau_{U_1} > t/2\right) + \mathbb{P}_x\left(X_{\tau_{U_1}} \in D \setminus U_1\right)$$

$$\leq \left(\left(\frac{2}{t}\mathbb{E}_x\left[\tau_{U_1}\right]\right) \wedge 1\right) + \mathbb{P}_x\left(X_{\tau_{U_1}} \in D \setminus U_1\right) \leq c_4\left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right). \qquad (3.7)$$

Now we deal with two cases separately.

Case 1: $|x - y| \leq \left(\sqrt{2dC_2} \vee (r_0/\sqrt{T})\right)\sqrt{t}$. By the semigroup property, symmetry and (1.7),

$$
\begin{aligned}
p_D(t, x, y) &= \int_D p_D(t/2, x, z)p_D(t/2, z, y)dz \\
&\leq \left(\sup_{z,w \in D} p(t/2, z, w)\right)\int_D p_D(t/2, x, z)dz \\
&\leq 2^{1+d/2}C_1 t^{-d/2}\mathbb{P}_x(\tau_D > t/2) \\
&\leq c_4 2^{1+d/2}C_1 t^{-d/2}\left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right),
\end{aligned}
$$

where in the last line (3.7) is used. Since

$$|x - y|^2/(4C_2 t) \leq (d/2) \vee (r_0^2/(4C_2 T)) \leq (d/2) \vee (r_0^2/(4T)), \qquad (3.8)$$

we have

$$p_D(t, x, y) \leq c_4 2^{1+d/2}C_1 e^{(d/2)\vee(r_0^2/(4T))} t^{-d/2} e^{-|x-y|^2/(4C_2 t)}\left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right).$$

Case 2: $|x - y| \geq \left(\sqrt{2dC_2} \vee (r_0/\sqrt{T})\right)\sqrt{t}$. Let

$$U_3 := \{z \in D : |z - x| > |x - y|/2\} \quad \text{and} \quad U_2 := D \setminus (U_1 \cup U_3). \qquad (3.9)$$

Since

$$|z - x| > \frac{|x - y|}{2} \geq \frac{r_0\sqrt{t}}{2\sqrt{T}} \quad \text{for } z \in U_3,$$

$\text{dist}(U_1, U_3) > 0$ and, for $u \in U_1$ and $z \in U_3$,

$$|u - z| \geq |z - x| - |x - x_0| - |x_0 - u| \geq |z - x| - \frac{r_0\sqrt{t}}{4\sqrt{T}} \geq |z - x|/2 \geq |x - y|/4. \qquad (3.10)$$

Thus,

$$\sup_{u \in U_1,\, z \in U_3} J(u, z) \leq c \sup_{(u,z):|u-z| \geq \frac{1}{4}|x-y|} j(|u - z|) \leq c_3 j(|x - y|/4). \qquad (3.11)$$

If $z \in U_2$,

$$\frac{3}{2}|x - y| \geq |x - y| + |x - z| \geq |z - y| \geq |x - y| - |x - z| \geq \frac{|x - y|}{2}. \qquad (3.12)$$

We remark here that up to this point, we have not used assumption (1.7) yet in this proof.

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Observe that, for any $\beta > 0$, the function $f(s) := s^{-d/2}e^{-\beta/s}$ is increasing on the interval $(0, 2\beta/d]$. By (1.7), (3.12) and the observation that $t \leq |x - y|^2/(2dC_2)$,

$$
\begin{aligned}
\sup_{s \leq t,\, z \in U_2} p(s, z, y) &\leq C_1 \sup_{s \leq t,\, z \in U_2} \left(s^{-d/2}e^{-|z-y|^2/C_2 s} + s^{-d/2} \wedge sJ(z/C_3, y/C_3) \right) \\
&\leq C_1 \sup_{s \leq t,\, |z-y| \geq |x-y|/2} \left(s^{-d/2}e^{-|z-y|^2/C_2 s} + sJ(z/C_3, y/C_3) \right) \\
&\leq C_1 \sup_{s \leq t} s^{-d/2}e^{-|x-y|^2/(4C_2 s)} + C_1 t j(|x - y|/(2C_3)) \\
&\leq C_1 t^{-d/2}e^{-|x-y|^2/(4C_2 t)} + c_5(t^{-d/2} \wedge t j(|x - y|/(2C_3))), \qquad (3.13)
\end{aligned}
$$

where in the last line (1.5) is used. In fact, since $|x - y| \geq (r_0/\sqrt{T})\sqrt{t}$, by (1.5),

$$
\begin{aligned}
t j(|x - y|/(2C_3)) &\leq t j((|x - y| \wedge r_0)/(2C_3)) \leq c_6 \left(\frac{t}{|x - y|^2 \wedge r_0^2} \right)^{1+d/2} t^{-d/2} \\
&\leq c_6 (T/r_0^2)^{1+d/2}\, t^{-d/2}, \qquad (3.14)
\end{aligned}
$$

where $c_6 > 0$ depends only on C_3.

By the same argument as that used to get (3.5), we can apply the boundary Harnack principle in [27] to get

$$
\mathbb{P}_x(X_{\tau_{U_1}} \in U_2) \leq c_7 \mathbb{P}_{x_1}(X_{\tau_{U_1}} \in U_2) \frac{\delta_D(x)}{\delta_D(x_1)} \leq c_8 \frac{\delta_D(x)}{\sqrt{t}}. \qquad (3.15)
$$

Applying (3.2), (3.6), (3.11), (3.13) and (3.15), we obtain

$$
\begin{aligned}
&p_D(t, x, y) \\
&\leq c_9 \left(t^{-d/2}e^{-|x-y|^2/(4C_2 t)} + t^{-d/2} \wedge t j\left(|x - y|/(2C_3)\right) \right) \frac{\delta_D(x)}{\sqrt{t}} + c_{10}\, t j(|x - y|/4) \frac{\delta_D(x)}{\sqrt{t}} \\
&\leq c_{11} \left(t^{-d/2}e^{-|x-y|^2/(4C_2 t)} + t^{-d/2} \wedge t j(|x - y|/(2C_3 \vee 4)) \right) \frac{\delta_D(x)}{\sqrt{t}},
\end{aligned}
$$

where in the last line (1.5) is used (see (3.14)). This combined with (1.7) completes the proof of this proposition. $\qquad \square$

Proposition 3.3 *Suppose that D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). Assume that (1.7) holds. For every $T > 0$, there exists $c = c(C_1, C_3, R_0, \Lambda_0, T, \psi) > 0$ such that for all $t \in (0, T]$ and all $x \in D$*

$$
\mathbb{P}_x(\tau_D > t) \leq c \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right).
$$

Proof. Fix $T > 0$. By Proposition 3.2 and (2.23) we have that for every $0 < t \leq T$ and x, z in D,

$$
\begin{aligned}
p_D(t, x, z) &\leq c_1 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left(t^{-d/2}e^{-|x-z|^2/C_2(4t)} + t^{-d/2} \wedge (t j\left(|x - z|/(4 \vee 2C_3)\right)) \right) \\
&\leq c_3 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) p(t, c_2 x, c_2 z),
\end{aligned}
$$

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where $c_2 := (\sqrt{C_2} \vee 4 \vee (2C_3))^{-1}$. Thus

$$\mathbb{P}_x(\tau_D > t) = \int_D p_D(t, x, z)dz \leq c_3 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right) \int_D p(t, c_2 x, c_2 z)dz \leq c_3 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right).$$

$\square$

Proof of Theorem 1.1: (i). This has already been established in Theorem 2.6.

(ii). Fix $T > 0$. There exists $r_0 = r_0(R_0, \Lambda_0) \in (0, R_0]$ such that D satisfies the uniform interior and uniform exterior ball conditions with radius r_0. Let $t \in (0, T]$ and $x, y \in D$. By Proposition 3.2, (1.7) and symmetry, we only need to prove (ii) for $\delta_D(x) \vee \delta_D(y) < r_0\sqrt{t}/(16\sqrt{T}) \leq r_0/(16)$, which we will assume throughout the remainder of the proof of (ii).

The proof of (ii) is along the line of the proof of Proposition 3.2 but using the estimate from Proposition 3.2 for the upper bound estimate of $p_D(t, x, y)$ on the right hand side of (3.1) rather than using (1.7). Define U_1, x_0 and x_1 in the same way as in the proof of Proposition 3.2 (see (3.3)–(3.4)), and consider the following two cases separately.

Case 1: $|x - y| \leq \left(\sqrt{8(d+1)C_2} \vee (r_0/\sqrt{T})\right)\sqrt{t}$. By the semigroup property, symmetry and Proposition 3.2,

$$\begin{aligned}
p_D(t, x, y) &= \int_D p_D(t/2, x, z)p_D(t/2, z, y)dz \\
&\leq \left(\sup_{z \in D} p_D(t/2, y, z)\right) \int_D p_D(t/2, x, z)dz \\
&\leq c_1 t^{-d/2}\left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right)\mathbb{P}_x(\tau_D > t/2) \\
&\leq c_1 t^{-d/2}\left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right) \\
&\leq c_2 t^{-d/2}e^{-|x-y|^2/(16C_2 t)}\left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right), \quad (3.16)
\end{aligned}$$

where in the third inequality, Proposition 3.3 is used.

Case 2: $|x - y| \geq \left(\sqrt{8(d+1)C_2} \vee (r_0/\sqrt{T})\right)\sqrt{t}$. Define U_2 and U_3 as in (3.9). Note that (3.11) holds. Moreover, for $z \in U_2$, as in (3.12),

$$3|x - y|/2 \geq |z - y| \geq |x - y|/2. \quad (3.17)$$

Observe that, for any $\beta > 0$, the function $f(s) := s^{-(d+1)/2}e^{-\beta/s}$ is increasing on the interval $(0, 2\beta/(d+1)]$. By Proposition 3.2 (instead of (1.7)), (3.17) and the observation that $t \leq |x - y|^2/(8(d+1)C_2)$,

$$\begin{aligned}
\sup_{s \leq t, z \in U_2} p_D(s, z, y) &\leq c_3 \sup_{s \leq t, z \in U_2} \left(s^{-d/2}e^{-|z-y|^2/(4C_2 s)} + s^{-d/2} \wedge sj(|z - y|/(4 \vee 2C_3))\right)\frac{\delta_D(y)}{\sqrt{s}} \\
&\leq c_3\delta_D(y) \sup_{s \leq t, |z-y| \geq |x-y|/2} \left(s^{-(d+1)/2}e^{-|z-y|^2/(4C_2 s)} + \sqrt{s}j(|z - y|/(4 \vee 2C_3))\right) \\
&\leq c_3\delta_D(y)\left(\sup_{s \leq t} s^{-(d+1)/2}e^{-|x-y|^2/(16C_2 s)} + \sqrt{t}j(|x - y|/(8 \vee 4C_3))\right)
\end{aligned}$$

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$$\leq \quad c_4 \frac{\delta_D(y)}{\sqrt{t}} \left(t^{-d/2} e^{-|x-y|^2/(16C_2 t)} + (t^{-d/2} \wedge tj(|x-y|/(8 \vee 4C_3))) \right), \qquad (3.18)$$

where in the last line we used an argument similar to that in (3.14). On the other hand, by Proposition 3.3 we have

$$\int_0^t \mathbb{P}_x(\tau_{U_1} > s)\mathbb{P}_y(\tau_D > t-s)ds \leq \int_0^t \mathbb{P}_x(\tau_D > s)\mathbb{P}_y(\tau_D > t-s)ds \leq c_5 \int_0^t \frac{\delta_D(x)}{\sqrt{s}} \frac{\delta_D(y)}{\sqrt{t-s}}ds$$

$$= c_5 \delta_D(x)\delta_D(y) \int_0^1 \frac{1}{\sqrt{r(1-r)}}dr = c_6 \delta_D(x)\delta_D(y). \qquad (3.19)$$

Combining (3.1), (3.11), (3.15), (3.18) and (3.19) all together, we conclude that

$$p_D(t,x,y)$$
$$\leq \quad c_7 \left(t^{-d/2} e^{-|x-y|^2/(16C_2 t)} + (t^{-d/2} \wedge tj(|x-y|/(8 \vee 4C_3))) \right) \frac{\delta_D(x)\delta_D(y)}{t}$$
$$\quad + c_8 \, tj(|x-y|/4) \frac{\delta_D(x)\delta_D(y)}{t}$$
$$\leq \quad c_9 \left(t^{-d/2} e^{-|x-y|^2/(16C_2 t)} + (t^{-d/2} \wedge tj(|x-y|/(8 \vee 4C_3))) \right) \frac{\delta_D(x)\delta_D(y)}{t}$$
$$= \quad c_9 \left(t^{-d/2} e^{-|x-y|^2/(16C_2 t)} + (t^{-d/2} \wedge tj(|x-y|/(8 \vee 4C_3))) \right) \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right),$$

where in the second inequality (1.5) is used (see (3.14)). This combined with (3.16) and Proposition 3.2 completes the proof of (ii).

(iii) and (iv). We first note that the path distance condition is satisfied in any bounded $C^{1,1}$ open set D with λ_0 depending only on R_0, Λ_0 and $\mathrm{diam}(D)$. Thus by (i) and (ii), it suffices to prove (iii) and (iv) for $T = 3$.

In view of (1.7), the transition semigroup $\{P_t^D, t > 0\}$ of X^D consists of Hilbert-Schmidt operators, and hence compact operators, in $L^2(D; dx)$. So P_t^D has discrete spectrum $\{e^{-\lambda_k t}; k \geq 1\}$ arranged in decreasing order and repeated according to their multiplicity. Let $\{\phi_k, k \geq 1\}$ be the corresponding eigenfunctions with unit L^2-norm, which forms an orthonormal basis for $L^2(D; dx)$. Clearly,

$$\int_D (1 \wedge \delta_D(x)) \, \phi_1(x) dx \leq |D|^{1/2} \|\phi_1\|_{L^2(D)} \leq |D|^{1/2}. \qquad (3.20)$$

By using the eigenfunction expansion of $p_D(t,x,y) = \sum_{k=1}^{\infty} e^{-\lambda_k t}\phi_k(x)\phi_k(y)$, we get

$$\int_{D \times D} (1 \wedge \delta_D(x)) \, p_D(t,x,y) \, (1 \wedge \delta_D(y)) \, dxdy = \sum_{k=1}^{\infty} e^{-t\lambda_k} \left(\int_D (1 \wedge \delta_D(x)) \, \phi_k(x) dx \right)^2. \qquad (3.21)$$

Noting that λ_k is increasing and $\|f\|_2^2 = \sum_{k=1}^{\infty} (\int_D f(z)\phi_k(z)dz)^2$, we have for all $t > 0$,

$$\int_{D \times D} (1 \wedge \delta_D(x)) \, p_D(t,x,y) \, (1 \wedge \delta_D(y)) \, dxdy \leq e^{-t\lambda_1} \int_D (1 \wedge \delta_D(x))^2 \, dx \leq e^{-t\lambda_1} |D|. \qquad (3.22)$$

On the other hand, by Theorem 1.1(ii) and (3.20), there exists $c_1 > 0$ so that for every $x \in D$,

$$\phi_1(x) \quad = \quad e^{\lambda_1} \int_D p_D(1,x,y)\phi_1(y)dy$$

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$$\leq \ c_1 e^{\lambda_1} \left(1 \wedge \delta_D(x)\right) \int_D \left(1 \wedge \delta_D(y)\right) \phi_1(y) dy \ \leq \ c_1 e^{\lambda_1} |D|^{1/2} \left(1 \wedge \delta_D(x)\right). \qquad (3.23)$$

It now follows from (3.21) that for every $t > 0$,

$$\int_{D \times D} \left(1 \wedge \delta_D(x)\right) p_D(t, x, y) \left(1 \wedge \delta_D(y)\right) \, dx dy$$

$$\geq \ e^{-t\lambda_1} \left(\int_D \left(1 \wedge \delta_D(x)\right) \phi_1(x) dx \right)^2$$

$$\geq \ e^{-t\lambda_1} \left(\int_D (c_1 e^{\lambda_1} |D|^{1/2})^{-1} \phi_1(x)^2 dx \right)^2 = c_1^{-2} |D|^{-1} e^{-(t+2)\lambda_1}. \qquad (3.24)$$

For $t \geq 3$ and $x, y \in D$, we have that

$$p_D(t, x, y) = \int_{D \times D} p_D(1, x, z) p_D(t - 2, z, w) p_D(1, w, y) dz dw. \qquad (3.25)$$

By Theorem 1.1(ii) and (3.22), there exist $c_i > 0$, $i = 2, 3$, so that for every $t \geq 3$ and $x, y \in D$,

$$p_D(t, x, y)$$

$$\leq \ c_2 \left(1 \wedge \delta_D(x)\right) \left(1 \wedge \delta_D(y)\right) \int_{D \times D} \left(1 \wedge \delta_D(z)\right) p_D(t - 2, z, w) \left(1 \wedge \delta_D(w)\right) dz dw$$

$$\leq \ c_2 |D| e^{-\lambda_1(t-2)} \left(1 \wedge \delta_D(x)\right) \left(1 \wedge \delta_D(y)\right) \ \leq c_3 e^{-\lambda_1 t} \left(1 \wedge \delta_D(x)\right) \left(1 \wedge \delta_D(y)\right). \qquad (3.26)$$

By (3.25), Theorem 2.6, the boundedness of D and (3.24) we have that there exist $c_i > 0$, $i = 4, 5$, so that for every $t \geq 3$ and $x, y \in D$,

$$p_D(t, x, y)$$

$$\geq \ c_4 \left(1 \wedge j(\mathrm{diam}(D))\right)^2 \left(1 \wedge \delta_D(x)\right) \left(1 \wedge \delta_D(y)\right) \int_{D \times D} \left(1 \wedge \delta_D(z)\right) p_D(t - 2, z, w) \left(1 \wedge \delta_D(w)\right) dz dw$$

$$\geq \ c_5 \left(1 \wedge j(\mathrm{diam}(D))\right)^2 |D|^{-1} \left(1 \wedge \delta_D(x)\right) \left(1 \wedge \delta_D(y)\right) e^{-t\lambda_1} = c_6 \left(1 \wedge \delta_D(x)\right) \left(1 \wedge \delta_D(y)\right) e^{-t\lambda_1}.$$

The theorem is now proved. $\qquad \square$

Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form of X on $L^2(\mathbb{R}^d; dx)$. It is known that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d; dx)$ with core $C_c^1(\mathbb{R}^d)$; see [4]. Moreover, for $u \in C_c^1(\mathbb{R}^d)$,

$$\mathcal{E}(u, u) := \int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla v(x) dx + \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))^2 J(x, y) dx dy \qquad (3.27)$$

and $\mathcal{F} := \overline{C_c^1(\mathbb{R}^d)}^{\mathcal{E}_1} \subset W^{1,2}(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d; dx) : \mathcal{E}(f, f) < \infty\}$. So we have the following Nash's inequality:

$$\|f\|_2^{2+4/d} \leq c_1 \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx \cdot \|f\|_1^{4/d} \leq c_2 \, \mathcal{E}(f, f) \|f\|_1^{4/d} \qquad \text{for } f \in \mathcal{F}. \qquad (3.28)$$

It follows then

$$p(t, x, y) \leq c_3 t^{-d/2} \quad \text{for } t > 0 \text{ and } x, y \in \mathbb{R}^d. \qquad (3.29)$$

Proof of Theorem 1.2. (i) There exists $r_0 = r_0(R_0, \Lambda_0) \in (0, R_0]$ so that D satisfies the uniform interior and uniform exterior ball conditions with radius r_0. Fix $T > 0$. We claim that there is a constant $c_0 > 0$ so that

$$p_D(t, x, y) \le c_0 p(t, |x - y|/4) \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) \quad \text{for every } (t, x, y) \in (0, T] \times D \times D. \quad (3.30)$$

In view of (1.6), $p_D(t, x, y) \le p(t, |x - y|) \le p(t, |x - y|/4)$. So it suffices to prove (3.30) when $\delta_D(x) < r_0\sqrt{t}/(16\sqrt{T}) \le r_0/16$. Define U_1, x_0 and x_1 in the same way as in the proof of Proposition 3.2 (see (3.3)–(3.4)). Hence (3.5)–(3.7) and (3.15) hold. We now prove (3.30) by considering the following two cases.

Case 1: $|x - y| \le (r_0/\sqrt{T})\sqrt{t}$. By the semigroup property, symmetry and (3.29),

$$
\begin{aligned}
p_D(t, x, y) &= \int_D p_D(t/2, x, z) p_D(t/2, z, y) dz \\
&\le \left(\sup_{z, w \in D} p(t/2, z, w) \right) \int_D p_D(t/2, x, z) dz \\
&\le c_1 t^{-d/2} \mathbb{P}_x(\tau_D > t/2) \le c_2 t^{-d/2} \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right),
\end{aligned}
$$

where in the last line (3.7) is used. Since $|x - y|^2/(64t) \le r_0^2/(64T)$, we have by Remark 2.7,

$$p_D(t, x, y) \le c_2 e^{r_0^2/(64T)} t^{-d/2} e^{-|x-y|^2/(64t)} \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \le c_3 p(t, |x - y|/4) \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right).$$

Case 2: $|x - y| \ge (r_0/\sqrt{T})\sqrt{t}$. Define U_2 and U_3 as in (3.9). Note that (3.11) and (3.12) hold. Observe that by (1.6)

$$\sup_{s \le t, \, z \in U_2} p(s, z, y) \le \sup_{s \le t, \, |z-y| \ge |x-y|/2} p(s, z, y) \le \sup_{s \le t} p(s, |x - y|/2). \quad (3.31)$$

By [28, Lemma 3.1], (1.14) implies

$$j(r) \le c_2 r^{-d} \psi(r^{-2}) \le c_1 c_2 \psi(1) r^{-d-2\delta} \quad \text{for all } r \in (0, 1].$$

Thus under assumption (1.14), according to [16, Theorem 1.3], the parabolic Harnack inequality holds for the subordinate Brownian motion X. Extend the definition of $p(t, r)$ by setting $p(t, r) = 0$ for $t < 0$ and $r \ge 0$. For each fixed $x, y \in \mathbb{R}^d$ and $t > 0$ with $|x - y| \ge (r_0/\sqrt{T})\sqrt{t}$, one can easily check that $(s, w) \mapsto p(s, |w - y|/2)$ is a parabolic function in $(-\infty, T] \times B(x, (r_0/\sqrt{T})\sqrt{t}/4)$. So by the parabolic Harnack inequality from [16, Theorem 1.3], there is a constant $c_3 = c_3(\psi) \ge 1$ so that for every $t \in (0, T]$,

$$\sup_{s \le t} p(s, |x - y|/2) \le c_3 p(t, |x - y|/2).$$

Hence we have

$$\sup_{s \le t, \, z \in U_2} p(s, z, y) \le c_3 p(t, |x - y|/2). \quad (3.32)$$

Applying (3.2), (3.6), (3.11), (3.15), (3.32), we obtain

$$p_D(t, x, y) \le \mathbb{P}_x\left(X_{\tau_{U_1}} \in U_2 \right) \left(\sup_{s < t, \, z \in U_2} p(s, z, y) \right) + \mathbb{E}_x[\tau_{U_1}] \left(\sup_{u \in U_1, \, z \in U_3} J(u, z) \right)$$

$$\leq c_5 p(t, |x-y|/2) \frac{\delta_D(x)}{\sqrt{t}} + c_5 \, tj(|x-y|/4) \frac{\delta_D(x)}{\sqrt{t}}. \tag{3.33}$$

Since $|x-y| \geq r_0 \sqrt{t}/\sqrt{T}$, by (1.5),

$$
\begin{aligned}
tj(|x-y|/4) &\leq tj((|x-y| \wedge r_0)/4) \leq c_4 \left(\frac{t}{|x-y|^2 \wedge r_0^2} \right)^{1+d/2} t^{-d/2} \\
&\leq c_4 (T/r_0^2)^{1+d/2} \, t^{-d/2}. \tag{3.34}
\end{aligned}
$$

Thus by the monotonicity of the transition density and (3.33) and (3.34) we obtain

$$
\begin{aligned}
p_D(t,x,y) &\leq c_5 p(t, |x-y|/4) \frac{\delta_D(x)}{\sqrt{t}} + c_6 \left(t^{-d/2} \wedge tj(|x-y|/4) \right) \frac{\delta_D(x)}{\sqrt{t}} \\
&\leq c_7 p(t, |x-y|/4) \frac{\delta_D(x)}{\sqrt{t}} \\
&\leq c_7 p(t, |x-y|/4) \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right),
\end{aligned}
$$

where in second inequality (2.22) is used.

Combining these two cases establishes the claim (3.30). Thus by the semigroup property and the symmetry of $p_D(t,x,y)$ in x and y, we conclude from (3.30) that for every $t \in (0,T]$ and $x,y \in D$,

$$
\begin{aligned}
p_D(t,x,y) &= \int_D p_D(t/2,x,z) p_D(t/2,z,y) dz \\
&\leq c_7^2 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) \int_D p(t/2, |x-z|/4) p(t/2, |z-y|/4) dz \\
&\leq c_7^2 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) \int_{\mathbb{R}^d} p(t/2, |x-z|/4) p(t/2, |z-y|/4) dz \\
&\leq c_8 \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) p(t, |x-y|/4).
\end{aligned}
$$

(ii) The lower bound in (1.15) is Theorem 1.1(iii). The proof of the upper bound in (1.15) is the same as that of Theorem 1.1(iv), the only difference is that we use part (i) of this theorem and (3.29), instead of Theorem 1.1(ii), so that $p_D(1,x,z) \leq c_1(1 \wedge \delta_D(x))(1 \wedge \delta_D(z))$ and $p_D(1,w,y) \leq c_1(1 \wedge \delta_D(w))(1 \wedge \delta_D(y))$. $\qquad \square$

4 Green function estimates

In this section we give the proof of Corollary 1.3.

Proof of Corollary 1.3: Put $T := \text{diam}(D)^2$. Recall that $g_D(x,y)$ is defined in (1.16). By an argument similar to that for [7, Corollary 1.2], one gets that (see the proofs of [29, Theorem 5.0.8] (for $d = 1,2$) and [24, Theorem 6.2] (for $d \geq 3$) for details)

$$\int_0^T \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) t^{-d/2} e^{-c_1 |x-y|^2/t} dt + \int_T^\infty e^{-\lambda_1 t} \delta_D(x) \delta_D(y) dt \asymp g_D(x,y).$$

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Thus, since D is bounded, by Theorem 1.1 (i) and (iii), we have $G_D(x, y) \geq c_2 g_D(x, y)$, which proves Corollary 1.3(i).

When the bounded $C^{1,1}$ open set D satisfies (1.7), by (1.5) and Theorem 1.1 (ii) and (iv), we have

$$G_D(x, y) \leq c_3 \left(g_D(x, y) + \int_0^T \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) \left(t^{-d/2} \wedge \frac{t}{|x-y|^{d+2}} \right) dt \right).$$

Therefore to prove Corollary 1.3(ii) it suffices to show that

$$\int_0^T \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) \left(t^{-d/2} \wedge \frac{t}{|x-y|^{d+2}} \right) dt \leq c_4 g_D(x, y). \tag{4.1}$$

By the change of variable $u = \frac{|x-y|^2}{t}$, we have

$$\int_0^T \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}} \right) \left(t^{-d/2} \wedge \frac{t}{|x-y|^{d+2}} \right) dt$$

$$= \frac{1}{|x-y|^{d-2}} \int_{|x-y|^2/T}^\infty \left(u^{d/2-2} \wedge u^{-3} \right) \left(1 \wedge \frac{\sqrt{u}\delta_D(x)}{|x-y|} \right) \left(1 \wedge \frac{\sqrt{u}\delta_D(y)}{|x-y|} \right) du. \tag{4.2}$$

Since for every $x, y \in D$ and $r > 0$,

$$\left(1 \wedge \frac{r\delta_D(x)}{|x-y|} \right) \left(1 \wedge \frac{r\delta_D(y)}{|x-y|} \right) \leq 1 \wedge \frac{r^2\delta_D(x)\delta_D(y)}{|x-y|^2}, \tag{4.3}$$

we have

$$\frac{1}{|x-y|^{d-2}} \int_1^\infty \left(u^{d/2-2} \wedge u^{-3} \right) \left(1 \wedge \frac{\sqrt{u}\delta_D(x)}{|x-y|} \right) \left(1 \wedge \frac{\sqrt{u}\delta_D(y)}{|x-y|} \right) du$$

$$= \frac{1}{|x-y|^{d-2}} \int_1^\infty u^{-2} \left(u^{-1/2} \wedge \frac{\delta_D(x)}{|x-y|} \right) \left(u^{-1/2} \wedge \frac{\delta_D(y)}{|x-y|} \right) du$$

$$\leq \frac{1}{|x-y|^{d-2}} \int_1^\infty u^{-2} \left(1 \wedge \frac{\delta_D(x)}{|x-y|} \right) \left(1 \wedge \frac{\delta_D(y)}{|x-y|} \right) du$$

$$\leq \frac{1}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \right). \tag{4.4}$$

(1) When $d \geq 3$, by (4.3), we have

$$\frac{1}{|x-y|^{d-2}} \int_{|x-y|^2/T}^1 \left(u^{d/2-2} \wedge u^{-3} \right) \left(1 \wedge \frac{\sqrt{u}\delta_D(x)}{|x-y|} \right) \left(1 \wedge \frac{\sqrt{u}\delta_D(y)}{|x-y|} \right) du$$

$$\leq \frac{1}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|} \right) \left(1 \wedge \frac{\delta_D(y)}{|x-y|} \right) \int_0^1 u^{d/2-2} du$$

$$\leq \frac{2}{d-2} \frac{1}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \right). \tag{4.5}$$

Combining (4.2), (4.4) and (4.5), we arrive at (4.1) for $d \geq 3$.

For the other cases, we define

$$u_0 := \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}. \tag{4.6}$$

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Clearly $1/u_0 \geq |x-y|^2/\mathrm{diam}(D)^2 = |x-y|^2/T$.

(2) Suppose $d = 2$. By (4.3) we have

$$\frac{1}{|x-y|^{d-2}} \int_{|x-y|^2/T}^1 \left(u^{d/2-2} \wedge u^{-3}\right) \left(1 \wedge \frac{\sqrt{u}\,\delta_D(x)}{|x-y|}\right) \left(1 \wedge \frac{\sqrt{u}\,\delta_D(y)}{|x-y|}\right) du$$

$$\leq \int_{|x-y|^2/T}^1 u^{-1} \left(1 \wedge \frac{u\,\delta_D(x)\delta_D(y)}{|x-y|^2}\right) du$$

$$= \int_{|x-y|^2/T}^1 u^{-1}\mathbf{1}_{\{u \geq 1/u_0\}} du + \int_{|x-y|^2/T}^1 u_0\mathbf{1}_{\{u < 1/u_0\}} du$$

$$= \log(u_0 \vee 1) + u_0\left(\frac{1}{u_0} \wedge 1 - \frac{|x-y|^2}{T}\right). \tag{4.7}$$

Thus by (4.2), (4.4) and (4.7),

$$\int_0^T \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right) \left(t^{-1} \wedge \frac{t}{|x-y|^4}\right) dt$$

$$\leq \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right) \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right) + \log(u_0 \vee 1) + u_0\left(\frac{1}{u_0} \wedge 1 - \frac{|x-y|^2}{T}\right)$$

$$\asymp 1 \wedge u_0 + \log(u_0 \vee 1) + u_0\left(\frac{1}{u_0} \wedge 1 - \frac{|x-y|^2}{T}\right)$$

$$\asymp 1 \wedge u_0 + \log(u_0 \vee 1) \asymp \log(1 + u_0) = \log\left(1 + \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right).$$

This proves (4.1) for $d = 2$.

(iii) Lastly we consider the case $d = 1$. By (4.3) and (4.6),

$$\frac{1}{|x-y|^{d-2}} \int_{|x-y|^2/T}^1 \left(u^{d/2-2} \wedge u^{-3}\right) \left(1 \wedge \frac{\sqrt{u}\,\delta_D(x)}{|x-y|}\right) \left(1 \wedge \frac{\sqrt{u}\,\delta_D(y)}{|x-y|}\right) du$$

$$\leq |x-y| \int_{|x-y|^2/T}^1 u^{-3/2} \left(1 \wedge \frac{u\,\delta_D(x)\delta_D(y)}{|x-y|^2}\right) du$$

$$= |x-y| \left(\int_{|x-y|^2/T}^1 u^{-3/2}\mathbf{1}_{\{u \geq 1/u_0\}} du + \int_{|x-y|^2/T}^1 u_0 u^{-1/2}\mathbf{1}_{\{u < 1/u_0\}} du\right)$$

$$= |x-y| \left(2\left((u_0 \vee 1)^{1/2} - 1\right) + 2u_0\left((u_0 \vee 1)^{-1/2} - \left(\frac{|x-y|^2}{T}\right)^{1/2}\right)\right).$$

Thus by (4.2), (4.4) and the last display, we have

$$\int_0^T \left(1 \wedge \frac{\delta_D(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)}{\sqrt{t}}\right) \left(t^{-1/2} \wedge \frac{t}{|x-y|^3}\right) dt$$

$$\leq |x-y|\,(1 \wedge u_0) + |x-y|\left(\left((u_0 \vee 1)^{1/2} - 1\right) + u_0\left((u_0 \vee 1)^{-1/2} - \left(\frac{|x-y|^2}{T}\right)^{1/2}\right)\right)$$

$$\asymp |x-y|\left(u_0^{1/2} \wedge u_0\right) = (\delta_D(x)\delta_D(y))^{1/2} \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|}.$$

This proves (4.1) for $d = 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

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Zhen-Qing Chen

Department of Mathematics, University of Washington, Seattle, WA 98195, USA

E-mail: zqchen@uw.edu

Panki Kim
Department of Mathematical Sciences and Research Institute of Mathematics,
Seoul National University, Building 27, 1 Gwanak-ro, Gwanak-gu Seoul 151-747,
Republic of Korea
E-mail: pkim@snu.ac.kr

Renming Song
Department of Mathematics, University of Illinois, Urbana, IL 61801, USA
E-mail: rsong@math.uiuc.edu