Stochastic Partial Differential Equations and Invariant Measures

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Ref.: BiBoS-Preprint Server, my homepage at Purdue University
(BiBoS = Bielefeld–Bonn–Stochastics Research Centre)

A – From ODE to PDE

in finitely many variables

A – ODE to PDE, finite dimensional

ODE

$$(1) \qquad \begin{aligned} \mathrm{d}X_t^x &= B(X_t^x)\,\mathrm{d}t \\ X_0^x &= x \in \mathbb{R}^d \end{aligned} \qquad \text{on } \mathbb{R}^d$$

$$X_t^x = x + \int_0^t B\big(X_s^x \quad\big)\,\mathrm{d}s$$

$$\Rightarrow \qquad \begin{aligned} &p_t f(x) := \quad f\big(X_t^x \quad\big) \\ &p_0 f = f \end{aligned}$$

semigroup
$\big(p_{t+s}f(x) = p_t(p_s f)(x)\big)$
by flow property

solves **PDE** (2)

$$\Rightarrow \qquad p_t f(x) := f\big(X_t^x\big)$$
$$p_0 f = f$$

semigroup
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solves **PDE**

$$(2) \qquad \frac{\partial}{\partial t} p_t f(x) = B(x) \cdot \nabla_x p_t f(x)$$
$$= B^i(x) \cdot \frac{\partial}{\partial e_i} p_t f(x)$$
$$=: \underset{\substack{\uparrow \\ \text{“generator”} \\ \text{of (1)}}}{L}(p_t f)(x)$$

Here $B = (B^1, \dots, B^d) : \mathbb{R}^d \to \mathbb{R}^d$ and $e_i = (0, \dots, \underset{\substack{\uparrow \\ i^{\text{th}}}}{1}, \dots, 0)$ canonical basis

ODE

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solves **PDE** (2)

A – SODE to PDE, finite dimensional

SODE

$$(1)\qquad \begin{aligned} \mathrm{d}X_t^x &= B(X_t^x)\,\mathrm{d}t + \mathrm{d}W_t \\ X_0^x &= x \in \mathbb{R}^d \end{aligned} \qquad \text{on } \mathbb{R}^d$$

$$X_t^x(\omega) = x + \int_0^t B\big(X_s^x(\omega)\big)\,\mathrm{d}s + \underbrace{W_t(\omega)}_{\substack{\text{Brownian motion}\\ \text{on } \mathbb{R}^d}}$$

$$\overset{\text{Kolmogorov}}{\Longleftrightarrow} \qquad \begin{aligned} &p_t f(x) := \int f\big(X_t^x(\omega)\big)\,\mathbb{P}(\mathrm{d}\omega) =: \mathbb{E}\big[f(X_t^x)\big] \\ &p_0 f = f \end{aligned}$$

semigroup
$\big(p_{t+s}f(x) = p_t(p_s f)(x)\big)$
by ~~flow property~~
Markov property

solves **PDE** (2) (heat equation in finitely many variables)

$$\Rightarrow \qquad p_t f(x) := \quad f\big(X_t^x \quad\big)$$
$$p_0 f = f$$

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$$(2) \qquad \frac{\partial}{\partial t} p_t f(x) \overset{\text{Itô}}{=} B(x) \cdot \nabla_x p_t f(x) + \frac{1}{2} \Delta_x p_t f(x)$$

$$= B^i(x) \cdot \frac{\partial}{\partial e_i} p_t f(x) + \qquad \frac{1}{2} \frac{\partial^2}{\partial e_i \partial e_i} p_t f(x)$$

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A – SODE to PDE, finite dimensional 3a

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$$

$$
X_t^x(\omega) = x + \int_0^t B\big(X_s^x(\omega)\big)\,\mathrm{d}s + \underbrace{W_t(\omega)}_{\substack{\text{Brownian motion} \\ \text{on } \mathbb{R}^d}}
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solves **PDE** (2) (heat equation in finitely many variables)

A – SODE to PDE, finite dimensional

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$$
(1) \qquad \begin{aligned} \mathrm{d}X_t^x &= B(X_t^x)\,\mathrm{d}t + \sigma(X_t^x)\,\mathrm{d}W_t \\ X_0^x &= x \in \mathbb{R}^d \end{aligned} \qquad \text{on } \mathbb{R}^d
$$

$$
X_t^x(\omega) = x + \int_0^t B\big(X_s^x(\omega)\big)\,\mathrm{d}s + \int_0^t \sigma\big(X_s^x(\omega)\big)\,\underbrace{\mathrm{d}W_{\cancel{t}\,s}(\omega)}_{\substack{\text{Brownian motion} \\ \text{on } \mathbb{R}^d}}
$$

$$
\overset{\text{Kolmogorov}}{\Longleftrightarrow} \qquad \begin{aligned} p_t f(x) &:= \int f\big(X_t^x(\omega)\big)\,\mathbb{P}(\mathrm{d}\omega) =: \mathbb{E}\big[f(X_t^x)\big] \\ p_0 f &= f \end{aligned}
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semigroup
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$$= B^i(x)\cdot \frac{\partial}{\partial e_i} p_t f(x) + \qquad \frac{1}{2}\frac{\partial^2}{\partial e_i \partial e_i} p_t f(x)$$

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$$= B^i(x) \cdot \frac{\partial}{\partial e_i} p_t f(x) + \big(\sigma^T(x)\sigma(x)\big)^{ij} \cdot \frac{1}{2} \frac{\partial^2}{\partial e_i \partial e_j} p_t f(x)$$

$$=: \underset{\substack{\uparrow \\ \text{“generator”} \\ \text{of (1)}}}{L}(p_t f)(x)$$

Here $B = (B^1, \dots, B^d) : \mathbb{R}^d \to \mathbb{R}^d$ and $e_i = (0, \dots, \underset{\substack{\uparrow \\ i^{\text{th}}}}{1}, \dots, 0)$ canonical basis

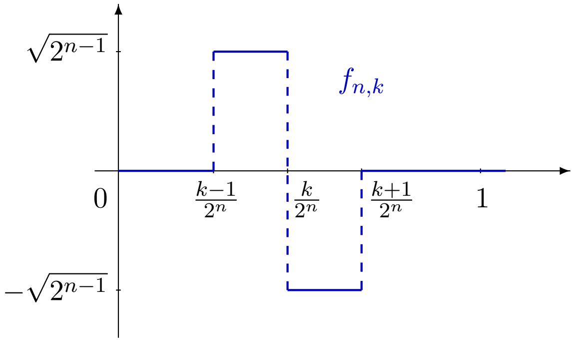
and $\sigma = (\sigma^{ij}) : \mathbb{R}^d \to \underbrace{M(d \times d)}_{d \times d\text{-matrices}}$

[Levy–Wiener–Ciesielski]

First ingredient: Haarbasis of $L^2\big([0,1],\, \mathrm{d}t\big)$:

$\uparrow$ Lebesgue measure

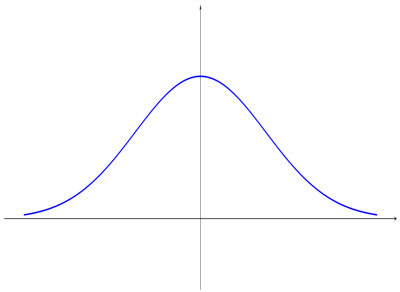
$f_{0,0} \equiv 1$, and for $n \in \mathbb{N}$, $0 < k < 2^n$, k odd,



$\big(f_{n,k}\big)_{\substack{0<k<2^n,\, k \text{ odd} \\ n\in\mathbb{N}}}$ is ONB of $L^2\big([0,1],\, \mathrm{d}t\big)$

Second ingredient: Standard normal distribution on $\mathbb{R}^\infty$

Standard normal distribution on $\mathbb{R}^1$:



$$\underset{\substack{\uparrow\\ \text{Gauss}}}{\gamma}(\mathrm{d}x) := \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \underset{\substack{\uparrow\\ \text{Lebesgue meas.}\\ \text{on } \mathbb{R}^1}}{\mathrm{d}x}$$

Set $\gamma_{n,k} := \gamma$.

$$\mathbb{P} := \bigotimes_{\substack{0<k<2^n\\ k \text{ odd}\\ n\in\mathbb{N}}} \gamma_{n,k} \qquad \text{product measure on } \mathbb{R}^\infty \ \left(= \mathbb{R}^{\{(n,k)|\dots\}}\right)$$

Define $\xi_{n,k} : \mathbb{R}^{\{(n,k)|n\in\mathbb{N},\, 0<k<2^n,\, k \text{ odd}\}\cup\{(0,0)\}} \longrightarrow \mathbb{R}$ (projection) and for $t \in [0,1]$

$$W_t(\omega) := \sum_{(n,k)} \xi_{n,k}(\omega) \int_0^t f_{n,k}(s)\,\mathrm{d}s \qquad \text{converges for } \mathbb{P}\text{-a.e. } \omega \in \mathbb{R}^\infty$$

$\uparrow$

Brownian motion on $\mathbb{R}^1$

B – From SODE to PDE

in infinitely many variables

SODE in E

$$(1) \qquad \mathrm{d}X_t^x = B(X_t^x)\,\mathrm{d}t + \sigma(X_t^x)\,\underset{\substack{\uparrow\\ \text{Brownian motion}\\ \text{on } E}}{\mathrm{d}W_t}$$

$$X_0^x = x \in \underset{\substack{\uparrow\\ \text{inner}\\ \text{product } \langle\ ,\ \rangle}}{E} \underset{\text{(e.g.)}}{:=} \text{separable Hilbert space} \quad \Big(\underset{\text{(e.g.)}}{=} \underset{\substack{\uparrow\\ \text{Lebesgue}\\ \text{measure}}}{L^2(\mathbb{R}^m, \mathrm{d}x)}, \quad \underset{\substack{\uparrow\\ \text{Sobolev}\\ \text{space}}}{H^k(\mathbb{R}^m, \mathrm{d}x)}, \quad \ldots \Big)$$

$\Longleftrightarrow$ ↙ under very strong conditions on B, σ

↑
OUR POINT!
("weak solution")

$$p_t f(x) := \int_\Omega f\big(X_t^x(\omega)\big)\,\mathbb{P}(\mathrm{d}\omega)\ , \qquad p_0 f = f$$

solves **PDE** (2) (heat equation in infinitely many variables)

$\Longleftrightarrow$

$$p_t f(x) := \int_\Omega f\big(X_t^x(\omega)\big)\, \mathbb{P}(\mathrm{d}\omega)\,, \qquad p_0 f = f$$

solves **PDE** (heat equation in infinitely many variables)

$$(2) \qquad \frac{\partial}{\partial t} p_t f(x) = \sum_{i=1}^{\infty} \langle B(x)\,,\ e_i \rangle\, \frac{\partial}{\partial e_i} p_t f(x) + \sum_{i,j=1}^{\infty} \langle \underbrace{\sigma^T(x)\sigma(x)}_{=:A(x)}\, e_i\,,\ e_j \rangle\, \frac{\partial^2}{\partial e_i \partial e_j} p_t f(x)$$

$$=: \underset{\substack{\uparrow \\ \text{“generator”} \\ \text{of (1)}}}{L}(p_t f)(x).$$

Here $B : E \to E$ and $\{e_i \mid i \in \mathbb{N}\}$ ONB of E and

$$\sigma : E \to \underset{\substack{\uparrow \\ \text{(bounded) linear operators} \\ \text{on } E}}{L(E)}$$

Briefly:

"strong": Find **MAPS** $G_x : C\big([0,\infty),E\big) \to C\big([0,\infty),E\big)$, $x \in E$, s.th. for any **GIVEN** BM $(W_t)_{t\geq 0}$ on any $(\Omega, \mathcal{F}, \mathbb{P})$ the process $X := G_x(W)$ solves (1) with initial condition $x \in E$, i.e.

$$X_t = x + \int_0^t B(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s\ , \quad t \geq 0,\ \mathbb{P}\text{-a.s.}$$

"weak": Find **MEASURES** $\mathbb{P}_x$ on $\Omega := C\big([0,\infty),E\big)$, $x \in E$, such that for $X_t(\omega) := \omega(t)$, $\omega \in \Omega$, and for

$$M_t := X_t - \int_0^t B(X_s^x)\,\mathrm{d}s\ , \quad t \geq 0,$$

we have:

$$M_t = \int_0^t \sigma(X_s)\,\mathrm{d}W_s\ , \quad t \geq 0,\ \mathbb{P}_x\text{-a.s. for } \textbf{SOME} \text{ BM } (W_t)_{t\geq 0} \text{ on } (\Omega, \mathcal{F}, \mathbb{P}_x)$$

and

$$X_0 = x \quad \mathbb{P}_x\text{-a.s.}$$

NOTE: If $\sigma \equiv 0$, then strong = weak.

B – SODE to PDE, infinite dimensional 6a

Let us first fix the class of SODE on a Hilbert space E we shall consider:

$$\mathrm{d}X_t = \Big[\underbrace{\overset{\substack{\text{generator}\\ \text{of semigroup on } E\\ \downarrow}}{H} X_t + \overset{\substack{\text{nonlinear}\\ \swarrow}}{F}(X_t)}_{=B(X_t)\text{ above}}\Big]\mathrm{d}t + \underset{\substack{\uparrow\\ (\sigma)(X_t)\text{ above}}}{\sqrt{\overset{\substack{\in L(E),\text{ pos.}\\ \swarrow}}{A}}}\,\mathrm{d}W_t \leftarrow \text{B.M. on } E \tag{1}$$

$$X_0 = x \in E \qquad = \text{sep. Hilbert space with } \langle\ ,\ \rangle$$

Associated generator (as above)

$$L\varphi(x) = \frac{1}{2}\sum_{i,j=1}^{N} \langle Ae_i, e_j\rangle \frac{\partial^2}{\partial e_i \partial e_j}\varphi(x) + \sum_{i=1}^{N} \big\langle Hx + F(x)\,,\ e_i\big\rangle \frac{\partial \varphi}{\partial e_i}(x)$$

$$= \frac{1}{2}\,\mathrm{Tr}(AD^2\varphi) + \big\langle Hx + F(x)\,,\ D\varphi(x)\big\rangle$$

$$\nwarrow \ \text{Fréchet derivatives} \ \nearrow$$

$$\text{for } x \in E \text{ and } \varphi = \underset{\substack{\uparrow\\ \in C_b^2(\mathbb{R}^N)}}{g}\Big(\langle \overset{\substack{\text{ONB of } E\\ \swarrow}}{e_1}, \cdot\rangle, \dots, \langle e_{\underset{\substack{\uparrow\\ \in\mathbb{N}\text{ arbitrary}}}{N}}, \cdot\rangle\Big) \leftarrow \text{all such } \mathcal{F}C_b^2$$

Let us first fix the class of SODE on a Hilbert space E we shall consider:

$$(1)\qquad \begin{aligned} dX_t &= \big[HX_t + F(X_t)\big]\,dt + \sqrt{A}\,dW_t \\ X_0 &= x \in E \end{aligned}$$

Associated generator

$$\begin{aligned} L\varphi(x) &= \frac{1}{2}\sum_{i,j=1}^{N}\langle Ae_i, e_j\rangle \frac{\partial^2}{\partial e_i \partial e_j}\varphi(x) + \sum_{i=1}^{N}\big\langle Hx + F(x)\,,\ e_i\big\rangle \frac{\partial \varphi}{\partial e_i}(x) \\ &= \frac{1}{2}\,\mathrm{Tr}(AD^2\varphi) + \big\langle Hx + F(x)\,,\ D\varphi(x)\big\rangle \end{aligned}$$

for $x \in E$ and $\varphi = g\big(\langle e_1, \cdot\rangle, \ldots, \langle e_N, \cdot\rangle\big)$.

The associated heat equation is also called **Kolmogorov (backward) equation**

$$(2)\qquad \frac{\partial}{\partial t}u(t,x) = Lu(t,x), \quad u(0,\cdot\,) = f,$$

where $f : E \to \mathbb{R}$.

Want to solve (2), then (1)!

C – Some examples and applications

(a) Stochastic quantization of $P(\phi)_2$

(b) Porous media equation

(c) Stochastic Navier–Stokes equation, $d = 2, 3$

(a) Stochastic quantization of $P(\phi)_2$ [Albeverio / R.: PTRF '91]

$$dX_t = \Big[\underbrace{(\Delta - 1)}_{H} X_t - \underbrace{\overbrace{:P(X_t):}^{\substack{\text{renorm. pol. of odd degree,}\\ \text{leading coeff. pos.}}}}_{F(X_t)}\Big]\,dt + dW_t$$

on $E \subset \mathcal{S}'(\mathbb{R}^2)$.

Remark: Typical for "weak approach": F allowed to be **very irregular**.

(b) Stochastic porous media equation [Da Prato / R.: JEE '04]
[Da Prato / R. / Barbu / Bogachev: JFA '06 (?)]

$$dX_t \underset{\substack{\uparrow\\ H\equiv 0}}{=} \underbrace{\Delta\Psi(X_t)}_{F(X_t)}\,dt + \underset{\substack{\uparrow\\ \text{trace class}}}{\sqrt{A}}\,dW_t$$

on $E := H^{-1}(\Lambda)$, $\Lambda \subset \mathbb{R}^d$, open

(c) Stochastic Navier–Stokes equation, $d = 2$ [Sobol / R.: Preprint '06], [Sobol / R.: Ann. Prob. '06] for $d = 1$

$$dX_t = [\overbrace{\nu\Delta_s}^{H} X_t \pm \overbrace{\langle X_t, \nabla\rangle X_t}^{F(X_t)}]\, dt + \sqrt{A}\, dW_t$$

viscosity Stokes–Laplacian gradient on $\mathbb{R}^2$

on

$$E := \{x \in L^2(\Lambda \to \mathbb{R}^2, dx) \mid \underbrace{\operatorname{div} x}_{\text{in the sense of distributions}} = 0\},$$

$\Lambda \subset \mathbb{R}^2$, open, bounded, $\partial\Lambda$ smooth

Remark: Existence of infinitesimally invariant measures also proved for $d = 3$ [Bogachev / R.: PTRF '00]

D – Strategy to solve the Kolmogorov equation

to solve

$$(2) \qquad \frac{\partial}{\partial t} u(t,x) = L u(t,x), \qquad u(0,\,\cdot\,) = f.$$

semigroup approach!

Construct

$$e^{tL} f(x) =: u(t,x), \qquad t \geq 0.$$

Need on some function space W over E:

L "dissipative" : $-L$ monotone (see below)

L "maximal" : $(1-L)(\mathcal{F}C_b^2)$ dense in W,
easier to achieve for a weak topology on W

Cannot take: $W = C_b(E)$, since coeff. of L not continuous in general.

Idea: $W = L^p(E,\mu)$ for suitable measures on E!

E – Programme to implement strategy

Step 1: Reference measures on E

Step 2: L^p-solutions for PDE (2)

Step 3: Regularity of L^p-solutions

Step 4: Weak solutions of SDE (1)

Step 5: Characterization of E_μ

Step 6: Calculation of p_t, $t > 0$

Step 1: Reference measures on E.

Solve $L^*\mu = 0$ "μ is L-infinitesimally invariant"

i.e. find probability measure μ on $\overbrace{\mathcal{B}(E)}^{\text{Borel } \sigma\text{-algebra}}$ such that $L\varphi \in L^1(E,\mu)$ and

$$\int L\varphi \,\mathrm{d}\mu = 0 \quad \forall\, \varphi \in \mathcal{F}C_b^2\,.$$

($\Rightarrow$ $(L, \mathcal{F}C_b^2)$ is dissipative on $L^p(E,\mu)$, so
has closure $\big(\bar{L}, D(\bar{L})\big)$ on $L^p(E,\mu)$ for all $p \in [1,\infty)$)

Step 2: L^p-solutions for PDE (2).

Show: $\exists\, e^{t\bar{L}}$, $t>0$, on $L^p(E,\mu)$ (i.e. $\bar{L}$ generates a semigroup on $L^p(E,\mu)$),

hence

$$L^p(E,\mu)\text{-}\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{e^{t\bar{L}}f}_{u(t,\,\cdot\,)} = \bar{L}(\underbrace{e^{t\bar{L}}f}_{u(t,\,\cdot\,)}), \quad t>0,\ f \in D(\bar{L}), \qquad \text{“solution in } L^p\text{”}$$

Remark. Then $\displaystyle\int e^{t\bar{L}}f \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu \ \ \forall\, t>0$ "μ invariant"

F – Model Case:

Stochastic Porous Medium Equation

For simplicity $\Psi(x) = x^3$. So,

$$\mathrm{d}X_t = \Delta(X_t^3)\,\mathrm{d}t + \sqrt{A}\,\mathrm{d}\underset{\substack{\uparrow \\ = (W_t^i e_i)_{i\in\mathbb{N}}}}{W_t} \qquad \text{(SPME)}$$

where W_t^i indep. B. motions on $\mathbb{R}^1$

on $E := H^{-1}(\Lambda)$ $\big(:=$ dual of $\underset{\substack{\uparrow \\ \text{Dirichlet bd. cond.}}}{H_0^1}(\Lambda)\big)$, $\Lambda \subset \mathbb{R}^d$, open, bdd., $\partial\Lambda$ smooth.

Have

$$H_0^1(\Lambda) \subset L^2(\Lambda) \subset H^{-1}(\Lambda) \xrightarrow[\text{bijection}]{\Delta^{-1}} H_0^1(\Lambda).$$

- $\{e_i \mid i \in \mathbb{N}\}$ = eigenbasis of Dirichlet Laplacian on $H^{-1}(\Lambda)$.
- $A \in L(H^{-1}, H^{-1})$, $Ae_i = \lambda_i e_i$ (“diagonal”)
- $\lambda_i \geq 0 \; \forall\, i \in \mathbb{N}$, and $\displaystyle\sum_{i=1}^{\infty} \lambda_i < \infty$ (“trace class”).

In this case for $\varphi \in \mathcal{F}C_b^2(H^{-1})$

$$L\varphi(x) = \frac{1}{2}\sum_{i=1}^{\infty} \lambda_i \, \frac{\partial^2}{\partial e_i^2}\varphi(x) + {}_{H^{-1}}\big\langle \Delta x^3 \, , \; D\varphi(x)\big\rangle_{H_0^1} \, ,$$

$x \in L^2(\Lambda)$ $\big(\subset H^{-1}(\Lambda)\big)$ s.th. $x^3 \in H_0^1$.
(So, only defined for *special* $x \in H^{-1}(\Lambda)$)

Let's implement (part of) "programme".

Step 1: Solve $L^*\mu = 0$.

Let $V_2 : H^{-1}(\Lambda) \to [0, \infty]$ "Lyapunov function"

$$V_2(x) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Lambda x^2(\xi) \, \mathrm{d}\xi, & x \in L^2(\Lambda), \\ +\infty, & \text{else.} \end{cases}$$

Then for $x \in L^2(\Lambda)$ $\big(\subset H^{-1}(\Lambda)\big)$ s.th. $x^2, x^3 \in H_0^1(\Lambda)$

$$LV_2(x) = \underbrace{\frac{1}{2}\sum_{i=1}^{\infty} \lambda_i \int_\Lambda e_i^2(\xi)\,\mathrm{d}\xi}_{=:C=\text{const.}} + \underbrace{{}_{H^{-1}}\langle \Delta x^3, x\rangle_{H_0^1}}_{\substack{=-\frac{3}{4}\int_\Lambda |\nabla x^2(\xi)|^2\,\mathrm{d}\xi \\ =:-\Theta_2(x)}} = \underbrace{C}_{\geq 0} - \underbrace{\Theta_2(x)}_{\geq 0}$$

Restrict to $x \in \operatorname{span}\{e_1, \ldots, e_N\}$

$$\underset{\substack{\uparrow \\ \text{operator} \\ \text{on } \mathbb{R}^N!}}{L_N} V_2(x) = \underbrace{\frac{1}{2}\sum_{i=1}^{N} \lambda_i \int_\Lambda e_i^2(\xi)\,\mathrm{d}\xi}_{\leq C} + \underbrace{{}_{H^{-1}}\big\langle P_N(\Delta x^3), x\big\rangle_{H_0^1}}_{=-\Theta_2(x)} \leq \underbrace{C - \Theta_2(x)}_{\substack{\text{independent} \\ \text{of } N!}}$$

Relatively easy to show: ([Bogachev / R.: Th. Prob. Appl. '00])

$\exists$ prob. measure μ_N on $\operatorname{span}\{e_1, \ldots, e_N\} \cong \mathbb{R}^N$ s.th. $L_N^* \mu_N = 0$, so

$$0 = \int L_N V_2 \,\mathrm{d}\mu_N \leq C - \int \Theta_2 \,\mathrm{d}\mu_N \qquad ❄$$

$$0 = \int L_N V_2 \, \mathrm{d}\mu_N \le C - \int \Theta_2 \, \mathrm{d}\mu_N \qquad ❄$$

Consider μ_N on $H^{-1}(\Lambda)$ $\big(\supset \operatorname{span}\{e_1, \dots, e_N\}\big)$, then

$$\sup_N \mu_N \big(\underbrace{\{\Theta_2 > R\}}_{\substack{\text{have compact} \\ \text{complements} \\ \text{in } H^{-1}(\Lambda)}} \big) \overset{\text{Chebychev}}{\le} \frac{1}{R} \cdot \sup_N \int \Theta_2 \, \mathrm{d}\mu_N \overset{❄}{\le} \frac{1}{R} \cdot C \xrightarrow{R \to \infty} 0$$

$$\overset{\text{Prokhorov}}{\Rightarrow} \exists \, \mu := \lim_{k \to \infty} \mu_{N_k} \qquad \text{in weak topology of measures on } H^{-1}(\Lambda)$$

$$\text{and} \quad \int \Theta_2 \, \mathrm{d}\mu \le C.$$

(Can show similarly: $\displaystyle\int |\nabla x^3|^2_{L^2} \, \mu(\mathrm{d}x) < \infty$,

so $\mu\big(\{x \in L^2(\Lambda) \mid x^2, x^3 \in H^1_0(\Lambda)\}\big) = 1$.)

Then show (again work!)

$$L^* \mu \quad \big(\overset{!}{=} \lim_{k \to \infty} L^*_{N_k} \mu_{N_k} \big) \quad = 0.$$

So far, by "programme":

weak solutions for SPME **("weak approach")**

Now:

strong solutions for SPME **("strong approach")**

[Da Prato / R. / Rosowskii / Wang, to appear in Comm. PDE '06]

Theorem. (formulated in special case: $\Psi(x) = x^3$, Λ bounded)

(i) $\exists$ (continuous) strong solution X^x $\big(= G_x(W)\big)$ for **SPME**.

(ii) $\mathbb{P}_x := \mathbb{P} \circ (X^x)^{-1}$, $x \in E$, form a Markov process.

(iii) For two solutions X, Y of **SPME** with (same) given BM $(W_t)_{t\geq 0}$

$$\|X_t - Y_t\|_E^2 \ \leq \ \|X_s - Y_s\|_E^2 \wedge \frac{2}{\gamma^2(t-s)}\ , \quad 0 \leq s \leq t,\ \mathbb{P}\text{-a.s.}$$

(in particular: **pathwise uniqueness**)

Here γ is the lowest eigenvalue of Δ on Λ.

(iv) $(p_t)_{t>0}$ has unique invariant measure μ (i.e. $\int p_t f \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu \; \forall \, t > 0$, $f : E \to \mathbb{R}_+$), and for some $C > 0$

$$\sup_{x \in E} \left| p_t f(x) - \int f \, \mathrm{d}\mu \right| \leq C \, \|f\|_{\mathrm{Lip}} \, t^{-1/2} \qquad \forall \, t > 0.$$

Here $p_t f(x) := \mathbb{E} f(X_t^x)$ ($= e^{t\bar{L}} f(x)$), $t \geq 0$, $x \in E$.

Remark. In [R. / Wang, Preprint '06] above theorem is generalized to increasing Ψ such that $N(s) := s \, \Psi(s)$ is a Young function.

Ex.: $\Psi(x) := \mathrm{sign}(s) \, s^{\alpha}$, $\alpha > 0$.

“Weak” versus “Strong” Approach

Theorem. [R., Preprint ’06] Let L be Kolmogorov operator corresponding to **SPME** on $E := H^{-1}(\Lambda)$. Then

$$L^*\mu = 0$$

has a unique solution (in space of probability measures μ on E).

Sketch of Proof for uniqueness. Let μ solve: $L^*\mu = 0$, μ prob. meas.
“weak approach” implies:

(1) $\exists\ \tilde{\mathbb{P}}_x$, $x \in E$, on $C\big([0,\infty), E\big)$, which are weak solutions for **SPME**.

(2) μ is invariant for $\tilde{\mathbb{P}}_x$, i.e.

$$\int_E \tilde{\mathbb{P}}_x \circ X_t^{-1}\ \mu(\mathrm{d}x) = \mu \quad \text{for all } t \geq 0.$$

(3) $\tilde{\mathbb{P}}_x$, $x \in E$, form a Markov process, which is (Lipschitz) Feller.

(4) $\tilde{\mathbb{P}}_x$, $x \in E$, is the only such Markov process. **(NEW)**

"strong approach" (see above) implies:

(5) $\mathbb{P}_x := \mathbb{P} \circ X^x$ is a Markov process, which is indeed Lipschitz Feller.

(6) **This** Markov process has a unique invariant measure μ_0.

HENCE,

$$\mu \overset{(2)}{=} \int_E \tilde{\mathbb{P}}_x \circ X_t^{-1}\, \mu(\mathrm{d}x) \overset{(3),\,(4),\,(5)}{=} \int_E \mathbb{P} \circ X_t^x\, \mu(\mathrm{d}x)\;, \quad \text{i.e. } \mu \text{ invariant for } \mathbb{P}_x,\, x \in E.$$

$$\overset{(6)}{\Rightarrow} \mu = \mu_0\;.$$

$\square$

Appendix I

Semigroup Property

Flow property $\Rightarrow$ semigroup property

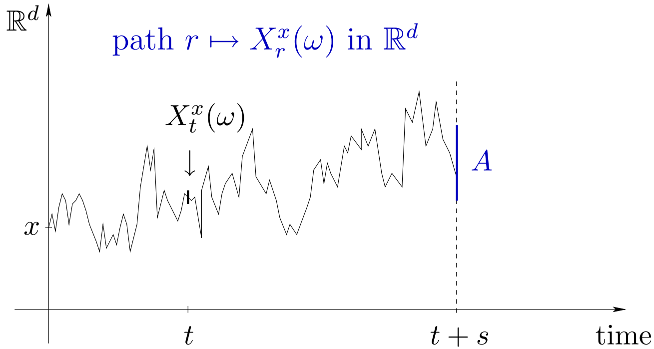
$$p_t(p_s f)(x) = p_s f(X_t^x) = f(X_s^{X_t^x}) \overset{\substack{\text{flow}\\ \text{property}}}{=} f(X_{t+s}^x) = p_{t+s} f(x)$$

Markov property $\Rightarrow$ semigroup property

$$\begin{aligned} p_t(p_s f)(x) &= \int p_s f\big(X_t^x(\omega)\big)\,\mathbb{P}(\mathrm{d}\omega) = \int \mathbb{E}\Big[f\big(X_s^{X_t^x(\omega)}\big)\Big]\,\mathbb{P}(\mathrm{d}\omega) \\ &\overset{\substack{\text{Markov}\\ \text{property}}}{=} \mathbb{E}\big[f(X_{t+s}^x)\big] = p_{t+s} f(x) \end{aligned}$$

Interpretation: $f := 1_A$, $A \subset \mathbb{R}^d$, Borel

$$\begin{aligned} p_t 1_A(x) &= \text{probability to be in } A \\ &\quad \text{after time } t \\ &\quad \text{after start in } x \\ &= \int 1_A\big(X_t^x(\omega)\big)\, \mathbb{P}(\mathrm{d}\omega) = \mathbb{E}\big[1_A(X_t^x)\big] = \mathbb{P}[X_t^x \in A] \end{aligned}$$



$$\mathbb{P}[X_{t+s}^x \in A] \underset{\substack{\uparrow \\ \text{Markov property}}}{=} \int \mathbb{P}\big[X_s^{X_t^x(\omega)} \in A\big]\, \mathbb{P}(\mathrm{d}\omega)$$

Appendix II

Itô's formula

$$f(X_t^x) - f(\underset{\substack{\uparrow \\ =x}}{X_0^x})$$

$$= \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial e_i}(X_s^x)\, \mathrm{d}(X_s^x)^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial e_i \partial e_j}(X_s^x)\, \mathrm{d}\big[(X_{\boldsymbol{\cdot}}^x)^i, (X_{\boldsymbol{\cdot}}^x)^j\big]_s \qquad \text{(Itô)}$$

$$f(X_t^x) - f(\underset{\substack{\uparrow \\ =x}}{X_0^x}) = \lim_{\substack{|\tau|\to 0, \\ \tau \text{ partition} \\ \text{of } [0,t]}} \sum_{k=1}^{N_\tau} \big(f(X_{s_{k+1}}^x) - f(X_{s_k}^x) \big)$$

$$\overset{\text{Taylor}}{=} \sum_{i=1}^{d} \underbrace{\int_0^t \frac{\partial f}{\partial e_i}(X_s^x)\, \mathrm{d}(X_s^x)^i}_{} + \frac{1}{2} \sum_{i,j=1}^{d} \underbrace{\int_0^t \frac{\partial^2 f}{\partial e_i \partial e_j}(X_s^x)\, \mathrm{d}\big[(X_\cdot^x)^i,\, (X_\cdot^x)^j\big]_s}_{} \qquad \text{(Itô)}$$

$$= \lim \sum_k \frac{\partial^2 f}{\partial e_i \partial e_j}(X_{s_{k+1}}^x)\big((X_{s_{k+1}}^x)^i - (X_{s_k}^x)^i\big)\big((X_{s_{k+1}}^x)^j - (X_{s_k}^x)^j\big)$$

$$= \lim \sum_k \frac{\partial f}{\partial e_i}(X_{s_k}^x)\big((X_{s_{k+1}}^x)^i - (X_{s_k}^x)^i\big)$$

$$= \int_0^t \big\langle \nabla f(X_s^x),\, \mathrm{d}W_s \big\rangle + \int_0^t \big\langle \nabla f(X_s^x),\, B(X_s^x) \big\rangle\, \mathrm{d}s + \frac{1}{2} \int_0^t \Delta f(X_s^x)\, \mathrm{d}s$$

$$\uparrow$$

$$\left(\begin{array}{ll} & \mathrm{d}X_s^x = \mathrm{d}W_s + B(X_s^x)\, \mathrm{d}s \\ \& & \mathrm{d}\big[(X_\cdot^x)^i,\, (X_\cdot^x)^j\big]_s = \mathrm{d}[W_\cdot^i,\, W_\cdot^j]_s = \delta_{ij}\, \mathrm{d}s \end{array} \right)$$

So,

$$
\begin{aligned}
&f(X_t^x) - f(x) \\
&\quad = \underbrace{\int_0^t \langle \nabla f(X_s^x),\ \mathrm{d}W_s \rangle}_{\text{martingale!}} + \underbrace{\int_0^t \langle \nabla f(X_s^x),\ B(X_s^x) \rangle \,\mathrm{d}s + \frac{1}{2}\int_0^t \Delta f(X_s^x)\,\mathrm{d}s}_{= \int_0^t Lf(X_s^x)\,\mathrm{d}s}
\end{aligned}
$$

Take expectation and obtain:

$$
p_t f(x) - f(x) = 0 + \int_0^t p_s(Lf)(x)\,\mathrm{d}s
$$

hence

$$
\frac{\partial}{\partial t} p_t f(x) = p_t(\underset{\uparrow \atop \frac{\partial}{\partial s} p_s f \restriction_{s=0}}{Lf})(x) \quad \overset{p_t p_s = p_{t+s} = p_s p_t \atop \downarrow}{=} \quad \frac{\partial}{\partial s} p_s(p_t f)(x)\restriction_{s=0} = L p_t f(x)
$$

Appendix III

The remaining steps

Step 3: Regularity of L^p-solutions.

Show: $\exists$ probability kernels p_t, $t > 0$

such that

$$e^{t\bar{L}} f(x) = \int f(y)\, p_t(x, \mathrm{d}y) =: p_t f(x)$$

for all $t > 0$, $f \in L^p(E, \mu)$, $x \in E_\mu \subset E$ with $\mu(E_\mu) = 1$.
Furthermore, try to prove

$$p_t\big(C_b^{(w)}(E_\mu)\big) \subset C_b^{(w)}(E_\mu) \qquad \text{"Feller property"}$$

or even

$$p_t\big(\mathcal{B}_b(E_\mu)\big) \subset C_b^{(w)}(E_\mu) \qquad \text{"strong Feller property"}$$

Step 4: Weak solutions of SDE (1).

Construct the Markov process solving (1)

Step 5: Characterization of E_μ.

Explicitly describe E_μ

E.g. $E_\mu = \text{supp } \mu$, then show $\text{supp } \mu = E$.

Step 6: Calculation of p_t, $t > 0$.

Construct finite dimensional approximations

Find p_t^N, $t > 0$, on $\mathbb{R}^N$ solving Kolmogorov equations with nice coefficients on $\mathbb{R}^N$ s.t.

$p_t^N \xrightarrow{N\to\infty} p_t$. "Galerkin"

Appendix IV

Step 2 for SPME

Step 2: Show $\exists\, e^{t\bar{L}}$, $t \geq 0$

Since $L^*\mu = 0$, have $\big(L, C_b^2(H^{-1})\big)$ dissipative on $L^2(H^{-1}, \mu)$, hence closable.
$\big(\bar{L}, D(\bar{L})\big) :=$ closure.

To show: $(1 - \bar{L})\big(D(\bar{L})\big) = L^2(H^{-1}, \mu)$ "m-dissipativity"

O.K. ! Use (nonlinear) Yosida approx. for Δx^3 and Gaussian mollifier on H^{-1}.

Define resolvent

$$G_\lambda := (\lambda - \bar{L})^{-1} = \int_0^\infty e^{t\lambda} \underbrace{e^{t\bar{L}}}_{\substack{\text{exists!} \\ \text{and positivity} \\ \text{preserving!}}} \, \mathrm{d}t$$

on $L^2(H^{-1}, \mu)$.

Step 3+5: Main point:

Find $g \in L^2(H^{-1}, \mu)$, $g \geq 0$, s.th. $G_1 g$ has compact level sets in H^{-1}
(Recall: $\big(G_1 g(X_t^0)\big)_{t \geq 0}$ supermartingale!)

Informally, for $V_4(x) := \frac{1}{4}|x|^4_{L^4(\Lambda)}$; $\Theta_4(x) := \big\||\nabla x^3|\big\|_{L^2(\Lambda)}$

$$\underset{\substack{\uparrow\\ \text{not in}\\ D(\bar{L})}}{(1-L)V_4} = V_4 - \underset{\substack{\uparrow\\ \text{positive(!)}\\ \text{function}}}{C_4} + \Theta_4 \leq \underbrace{V_4 + \Theta_4}_{=:g(\geq 0)}$$

Apply G_1 to get (${G_1}$ positivity preserving!)

$$\underbrace{\frac{1}{4} \cdot |x|^4_{L^4(\Lambda)}}_{\substack{\text{has compact}\\ \text{level sets}\\ \text{in } H^{-1}}} = V_4(x) \leq G_1 g(x)$$

Can be made rigorous, using Moreau approximation of V_4 and a Gaussian mollifier on H^{-1}.

Summary: All the programme has been implemented.

For $A := (-\Delta)^{-\sigma}$, $\sigma > \frac{d}{2}$, $\operatorname{supp} \mu = H^{-1}$, and if $d = 1$ (i.e. $\Lambda \subset \mathbb{R}^1$) and $\frac{1}{2} < \sigma \leq 1$, then semigroup is strong Feller.