Optimal Transportation, Ricci Curvature and Diffusions on the L^2-Wasserstein Space

We introduce and analyze generalized Ricci curvature bounds for metric measure spaces (M, d, m), based on convexity properties of the relative entropy $Ent(.|m)$. For Riemannian manifolds, $Curv(M, d, m) \geq K$ if and only if $Ric_M \geq K$ on M; for the Wiener space, $Curv(M, d, m) = 1$.
One of the main results is that these lower curvature bounds are stable under (e.g. measured Gromov-Hausdorff) convergence. This solves one of the basic problems in this field, open for many years.

Furthermore, we introduce a (more restrictive) curvature-dimension condition $\mathrm{CD}(K, N)$ which implies sharp versions of the Brunn-Minkowski inequality, of the Bishop-Gromov volume comparison theorem and of the Bonnet-Myers theorem. Moreover, it allows to construct a canonical Dirichlet form with Gaussian bounds for the corresponding heat kernel.

Finally, we indicate how to construct a canonical reversible process on the L^2-Wasserstein space of probability measures $\mathcal{P}(\mathbb{R})$, regarded as an infinite dimensional Riemannian manifold. This process has an invariant measure $\mathbb{P}^\beta$ which may be characterized as the 'uniform distribution' on $\mathcal{P}(\mathbb{R})$ with weight function $\frac{1}{Z}\exp(-\beta \cdot Ent(.|m))$ where m denotes a given finite measure on $\mathbb{R}$. One of the key results is the quasi-invariance of this measure $\mathbb{P}^\beta$ under push forwards $\mu \mapsto h_*\mu$ by means of smooth diffeomorphisms h of $\mathbb{R}$.