

ON FELLER'S BOUNDARY PROBLEM
AND DARNING COUNTABLE HOLES FOR MARKOV PROCESSES

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Let E be a locally compact separable metric space and m be a σ -finite Borel measure on E . We consider a pair of Borel standard processes $X = \{X_t, \mathbf{P}_x\}$, $\widehat{X} = \{\widehat{X}_t, \widehat{\mathbf{P}}_x\}$ on E which are in weak duality with respect to m and satisfy the following:

(X.1) Every semi-polar set is m -polar for X .

Let $F = \{a_1, a_2, \dots\}$ be a finite or countable closed subset of E and $E_0 := E \setminus F$.

(X.2) $m(F) = 0$.

(X.3) $u_\alpha^{(i)}(x) := \mathbf{E}_x(e^{-\alpha\sigma_F}; X_{\sigma_F} = a_i) > 0$, $\widehat{u}_\alpha^{(i)}(x) := \widehat{\mathbf{E}}_x(e^{-\alpha\sigma_F}; \widehat{X}_{\sigma_F} = a_i) > 0$. for q.e. $x \in E_0$.

(X.4) X and \widehat{X} admit no jumps from E_0 to F .

Let $X^0 = \{X_t^0, \mathbf{P}_x^0\}$ and $\widehat{X}^0 = \{\widehat{X}_t^0, \widehat{\mathbf{P}}_x^0\}$ be the part processes of X and \widehat{X} , respectively, killed upon leaving E_0 . X^0 and \widehat{X}^0 are in weak duality with respect to m_0 the restriction of m to E_0 . We view X and \widehat{X} as most general duality preserving extensions of X^0 and \widehat{X}^0 , respectively, from E_0 to $E = E_0 \cup F$. Our objectives are to characterize those extensions at the resolvent level as well as at the generator level using only the quantities intrinsic to X^0 and \widehat{X}^0 , thus solving in the duality setting the *boundary problem of Markov processes* going back to W.Feller [6].

(i) Let $U_{ij}, V_i, U_\alpha^{ij}$ be the Feller measures on F determined by X^0, \widehat{X}^0 and let J_{ij}, κ_i be the restrictions to F of the jumping and killing measures of X . Let $\check{\mathcal{A}}$ be the matrix with entries given by

$$\check{\mathcal{A}}_{ij} = U_{ij} + J_{ij} \quad \text{for } i \neq j \quad \text{and} \quad \check{\mathcal{A}}_{ii} = - \sum_{j \neq i} (U_{ij} + J_{ij}) + V_i + \kappa_i.$$

By using a general result in [3], we get the following: If F is finite, then

$$G_\alpha v = G_\alpha^0 v - \mathbf{H}_\alpha (\check{\mathcal{A}} - U_\alpha)^{-1} \widehat{\mathcal{H}}_\alpha v \quad \text{for } v \in \mathcal{B}_b(E_0), \quad (1)$$

where G_α, G_α^0 are resolvent of X, X^0 , and $\mathbf{H}_\alpha, \widehat{\mathcal{H}}_\alpha v$ are vectors with entries $u_\alpha^{(i)}, (\widehat{u}_\alpha^{(i)}, v)_{m_0}$, respectively. If F is infinite, then we have the expression of G_α as the limit of (1).

(ii) Consider the symmetric case: $X = \widehat{X}$. Then $X^0 = \widehat{X}^0$. Define the function space on F by

$$\mathcal{F}_F = \{f : \mathcal{E}^F(f, f) < \infty, U_1(|f|, |f|) < \infty\}, \quad (2)$$

with $\mathcal{E}^F(f, f)$ being defined by

$$\mathcal{E}^F(f, f) = \frac{1}{2} \sum_{i, j \geq 1, i \neq j} (f(a_i) - f(a_j))^2 (U_{ij} + J_{ij}) + \sum_{i \geq 1} f(a_i)^2 (V_i + \kappa_i) \quad (\leq \infty). \quad (3)$$

Let $(\mathcal{F}^0, \mathcal{E}^0)$ be the Dirichlet space of X^0 on $L^2(E_0, m_0)$ and $((\mathcal{F}^0)^{\text{ref}}, \mathcal{E}^{\text{ref}})$ be its reflected Dirichlet space. Let us introduce a linear operator \mathcal{L} on $L^2(E_0; m_0)$ specified by

$$f \in \mathcal{D}(\mathcal{L}), \quad \mathcal{L}f = g \quad (\in L^2(E_0; m_0)) \iff f \in (\mathcal{F}^0)_a^{\text{ref}}, \quad \mathcal{E}^{\text{ref}}(f, v) = -(g, v), \quad \forall v \in \mathcal{F}^0.$$

We also introduce the *flux* $\mathcal{N}(f)(a_i)$ of $f \in \mathcal{D}(\mathcal{L})$ at a_i by

$$\mathcal{N}(f)(a_i) = \mathcal{E}^{\text{ref}}(f, u_\alpha^{(i)}) + (\mathcal{L}f, u_\alpha^{(i)}),$$

which turns out to be independent of $\alpha > 0$.

Let \mathcal{A} be the $L^2(E; m)$ -infinitesimal generator of X . Using a general result in [2], we see that \mathcal{L} is an extension of \mathcal{A} and furthermore, under certain conditions on X^0 and F , we have the next characterization: $f \in \mathcal{D}(\mathcal{A})$ if and only if

$$f \in \mathcal{D}(\mathcal{L}), \quad f \text{ admits an } X^0\text{-fine limit function } \gamma f \in \mathcal{F}'_F, \quad (4)$$

and

$$\mathcal{N}(f)(a_i) + \sum_{j \geq 1, j \neq i} ((\gamma f)(a_i) - (\gamma f)(a_j)) J_{ij} + (\gamma f)(a_i) \kappa_i = 0, \quad i \geq 1. \quad (5)$$

(iii) Let $K = \bigcup_i K_i$ be a disjoint union of compact sets which are locally finite, $E_0 = E \setminus K$ and $E^* = E_0 \cup \{a_1, a_2, \dots\}$ be the topological extension of E_0 obtained by regarding each set K_i as a one point a_i . Given a pair of standard processes X, \widehat{X} on E which are in weak duality with respect to a σ -finite measure m on E , we consider their part processes X^0, \widehat{X}^0 on E_0 which are in weak duality in m_0 . m_0 is extended to E^* by setting $m(\{a_i\}) = 0$. Under certain conditions on X^0, \widehat{X}^0 , we can repeat the procedure of darning each hole K_i as has been performed in [4],[5] to construct duality preserving extensions X^*, \widehat{X}^* of X^0, \widehat{X}^0 to E^* . They admit no jumps from F to F but they may admit killings on F . Their resolvents and generators can be characterized as in (i), (ii).

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