SUPERCRITICAL SDES DRIVEN BY MULTIPLICATIVE STABLE-LIKE LÉVY PROCESSES

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Abstract. In this paper, we study the following time-dependent stochastic differential equation (SDE) in $\mathbb{R}^d$:

$$\mathrm{d}X_t = \sigma(t, X_{t-})\mathrm{d}Z_t + b(t, X_t)\mathrm{d}t, \quad X_0 = x \in \mathbb{R}^d,$$

where Z is a d-dimensional non-degenerate α-stable-like process with $\alpha \in (0, 2)$, and uniform in $t \geqslant 0$, $x \mapsto \sigma(t, x) : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is β-order Hölder continuous and uniformly elliptic with $\beta \in ((1-\alpha)^+, 1)$, and $x \mapsto b(t, x)$ is β-order Hölder continuous. The Lévy measure of the Lévy process Z can be anisotropic or singular with respect to the Lebesgue measure on $\mathbb{R}^d$ and its support can be a proper subset of $\mathbb{R}^d$. We show in this paper that for every starting point $x \in \mathbb{R}^d$, the above SDE has a unique weak solution. We further show that the above SDE has a unique strong solution if $x \mapsto \sigma(t, x)$ is Lipschitz continuous and $x \mapsto b(t, x)$ is β-order Hölder continuous with $\beta \in (1-\alpha/2, 1)$. When $\sigma(t, x) = \mathbb{I}_{d\times d}$, the $d \times d$ identity matrix, and Z is an arbitrary non-degenerate α-stable process with $0 < \alpha < 1$, our strong well-posedness result in particular gives an affirmative answer to the open problem in [22].

Keywords: Stochastic differential equation, Lévy process, Besov space, Zvonkin's transform

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1. Introduction

The main purpose of this paper is to establish the strong well-posedness as well as weak well-posedness for a class of stochastic differential equations driven by any non-degenerate α-stable-like Lévy process with $\alpha \in (0, 2)$, and with time-dependent Hölder drift b. More precisely, we are mainly concerned with the following time-dependent SDE:

$$\mathrm{d}X_t = \sigma(t, X_{t-})\mathrm{d}Z_t + b(t, X_t)\mathrm{d}t, \quad X_0 = x \in \mathbb{R}^d, \tag{1.1}$$

where $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ are two Borel measurable functions, and Z is a pure jump Lévy process on $\mathbb{R}^d$ whose Lévy measure ν when restricted to the unit ball centered at the origin is bounded between the Lévy measures of two α-stable Lévy processes with $\alpha \in (0, 2)$. When $\sigma(t, x)$ and $b(t, x)$ are Lipschitz continuous in $x \in \mathbb{R}^d$, it is well known that by first removing large jumps of Z and applying Picard's iteration method, one can show that SDE (1.1) has a unique strong solution. This paper is concerned with the strong existence and strong uniqueness of solution to SDE (1.1) when $b(t, x)$ is only Hölder continuous

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in x, as well as weak existence and uniqueness for solutions of SDE (1.1) when both $\sigma(t,x)$ and $b(t,x)$ are only Hölder continuous in x.

To be precise, in this paper Z is a Lévy process on $\mathbb{R}^d$ with Lévy exponent

$$\psi(\xi) := \log \mathbb{E}\left[\mathrm{e}^{-\mathrm{i}\xi\cdot Z_1}\right] = \int_{\mathbb{R}^d} \left(\mathrm{e}^{\mathrm{i}\xi\cdot z} - 1 - \mathrm{i}\xi\cdot z\mathbf{1}_{\{|z|\leqslant 1\}}\right)\nu(\mathrm{d}z), \tag{1.2}$$

where ν is a positive measure on $\mathbb{R}^d\setminus\{0\}$ so that $\int_{\mathbb{R}^d}\min\{|z|^2,1\}\nu(\mathrm{d}z)<\infty$. To state our condition on Lévy measure ν, for $\alpha\in(0,2)$, denote by $\mathbb{L}^{(\alpha)}_{non}$ the space of all non-degenerate α-stable Lévy measures $\nu^{(\alpha)}$; that is,

$$\nu^{(\alpha)}(A) = \int_0^\infty \left(\int_{\mathbb{S}^{d-1}} \frac{\mathbf{1}_A(r\theta)\Sigma(\mathrm{d}\theta)}{r^{1+\alpha}}\right)\mathrm{d}r, \quad A\in\mathscr{B}(\mathbb{R}^d), \tag{1.3}$$

where Σ is a finite measure over the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ with

$$\int_{\mathbb{S}^{d-1}} |\theta_0\cdot\theta|\,\Sigma(\mathrm{d}\theta)>0 \quad \text{for every } \theta_0\in\mathbb{S}^{d-1}. \tag{1.4}$$

Since the left hand side of the above is a continuous function of $\theta_0\in\mathbb{S}^{d-1}$, condition (1.4) is equivalent to

$$\inf_{\theta_0\in\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}} |\theta_0\cdot\theta|\,\Sigma(\mathrm{d}\theta)>0.$$

For $R>0$, denote by B_R the closed ball in $\mathbb{R}^d$ centered at the origin with radius R. We assume that there are $\nu_1,\nu_2\in\mathbb{L}^{(\alpha)}_{non}$, so that

$$\nu_1(A)\leqslant\nu(A)\leqslant\nu_2(A) \quad \text{for } A\in\mathscr{B}(B_1). \tag{1.5}$$

For the drift coefficient $b(t,x)$ and diffusion matrix $\sigma(t,x)$, we assume that there are constants $\beta,\theta\in((1-\alpha)^+,1]$ and $\Lambda>0$ so that for all $t\geqslant 0$ and $x,y\in\mathbb{R}^d$,

$$|b(t,x)|\leqslant\Lambda \quad \text{and} \quad |b(t,x)-b(t,y)|\leqslant\Lambda|x-y|^\beta, \tag{1.6}$$

$$\Lambda^{-1}|\xi|\leqslant|\sigma(t,x)\xi|\leqslant\Lambda|\xi|, \quad \|\sigma(t,x)-\sigma(t,y)\|\leqslant\Lambda|x-y|^\theta, \tag{1.7}$$

where $\|\cdot\|$ denotes the Hilbert-Schmidt norm of a matrix, and $|\cdot|$ denotes the Euclidean norm. We call a pure jump Lévy process Z whose Lévy measure ν satisfies condition (1.5) an α-stable-like Lévy process. The following is the main result of this paper.

Theorem 1.1. *Under conditions (1.5), (1.6) and (1.7), for each $x\in\mathbb{R}^d$, there is a unique weak solution to SDE (1.1). Moreover, if $\beta\in(1-\alpha/2,1]$ in (1.6) and $\theta=1$ in (1.7), then there is a unique strong solution to SDE (1.1).*

Remark 1.2. Condition (1.4) is clearly satisfied if $Z_t=(Z_t^{(1)},\cdots,Z_t^{(d)})$ is a cylindrical α-stable process, that is, each component is an independent copy of a non-degenerate one-dimensional (possibly asymmetric) α-stable process. Note that the Lévy measure of a cylindrical α-stable process Z_t is singular with respect to the Lebesgue measure on $\mathbb{R}^d$. In this case, Theorem 1.1 in particular answers affirmatively an open question from [22], and improves a result in [11, Corollary 1.4(iii)] for cylindrical α-stable processes from $\alpha\in(2/3,2)$ to all $\alpha\in(0,2)$.

Remark 1.3. It follows from Theorem 1.1 and a standard localization argument (see, e.g., [14, 25]) that if conditions (1.6) and (1.7) are satisfied locally on each ball B_R with Λ depending on R, then for each $x\in\mathbb{R}^d$, there exists a unique strong solution to SDE (1.1) up to the explosion time ζ with $\lim_{t\uparrow\zeta}X_t=\infty$.

The new feature or contributions of this paper are

- the driving Lévy process Z is any non-degenerate α-stable-like Lévy process on $\mathbb{R}^d$ with $\alpha \in (0,2)$ whose Lévy measure can be singular with respect to the Lebesgue measure on $\mathbb{R}^d$ and its support can be a proper subset of $\mathbb{R}^d$;
- the SDE (1.1) has variable diffusion matrix $\sigma(t,x)$;
- weak existence and uniqueness of solutions to (1.1) are established for β-Hölder continuous multiplicative coefficients $\sigma(t,x)$ and β-Hölder continuous drift $b(t,x)$ with $\beta \in ((1-\alpha)^+, 1)$.

Note that the (time-dependent) infinitesimal generator corresponding to the solution X of SDE (1.1) is $\mathscr{L}_t + b(x)\cdot\nabla$, where

$$\mathscr{L}_t u(x) := \int_{\mathbb{R}^d} \left(u(x+\sigma(t,x)z) - u(x) - \mathbf{1}_{\{|z|\leqslant 1\}}\sigma(t,x)z\cdot\nabla u(x)\right)\nu(\mathrm{d}z), \tag{1.8}$$

which is a nonlocal operator of order α under assumption (1.5). When $\alpha > 1$, $\mathscr{L}_t$ is the dominant term, which is called the subcritical case. When $\alpha \in (0,1)$, the gradient ∇ is of higher order than the nonlocal operator $\mathscr{L}_t$ so the corresponding SDE (1.1) is called supercritical. The critical case corresponds to $\alpha = 1$. Strong solution and pathwise uniqueness for SDEs driven by Lévy processes with non-Lipschitz drifts, especially for supercritical SDEs, is known to be a challenging problem; see [11, 21, 22].

The study of weak and strong well-posedness of SDE (1.1) with irregular coefficients has a long history and there is a large amount of literatures devoted to this topic especially when Z is a Brownian motion. When Z is a standard d-dimensional Brownian motion, $\sigma(t,x) = \mathbb{I}_{d\times d}$ and b is bounded measurable, Veretennikov [27] proved that SDE (1.1) has a unique strong solution, which extended a result of Zvonkin [32] in one-dimension. Using Girsanov's transformation and results from PDEs, Krylov and Röckner [17] obtained the existence and uniqueness of strong solutions to SDE (1.1) when σ is the identity matrix and b satisfies

$$\|b\|_{L^q_T(L^p(\mathbb{R}^d))} := \left[\int_0^T \left(\int_{\mathbb{R}^d} |b(t,x)|^p \mathrm{d}x\right)^{q/p} \mathrm{d}t\right]^{1/q} < \infty, \quad \frac{2}{q} + \frac{d}{p} < 1.$$

These results have been extended to SDEs with Sobolev diffusion coefficients and singular drifts in [28, 29] by using Zvonkin's idea.

However, things become quite different when Z is a pure jump Lévy process. In one-dimensional case, Tanaka, Tsuchiya and Watanabe [26] proved that if Z is a symmetric α-stable process with $\alpha \in [1,2)$, $\sigma(t,x) \equiv 1$ and $b(t,x) = b(x)$ is bounded continuous when $\alpha = 1$ or bounded measurable when $\alpha \in (1,2)$, then SDE (1.1) has a unique pathwise strong solution for every $x \in \mathbb{R}$. They further showed that for $\alpha \in (0,1)$, the SDE (1.1) has a unique weak solution when $b(x)$ is a bounded non-decreasing function that is β-Hölder continuous with $\beta > 1-\alpha$, and for any $\beta \in (0, 1-\alpha)$ there is a bounded β-Hölder continuous function $b(x)$ so that both strong and weak uniqueness fails. For one-dimensional multiplicative noise case where $\sigma(t,x) = \sigma(x)$, see [2] and [16, Theorem 1]. For multidimensional case, Priola [21] proved pathwise uniqueness for (1.1) when $\sigma(t,x) = \mathbb{I}_{d\times d}$, Z is a non-degenerate symmetric but possibly non-isotropic α-stable process with $\alpha \in [1,2)$ and $b(t,x) = b(x) \in C^\beta(\mathbb{R}^d)$ with $\beta \in (1-\alpha/2, 1)$ is time-independent. Priola's result was extended to drift b in some fractional Sobolev spaces in the subcritical case in Zhang [30] and to more general Lévy processes in the subcritical and critical cases in Priola [22]. Recently, for a large class of Lévy processes, Chen, Song and

Zhang in [11] established strong existence and pathwise uniqueness for SDE (1.1) when $\sigma(t,x) = \mathbb{I}_{d\times d}$ and $b(t,x)$ is time-dependent, Hölder continuous in x. Therein, the authors not only extend the main results of [21] and [22] for the subcritical and critical case ($\alpha \in [1,2)$) to more general Lévy processes and time-dependent drifts $b \in L^\infty([0,T];C^\beta)$ with $\beta \in (1-\frac{\alpha}{2},1)$, but also establish strong existence and pathwise uniqueness for the supercritical case ($\alpha \in (0,1)$) with $b \in L^\infty([0,T];C^\beta)$ for any $\beta \in (1-\frac{\alpha}{2},1)$. It partially answers an open question posted in [22] on the pathwise well-posedness of SDE (1.1) in the supercritical case. However, when Z is a *cylindrical* $\alpha-$stable process, the result of [11] requires $\alpha > 2/3$. As mentioned in [11], it is a quite interesting question whether the constraint $\alpha > 2/3$ can be dropped. Theorem 1.1 of this paper not only gives an affirmative answer to the above question but moreover it is done for the multiplicative noise setting and for a large class of Lévy processes. We remark that except in the one-dimensional case, almost all the known results in literature on strong well-posedness of SDE (1.1) driven by pure jump Lévy process Z requires $\sigma(t,x) = \mathbb{I}_{d\times d}$. On the other hand, for $d \geqslant 1$ and Z being a rotationally symmetric α-stable process in $\mathbb{R}^d$ with $\alpha \in (1,2)$, it is shown in [20] when $d=1$ and in [19] for $d \geqslant 2$ that SDE (1.1) with $\sigma(t,x) = \mathbb{I}_{d\times d}$ has a weak unique solution for any $b(t,x) = b(x) \in L^p(\mathbb{R}^d)$ with $p > d/(\alpha-1)$. The above result is extended in [8] to any $b(t,x) = b(x)$ in a Kato class that includes any function that can be written as the sum of a bounded function and an L^p-integrable function with $p > d/(\alpha-1)$. In a recent paper [18], SDE (1.1) is shown to have a unique weak solution for $\sigma(t,x) = \sigma(x)\mathbb{I}_{d\times d}$ and Z being a rotationally invariant symmetric α, where $\sigma(x)$ is a scale Hölder continuous function on $\mathbb{R}^d$ that bounded between two positive constants, and β-Hölder continuous drift $b(t,x) = b(x)$ with $\beta \in ((1-\alpha)^+,1)$. In another recent work [31], the above weak well-posenedss result has been obtained for a subclass of α-stable processes Z with $\alpha \in (0,1)$, $\sigma(t,x) = \mathbb{I}_{d\times d}$ and β-Hölder continuous drift $b(t,x) = b(x)$ with $\beta > 1-\alpha$ (see also [6] for the case of $\alpha > 1/2$). Our weak well-posedness result in Theorem 1.1 holds for any α-stable process Z and for any β-Hölder continuous drift $b(t,x) = b(x)$ with $\beta > 1-\alpha$, and thus extending the results of [18, 31].

We now describe the approach of this paper. For the strong well-posedness of SDE (1.1), we shall use a Zvonkin type of change variables to remove the drift term and convert the SDE (1.1) to a new SDE whose strong existence and pathwise uniqueness can be readily established. This requires a deep understanding for the following nonlocal PDE (Kolmogorov's equation):

$$\partial_t u = \mathscr{L}_t u + b\cdot\nabla u - \lambda u + f \quad \text{with } u(0,x) = 0, \tag{1.9}$$

and establish some a priori regularity estimates for its solution. Here $\mathscr{L}_t u$ is defined in (1.8). When $\sigma(t,x) = \mathbb{I}_{d\times d}$ and $b(t,x) = b(x)$ is time-independent, a priori regularity estimates have been obtained in [21, 22] for a class of α-stable type Lévy processes Z with $\alpha \geqslant 1$ under certain derivative condition on the transition semigroup of the Lévy process Z. The supercritical case $\alpha \in (0,1)$ is much harder. When $\mathscr{L}_t$ is the usual fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ with $\alpha \in (0,1)$, that is, when $\nu(\mathrm{d}z) = c(d,\alpha)|z|^{-d-\alpha}\mathrm{d}z$ for some constant $c(d,\alpha) > 0$ and $\sigma = \mathbb{I}_{d\times d}$ in the above definition, and $b \in L^\infty([0,T];C^\beta)$ with $\beta \in (1-\alpha,1)$, Silvestre [23] obtained the following a priori interior estimate for any solution u of (1.9):

$$\|u\|_{L^\infty([0,1];C^{\alpha+\beta}(B_1))} \leqslant C\Big(\|u\|_{L^\infty([0,2]\times B_2)} + \|f\|_{L^\infty([0,2];C^\beta(B_2))}\Big). \tag{1.10}$$

Such an interior estimate suggests that one could solve the supercritical SDE (1.1) uniquely when Z is a rotationally symmetric α-stable process with $\alpha \in (0,1)$ and $b \in L^\infty([0,T];C^\beta)$ with $\beta \in (1-\alpha/2, 1)$ (see [22]). However the approach of [23] strongly depends on realizing the fractional Laplacian in $\mathbb{R}^d$ as the boundary trace of an elliptic operator in upper half space of $\mathbb{R}^{d+1}$. Extending Silvestre's argument to general α-stable-type operators would be very hard, if not impossible at all. In [11, Theorem 2.3], a new approach of establishing estimates analogous to (1.10) is developed under the assumption that $\sigma(t,x) = \mathbb{I}_{d\times d}$ for a large class of Lévy processes Z. Probabilistic consideration played a key role in that approach. As mentioned above, when Z is a cylindrical α-stable process, the approach of [11] requires $\alpha > 2/3$. So new ideas are needed for the study of SDE (1.1) with general Lévy process Z and variable diffusion matrix $\sigma(t,x)$.

Our approach in the study of (1.9) is based on the Littlewood-Paley decomposition and some Bernstein's type inequalities. This approach seems to be new and allows us to handle a large class of Lévy's type operator in a unified way, including Lévy's type operators with singular Lévy measures, see Theorem 3.3 below. When $\sigma(t,x) = \sigma(t)$ is spatially independent and real part of the symbol $\psi_t(\xi)$ of $\mathscr{L}_t$ (that is, $\psi_t(\xi) = \int_{\mathbb{R}^d}\left(\mathrm{e}^{\mathrm{i}\xi\cdot\sigma(t)z} - 1 - \mathbf{1}_{\{|z|\leqslant 1\}}\mathrm{i}\sigma(t)z\cdot\xi\right)\nu(\mathrm{d}z))$ is bounded from above by $-c_0|\xi|^\alpha$, we show the following a priori estimate for solutions of (1.9): for every $p > d/(\alpha+\beta-1)$, there is a constant $C > 0$ depending only on T, d, p, α, β and $\|b\|_{L^\infty([0,T];B^\beta_{p,\infty})}$ so that

$$\|u\|_{L^\infty([0,T];B^{\alpha+\beta}_{p,\infty})} \leqslant C\|f\|_{L^\infty([0,T];B^\beta_{p,\infty})}, \tag{1.11}$$

where $B^\beta_{p,\infty}$ is the usual Besov space (see Definition 2.1 below). The above a priori estimate is the key in our solution to the pathwise well-posedness problem of SDE (1.1) when $\sigma(t,x) = \sigma(t)$ is spatially independent. The general case with variable coefficient $\sigma(t,x)$ is much more delicate. First of all, in general $x \mapsto \int_{\{|z|>1\}} f(x+\sigma(t,x)z)\nu(\mathrm{d}z)$ may not be smooth even if $f(x)$ and $\sigma(t,x)$ are smooth. Thus to treat the general case, we have to first remove the large jumps from the Lévy process Z. Next we need to impose a small condition on the oscillation of σ by using a perturbation argument and establish an estimate analogous to (1.11) but for solutions u of (1.9) where the operator $\mathscr{L}_t$ of (1.8) being redefined with $\mathbf{1}_{\{|z|\leqslant 1\}}\nu(\mathrm{d}z)$ in place of $\nu(\mathrm{d}z)$; see Theorem 3.6. This new estimate is also the key for our weak well-posedness result for SDE (1.1). Then remove the small oscillation on σ and add back large jumps from the driving Lévy process Z through a localization and patching together procedure.

The rest of this paper is organized as follows: In Section 2, we recall some well-known facts from Littlewood-Paley theory, in particular, the Bony's decomposition and Bernstein's inequalities, and establish a useful commutator estimate. In Section 3, we study the nonlocal advection equation (1.9) with irregular drift b, and obtain some a priori estimates in Besov spaces. In Section 4, we establish strong well-posedness result by utilizing these estimates, Zvonkin's transform and a suitable patching together technique. In Section 5, we first obtain the well-posedness for the martingale problem corresponding to SDE (1.1) driven by truncated Lévy process

$\widetilde{Z}$ obtained by removing large jumps from Z. We then, through a conditioning and piecing together procedure, establish the well-posedness for the martingale problem corresponding to SDE (1.1) driven by Z. The latter result will yield the weak existence and uniqueness for solutions of (1.1).

We close this section by mentioning some notations used throughout this paper: We use $:=$ as a way of definition. For $a,b\in\mathbb{R}$, $a\vee b:=\max\{a,b\}$, $a\wedge b:=\min\{a,b\}$, and $a^+:=a\vee 0$. On $\mathbb{R}^d$, $\nabla:=(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_d})$ and $\Delta:=\sum_{k=1}^d\frac{\partial^2}{\partial x_k^2}$. The letter c or C with or without subscripts stands for an unimportant constant, whose value may change in difference places. We use $A\asymp B$ to denote that A and B are comparable up to a constant, and use $A\lesssim B$ to denote $A\leqslant CB$ for some constant $C>0$. For two functions f and g on $\mathbb{R}^d$, we use $f*g$ to denote its convolution

$$f*g(x):=\int_{\mathbb{R}^d}f(x-y)g(y)\mathrm{d}y,\quad x\in\mathbb{R}^d,$$

whenever it is defined, and $\mathrm{supp}[f]$ the support of the function f on $\mathbb{R}^d$. For $p\in[1,\infty]$, we use L^p to denote the L^p space on $\mathbb{R}^d$ with respect to the Lebesgue measure $\mathrm{d}x$, and $\|f\|_p:=\big(\int_{\mathbb{R}^d}|f(x)|^p\mathrm{d}x\big)^{1/p}$ for $p\in[1,\infty)$ and $\|f\|_\infty$ the essential supremum of $|f|$.

2. Preliminary

In this section, we recall some basic facts from Littlewood-Paley theory, especially Bernstein's inequalities (see [1]). We then establish a commutator estimate, which plays an important role in our approach.

Let $\mathscr{S}(\mathbb{R}^d)$ be the Schwartz space of all rapidly decreasing functions, and $\mathscr{S}'(\mathbb{R}^d)$ the dual space of $\mathscr{S}(\mathbb{R}^d)$ called Schwartz generalized function (or tempered distribution) space. For $f\in L^1(\mathbb{R}^d)$, its Fourier transform $\mathcal{F}f=\hat f$ and inverse transform $\mathcal{F}^{-1}f=\check f$ are defined as

$$\hat f(\xi):=(2\pi)^{-d/2}\int_{\mathbb{R}^d}\mathrm{e}^{-\mathrm{i}\xi\cdot x}f(x)\mathrm{d}x,\quad \check f(\xi):=(2\pi)^{-d/2}\int_{\mathbb{R}^d}\mathrm{e}^{\mathrm{i}\xi\cdot x}f(x)\mathrm{d}x.$$

Using Schwartz's duality, the definition of Fourier transform and inverse Fourier transform can be extended to tempered distributions as follows. For any $f\in\mathscr{S}'(\mathbb{R}^d)$, $\mathcal{F}f=\hat f$ and $\mathcal{F}^{-1}f=\check f$ are the unique elements in $\mathscr{S}'(\mathbb{R}^d)$ so that

$$\mathcal{F}f(\psi)=f(\mathcal{F}\psi)\quad\text{and}\quad \mathcal{F}^{-1}f(\psi)=f(\mathcal{F}^{-1}\psi)\quad\text{for every }\psi\in\mathscr{S}(\mathbb{R}^d).$$

See, e.g., [1, §1.2.2].

For $R,R_1,R_2\geqslant 0$ with $R_1<R_2$, denote

$$B_R:=\{x\in\mathbb{R}^d:|x|\leqslant R\}\quad\text{and}\quad D_{R_1,R_2}:=\{x\in\mathbb{R}^d:R_1\leqslant|x|\leqslant R_2\}.$$

The following simple fact will be used frequently. For any two $f\in L^p(\mathbb{R}^d)$ and $g\in L^q(\mathbb{R}^d)$ with $\frac1p+\frac1q=1$ whose supports are in B_{R_0} and D_{R_1,R_2}, respectively,

$$\mathrm{supp}[f*g]\subset D_{(R_1-R_0)^+,R_2+R_0}. \tag{2.1}$$

Let $\chi:\mathbb{R}^d\to[0,1]$ be a smooth radial function so that

$$\chi(\xi)=\begin{cases}1 & \text{when }|\xi|\leqslant 1,\\ 0 & \text{when }|\xi|\geqslant 3/2.\end{cases}$$

Define

$$\varphi(\xi):=\chi(\xi)-\chi(2\xi).$$

It is easy to see that $\varphi \geqslant 0$, $\mathrm{supp}[\varphi] \subset B_{3/2} \setminus B_{1/2}$ and for each $\xi \in \mathbb{R}^d$,

$$\chi(2\xi) + \sum_{j=0}^{k} \varphi(2^{-j}\xi) = \chi(2^{-k}\xi) \to 1 \quad \text{as } k \to \infty. \tag{2.2}$$

Moreover,

$$\mathrm{supp}\left[\varphi(2^{-j}\cdot)\right] \cap \mathrm{supp}\left[\varphi(2^{-k}\cdot)\right] = \emptyset \quad \text{if } |j-k| \geqslant 2. \tag{2.3}$$

From now on, we shall fix such χ and φ, and introduce the following definitions.

Definition 2.1. *The dyadic block operator Π_j is defined by*

$$\Pi_j f := \begin{cases} \mathcal{F}^{-1}(\chi(2\cdot)\mathcal{F}f), & j = -1, \\ \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}f), & j \geqslant 0. \end{cases}$$

For $s \in \mathbb{R}$ and $p, q \in [1, \infty]$, the Besov space $B^s_{p,q}$ is defined as the set of all $f \in \mathscr{S}'(\mathbb{R}^d)$ with

$$\|f\|_{B^s_{p,q}} := \mathbf{1}_{\{q<\infty\}} \left(\sum_{j \geqslant -1} 2^{jsq} \|\Pi_j f\|_p^q \right)^{1/q} + \mathbf{1}_{\{q=\infty\}} \left(\sup_{j \geqslant -1} 2^{js} \|\Pi_j f\|_p \right) < \infty.$$

Some literature, e.g., [1, 7], uses notation Δ_j for the dyadic block operator Π_j defined above. We choose to use notation Π_j in this paper out of two considerations: (i) the dyadic block operator is a projection operator in the L^2-space, and (ii) we want to avoid possible confusion with the Laplacian operator Δ on $\mathbb{R}^d$.

For $s \geqslant 0$ and $p \in [1, \infty)$, let $H^s_p := (I - \Delta)^{-s/2}(L^p)$ be the usual Bessel potential space with norm

$$\|f\|_{H^s_p} := \|(I - \Delta)^{s/2} f\|_p.$$

Note that

$$\|f\|_{H^s_p} \asymp \|f\|_p + \|(-\Delta)^{s/2} f\|_p \quad \text{for } f \in H^s_p.$$

It should be observed that if $s > 0$ is not an integer, then the Besov space $B^s_{\infty,\infty}$ is just the usual Hölder space C^s. Moreover, Besov spaces have the following embedding relations: For any $s, s', s'' \in \mathbb{R}$ and $p, p', q, q' \in [1, \infty]$ with

$$p \leqslant p', \ q \leqslant q', \ s < s'' \ \text{ and } s - d/p = s' - d/p',$$

it holds that (cf. [4])

$$B^{s''}_{p,1} \subset H^{s''}_p \subset B^{s''}_{p,\infty} \subset B^s_{p,q} \subset B^{s'}_{p',q'}. \tag{2.4}$$

Let $h = \mathcal{F}^{-1}\chi$ be the inverse Fourier transform of χ. Define

$$h_{-1}(x) := \mathcal{F}^{-1}\chi(2\cdot)(x) = 2^{-d} h(2^{-1}x) \in \mathscr{S}(\mathbb{R}^d),$$

and for $j \geqslant 0$,

$$h_j(x) := \mathcal{F}^{-1}\varphi(2^{-j}\cdot)(x) = 2^{jd} h(2^j x) - 2^{(j-1)d} h(2^{j-1}x) \in \mathscr{S}(\mathbb{R}^d). \tag{2.5}$$

It follows from the definition that

$$\Pi_j f(x) = (h_j * f)(x) = \int_{\mathbb{R}^d} h_j(x-y) f(y) \mathrm{d}y, \quad j \geqslant -1. \tag{2.6}$$

The cut-off low frequency operator S_k is defined by

$$S_k f := \sum_{j=-1}^{k-1} \Pi_j f = 2^{(k-1)d} \int_{\mathbb{R}^d} h(2^{k-1}(x-y)) f(y) \mathrm{d}y.$$

It is easy to see that

$$\|S_k f\|_p \leqslant \|h\|_1 \|f\|_p \quad \text{and} \quad \|S_k f\|_{B^s_{p,q}} \leqslant \|h\|_1 \|f\|_{B^s_{p,q}}. \tag{2.7}$$

Moreover, one has by (2.2) that

$$\widehat{S_k f} = \chi(2^{1-k}\cdot)\hat{f}, \qquad f = \lim_{k\to\infty} S_k f = \sum_{j=-1}^{\infty} \Pi_j f. \tag{2.8}$$

For $f, g \in \mathscr{S}'(\mathbb{R}^d)$, define

$$T_f g = \sum_{k=0}^{\infty} (S_k f)(\Pi_{k+1} g), \qquad R(f,g) := \sum_{k=-1}^{\infty} \sum_{|i|\leqslant 1} (\Pi_k f)(\Pi_{k-i} g)$$

with the convention that $\Pi_{-2} := 0$. Clearly,

$$fg = T_f g + T_g f + R(f,g) \quad \text{for } f, g \in \mathscr{S}'(\mathbb{R}^d).$$

This identity is called Bony's (paraproduct) decomposition of fg.

We first recall the following Bernstein's type inequality.

Lemma 2.2. *(Bernstein's type inequality) Let $1 \leqslant p \leqslant q \leqslant \infty$. For any $k = 0, 1, \cdots$ and $\beta \in (-1, 2)$, there is a constant C such that for all $f \in \mathscr{S}'(\mathbb{R}^d)$ and $j \geqslant -1$,*

$$\|\nabla^k \Pi_j f\|_q \leqslant C 2^{(k + d(\frac{1}{p} - \frac{1}{q}))j} \|\Pi_j f\|_p, \tag{2.9}$$

and for any $j \geqslant 0$,

$$\|(-\Delta)^{\beta/2} \Pi_j f\|_q \leqslant C 2^{(\beta + d(\frac{1}{p} - \frac{1}{q}))j} \|\Pi_j f\|_p, \tag{2.10}$$

and for any $2 \leqslant p < \infty$, $j \geqslant 0$ and $\alpha \in (0, 2)$, there is a constant $c > 0$ such that for all $f \in \mathscr{S}'(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \left| (-\Delta)^{\alpha/4} |\Pi_j f|^{p/2} \right|^2 \mathrm{d}x \geqslant c 2^{\alpha j} \|\Pi_j f\|_p^p. \tag{2.11}$$

Proof. Estimates (2.9) and (2.11) can be found in [1, Lemma 2.1] and [7, Theorem 1.1]), respectively. For (2.10), its proof is essentially the same as that of [1, Lemma 2.1]. Indeed, by dilation, it suffices to prove (2.10) for $j = 0$. Let $\widetilde{\varphi} \in \mathscr{S}(\mathbb{R}^d \backslash \{0\})$ with value 1 on $\{1/2 \leqslant |x| \leqslant 3/2\}$, and $\widetilde{h} := \mathscr{F}^{-1}\widetilde{\varphi}$. Define $\widetilde{\Pi}_0 f := \mathscr{F}^{-1}(\widetilde{\varphi}\mathscr{F} f) = \widetilde{h} * f$. Since $\widetilde{\Pi}_0 \Pi_0 = \Pi_0$ as $\widetilde{\varphi}\varphi = \varphi$, we have by Young's inequality,

$$\|(-\Delta)^{\beta/2} \Pi_0 f\|_q \leqslant \|(-\Delta)^{\beta/2} \widetilde{\Pi}_0 \Pi_0 f\|_q \leqslant \|(-\Delta)^{\beta/2} \widetilde{h}\|_r \|\Pi_0 f\|_p,$$

where $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}$. Since $|\xi|^\beta \widetilde{\varphi}(\xi) \in \mathscr{S}(\mathbb{R}^d)$, we have $(-\Delta)^{\beta/2}\widetilde{h} \in \mathscr{S}(\mathbb{R}^d)$ and so $\|(-\Delta)^{\beta/2}\widetilde{h}\|_r < \infty$. This establishes (2.10) for $j = 0$, and consequently for all $j \geqslant 1$ by dilation. □

The following commutator estimate plays an important role in this paper.

Lemma 2.3. *Let $p, p_1, p_2, q_1, q_2 \in [1, \infty]$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1$. For any $\beta_1 \in (0, 1)$ and $\beta_2 \in [-\beta_1, 0]$, there is a constant $C > 0$ depending only on $d, p, p_1, p_2, \beta_1, \beta_2$ such that*

$$\|[\Pi_j, f]g\|_p \leqslant C 2^{-j(\beta_1 + \beta_2)} \begin{cases} \|f\|_{B^{\beta_1}_{p_1,\infty}} \|g\|_{p_2}, & \text{if } \beta_2 = 0, \\ \|f\|_{B^{\beta_1}_{p_1,\infty}} \|g\|_{B^{\beta_2}_{p_2,\infty}}, & \text{if } \beta_1 + \beta_2 > 0, \\ \|f\|_{B^{\beta_1}_{p_1,q_1}} \|g\|_{B^{\beta_2}_{p_2,q_2}}, & \text{if } \beta_1 + \beta_2 = 0, \end{cases}$$

where $[\Pi_j, f]g := \Pi_j(fg) - f\Pi_j g$.

Proof. We first consider the case $\beta_2 = 0$. In this case, by (2.6),

$$[\Pi_j, f]g(x) = \int_{\mathbb{R}^d} h_j(y)(f(x-y) - f(x))g(x-y)\mathrm{d}y.$$

For any $p \in [1, \infty]$ and $s \in (0, 1)$, by Theorem 2.36 of [1],

$$\|f(\cdot - y) - f(\cdot)\|_p \leqslant C|y|^s \|f\|_{B^s_{p,\infty}}. \tag{2.12}$$

Using Hölder's inequality and (2.5), we have

$$\begin{aligned} \|[\Pi_j, f]g\|_p &\leqslant \int_{\mathbb{R}^d} h_j(y)\|f(\cdot - y) - f(\cdot)\|_{p_1}\|g\|_{p_2}\mathrm{d}y \\ &\lesssim \|f\|_{B^{\beta_1}_{p_1,\infty}}\|g\|_{p_2} \int_{\mathbb{R}^d} |h_j(y)|\,|y|^{\beta_1}\mathrm{d}y \\ &= \|f\|_{B^{\beta_1}_{p_1,\infty}}\|g\|_{p_2} 2^{-(j-1)\beta_1} \int_{\mathbb{R}^d} |2^d h(2y) - h(y)|\,|y|^{\beta_1}\mathrm{d}y \\ &\lesssim 2^{-j\beta_1}\|f\|_{B^{\beta_1}_{p_1,\infty}}\|g\|_{p_2}. \end{aligned} \tag{2.13}$$

Next we consider the case $\beta_2 \in [-\beta_1, 0)$. By using Bony's decomposition, we can write

$$[\Pi_j, f]g = [\Pi_j, T_f]g + \Pi_j(T_g f) - T_{\Pi_j g} f + \Pi_j R(f, g) - R(f, \Pi_j g).$$

It follows from (2.8) and (2.3) that

$$\mathcal{F}(\Pi_j(S_{k-1}f\Pi_k g)) = \varphi(2^{-j}\cdot)\Big((\chi(2^{2-k}\cdot)\hat{f}) * (\varphi(2^{-k}\cdot)\hat{g})\Big) = 0 \quad \text{for } |k-j| > 2,$$

and

$$\Pi_j\Pi_k = 0 \quad \text{for } |k-j| \geqslant 2.$$

Therefore, by (2.7) and (2.13) we have

$$\begin{aligned} \|[\Pi_j, T_f]g\|_p &= \Big\|\sum_{|k-j|\leqslant 2}\Big(\Pi_j(S_{k-1}f\Pi_k g) - S_{k-1}f\Pi_j\Pi_k g\Big)\Big\|_p \\ &\leqslant \sum_{|k-j|\leqslant 2}\big\|[\Pi_j, S_{k-1}f]\Pi_k g\big\|_p \\ &\lesssim 2^{-j\beta_1}\sum_{|k-j|\leqslant 2}\|S_{k-1}f\|_{B^{\beta_1}_{p_1,\infty}}\|\Pi_k g\|_{p_2} \\ &\lesssim 2^{-j\beta_1}\|f\|_{B^{\beta_1}_{p_1,\infty}}\sum_{|k-j|\leqslant 2} 2^{-k\beta_2}\|g\|_{B^{\beta_2}_{p_2,\infty}} \\ &\lesssim 2^{-j(\beta_1+\beta_2)}\|f\|_{B^{\beta_1}_{p_1,\infty}}\|g\|_{B^{\beta_2}_{p_2,\infty}}. \end{aligned}$$

Similarly, we have by Hölder's inequality and $\beta_2 < 0$,

$$\begin{aligned} \|\Pi_j(T_g f)\|_p &= \Big\|\sum_{|k-j|\leqslant 2}\Pi_j(S_{k-1}g\Pi_k f)\Big\|_p \leqslant \sum_{|k-j|\leqslant 2}\|\Pi_j(S_{k-1}g\Pi_k f)\|_p \\ &\lesssim \sum_{|k-j|\leqslant 2}\|S_{k-1}g\Pi_k f\|_p \leqslant \sum_{|k-j|\leqslant 2}\sum_{m\leqslant k-2}\|\Pi_m g\,\Pi_k f\|_p \\ &\lesssim \|g\|_{B^{\beta_2}_{p_2,\infty}}\|f\|_{B^{\beta_1}_{p_1,\infty}}\sum_{|k-j|\leqslant 2}\sum_{m\leqslant k-2} 2^{-m\beta_2}2^{-k\beta_1} \\ &\lesssim \|g\|_{B^{\beta_2}_{p_2,\infty}}\|f\|_{B^{\beta_1}_{p_1,\infty}} 2^{-j(\beta_2+\beta_1)}, \end{aligned}$$

and

$$\begin{aligned}\|T_{\Pi_j g}f\|_p &\leqslant \sum_{k\geqslant j-2}\|S_{k-1}\Pi_j g\Pi_k f\|_p \leqslant \sum_{k\geqslant j-2}\|\Pi_k f\|_{p_1}\|S_{k-1}\Pi_j g\|_{p_2}\\ &\leqslant \sum_{k\geqslant j-2}2^{-k\beta_1}\|f\|_{B^{\beta_1}_{p_1,\infty}}\|\Pi_j g\|_{p_2}\leqslant C2^{-j(\beta_1+\beta_2)}\|f\|_{B^{\beta_1}_{p_1,\infty}}\|g\|_{B^{\beta_2}_{p_2,\infty}}.\end{aligned}$$

Finally, we have

$$\begin{aligned}\|\Pi_j R(f,g)\|_p &= \Big\|\sum_{|i|\leqslant 1,k\geqslant j-4}\Pi_j(\Pi_k f\Pi_{k-i}g)\Big\|_p \lesssim \sum_{|i|\leqslant 1,k\geqslant j-4}\|\Pi_k f\|_{p_1}\|\Pi_{k-i}g\|_{p_2}\\ &\lesssim \sum_{|i|\leqslant 1,k\geqslant j-4}2^{-k(\beta_1+\beta_2)}\Big(2^{k\beta_1}\|\Pi_k f\|_{p_1}\Big)\Big(2^{k\beta_2}\|\Pi_{k-i}g\|_{p_2}\Big)\\ &\lesssim 2^{-j(\beta_1+\beta_2)}\begin{cases}\|f\|_{B^{\beta_1}_{p_1,\infty}}\|g\|_{B^{\beta_2}_{p_2,\infty}}, & \beta_1+\beta_2>0,\\ \|f\|_{B^{\beta_1}_{p_1,q_1}}\|g\|_{B^{\beta_2}_{p_2,q_2}}, & \beta_1+\beta_2=0,\end{cases}\end{aligned}$$

where $\frac{1}{q_1}+\frac{1}{q_2}=1$, and

$$\|R(f,\Pi_j g)\|_p = \Big\|\sum_{|i|\leqslant 1,|k-j|\leqslant 1}\Pi_{k-i}f\Pi_k\Pi_j g\Big\|_p \lesssim \|f\|_{B^{\beta_1}_{p_1,\infty}}\|g\|_{B^{\beta_2}_{p_2,\infty}}2^{-j(\beta_1+\beta_2)}.$$

Combining the above calculations and noticing $\|f\|_{B^s_{p,\infty}}\leqslant\|f\|_{B^s_{p,q}}$, we complete the proof. □

3. Nonlocal parabolic equations

In this section we study the solvability and regularity of nonlocal parabolic equation (1.9) with Hölder drift b. Let σ be a constant $d\times d$-matrix and ν a measure on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d\setminus\{0\}}(|z|^2\wedge 1)\nu(\mathrm{d}z)<\infty.$$

We define a Lévy-type operator $\mathcal{L}^\nu_\sigma$ by

$$\mathcal{L}^\nu_\sigma f(x):=\int_{\mathbb{R}^d}\Big(f(x+\sigma z)-f(x)-\mathbf{1}_{\{|z|\leqslant 1\}}\sigma z\cdot\nabla f(x)\Big)\nu(\mathrm{d}z),\ \ f\in\mathscr{S}(\mathbb{R}^d).$$

By Fourier's transform, we have

$$\widehat{\mathcal{L}^\nu_\sigma f}(\xi)=\psi^\nu_\sigma(\xi)\hat f(\xi),$$

where the symbol $\psi^\nu_\sigma(\xi)$ is given by

$$\psi^\nu_\sigma(\xi)=\int_{\mathbb{R}^d}\left(\mathrm{e}^{\mathrm{i}\xi\cdot\sigma z}-1-\mathbf{1}_{\{|z|\leqslant 1\}}\mathrm{i}\sigma z\cdot\xi\right)\nu(\mathrm{d}z).$$

Now let $\sigma(t,x):\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^d\otimes\mathbb{R}^d$ be a Borel measurable function. Define a time-dependent Lévy-type operator

$$\mathscr{L}_t f(x):=\mathcal{L}^\nu_{\sigma(t,x)}f(x).$$

In this section, for $\lambda\geqslant 0$, we study the solvability of the following equation with Besov drift $b(t,x):\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^d$,

$$\partial_t u=(\mathscr{L}_t-\lambda)u+b\cdot\nabla u+f\quad\text{with } u(0)=0. \tag{3.1}$$

For a space-time function $f:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}$ and $T>0$, define

$$\|f\|_{L^\infty_T(B^s_{p,q})}:=\sup_{t\in[0,T]}\|f(t,\cdot)\|_{B^s_{p,q}}.$$

3.1. **Constant diffusion matrix case.** In this subsection we consider equation (3.1) with time dependent constant coefficient $\sigma(t,x)=\sigma(t)$. First of all, we establish the following Bernstein's type inequality for nonlocal operator $\mathcal{L}^\nu_\sigma$, which plays a crucial role in the sequel.

Lemma 3.1. *Suppose* $\mathrm{Re}(\psi^\nu_\sigma(\xi))\leqslant -c_0|\xi|^\alpha$ *for some* $c_0>0$. *Then for any* $p>2$, *there is a constant* $c_p=c(c_0,p)>0$ *such that for* $j=0,1,\cdots$,

$$\int_{\mathbb{R}^d}|\Pi_j f|^{p-2}(\Pi_j f)\mathcal{L}^\nu_\sigma\Pi_j f\mathrm{d}x\leqslant -c_p2^{\alpha j}\|\Pi_j f\|_p^p, \tag{3.2}$$

and for $j=-1$,

$$\int_{\mathbb{R}^d}|\Pi_{-1} f|^{p-2}(\Pi_{-1} f)\mathcal{L}^\nu_\sigma\Pi_{-1} f\mathrm{d}x\leqslant 0.$$

Proof. For $p\geqslant 2$, by the elementary inequality $|r|^{p/2}-1\geqslant\frac{p}{2}(r-1)$ for $r\in\mathbb{R}$, we have

$$|a|^{p/2}-|b|^{p/2}\geqslant\tfrac{p}{2}(a-b)b|b|^{p/2-2},\quad a,b\in\mathbb{R}.$$

Letting g be a smooth function, by definition we have

$$\begin{aligned}\mathcal{L}^\nu_\sigma|g|^{p/2}(x)&=\int_{\mathbb{R}^d}\Big(|g(x+\sigma z)|^{p/2}-|g(x)|^{p/2}-\mathbf{1}_{|z|\leqslant 1}\sigma z\cdot\nabla_x|g(x)|^{p/2}\Big)\nu(\mathrm{d}z)\\&\geqslant\frac{p}{2}|g(x)|^{p/2-2}g(x)\int_{\mathbb{R}^d}\Big(g(x+\sigma z)-g(x)-\mathbf{1}_{|z|\leqslant 1}\sigma z\cdot\nabla g(x)\Big)\nu(\mathrm{d}z)\\&=\frac{p}{2}|g(x)|^{p/2-2}g(x)\mathcal{L}^\nu_\sigma g(x).\end{aligned}$$

Multiplying both sides by $|g|^{p/2}$ and then integrating in x over $\mathbb{R}^d$, by Plancherel's formula, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}|g|^{p-2}g\mathcal{L}^\nu_\sigma g\mathrm{d}x&\leqslant\frac{2}{p}\int_{\mathbb{R}^d}|g|^{p/2}\mathcal{L}^\nu_\sigma|g|^{p/2}\mathrm{d}x=\frac{2}{p}\int_{\mathbb{R}^d}|\widehat{|g|^{p/2}}(\xi)|^2\psi^\nu_\sigma(\xi)\mathrm{d}\xi\\&=\frac{2}{p}\int_{\mathbb{R}^d}|\widehat{|g|^{p/2}}(\xi)|^2\mathrm{Re}(\psi^\nu_\sigma(\xi))\mathrm{d}\xi\leqslant-\frac{2c_0}{p}\int_{\mathbb{R}^d}|\widehat{|g|^{p/2}}(\xi)|^2|\xi|^\alpha\mathrm{d}\xi\\&\leqslant-\frac{2c_0}{p}\int_{\mathbb{R}^d}|(-\Delta)^{\alpha/4}|g|^{p/2}|^2\mathrm{d}x,\end{aligned}$$

which in turn gives the desired estimate by taking $g=\Pi_j f$ and (2.11). □

We introduce the following assumptions about drift $b(t,x)$:

($\mathbf{H}^{\beta,p}_b$) $b=b_1+b_2$ satisfies that for some $\beta\in[0,1]$ and $p>1$,

$$\|b_1\|_{L^\infty_T(B^\beta_{p,\infty})}+\|b_2\|_{L^\infty_T(B^\beta_{\infty,\infty})}\leqslant\kappa<\infty. \tag{3.3}$$

Remark 3.2. The reason to consider b of the form b_1+b_2 satisfying condition (3.3) is the need in our study of strong well-posedness of solutions to SDE (1.1). In Theorem 4.1, we will apply Theorem 3.6 below with $b_1=b(t,x)$ and $b_2=-\mathbf{1}_{\alpha\in(0,1)}\sigma(t,x)\int_{|z|\leqslant 1}z\nu(\mathrm{d}z)$ to show that the SDE driven by the truncated Lévy process $\widetilde{Z}$ obtained from Z by removing jumps of size larger than 1 has a unique strong solution. Moreover, $B^\beta_{q,\infty}\subset B^\beta_{p,\infty}\cap B^\beta_{\infty,\infty}$ for any $q>p$.

We introduce the following parameter set for later use:

$$\Theta := (c_0, T, d, p, \alpha, \beta, \kappa).$$

Now we can state the following main a priori estimate of this subsection.

Theorem 3.3. *Let $\beta \in (0,1)$ and $\alpha \in (0,2)$ with $\alpha+\beta>1$. Let $T>0$ and $p \in (\frac{d}{\alpha+\beta-1} \vee 2, \infty)$. Suppose that $(\mathbf{H}_b^{\beta,p})$ holds and for some $c_0>0$,*

$$\mathrm{Re}(\psi^\nu_{\sigma(t)}(\xi)) \leqslant -c_0|\xi|^\alpha, \quad \xi \in \mathbb{R}^d, \quad t \in [0,T].$$

For any $\eta \in [0,\beta]$ and $q \in [2,p]$, there is a constant $C = C(\eta, q, \Theta) > 0$ independent of λ so that for any classical solution u to the nonlocal PDE (3.1) with $f \in L^\infty_T(B^\eta_{q,\infty})$ and $\lambda \geqslant 0$,

$$\|u\|_{L^\infty_T(B^{\alpha+\eta}_{q,\infty})} \leqslant C\|f\|_{L^\infty_T(B^\eta_{q,\infty})}. \tag{3.4}$$

Moreover, for any $\gamma \in [0, \alpha+\eta)$,

$$\|u\|_{L^\infty_T(B^{\gamma}_{q,\infty})} \leqslant c_\lambda\|f\|_{L^\infty_T(B^\eta_{q,\infty})}, \tag{3.5}$$

where $c_\lambda = c(\lambda,\gamma,\eta,q,\Theta) \to 0$ as $\lambda \to \infty$.

Proof. Applying the operator Π_j on both sides of (3.1), we have

$$\partial_t \Pi_j u = (\mathscr{L}_t - \lambda)\Pi_j u + \Pi_j(b\cdot\nabla u) + \Pi_j f.$$

For $q \geqslant 2$, by the chain rule or multiplying both sides by $|\Pi_j u|^{q-2}\Pi_j u$ and then integrating in x, we obtain

$$\begin{aligned}
\frac{\partial_t \|\Pi_j u\|_q^q}{q} &= \int_{\mathbb{R}^d}\Big(|\Pi_j u|^{q-2}(\Pi_j u)\big(\mathscr{L}_t\Pi_j u + \Pi_j(b\cdot\nabla u) + \Pi_j f - \lambda\Pi_j u\big)\Big)\mathrm{d}x\\
&= \int_{\mathbb{R}^d}|\Pi_j u|^{q-2}(\Pi_j u)\mathscr{L}_t\Pi_j u\mathrm{d}x + \int_{\mathbb{R}^d}|\Pi_j u|^{q-2}(\Pi_j u)\,[\Pi_j, b\cdot\nabla]u\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^d}|\Pi_j u|^{q-2}(\Pi_j u)\,(b\cdot\nabla)\Pi_j u\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^d}|\Pi_j u|^{q-2}(\Pi_j u)\Pi_j f\mathrm{d}x - \lambda\|\Pi_j u\|_q^q\\
&=: I_j^{(1)} + I_j^{(2)} + I_j^{(3)} + I_j^{(4)} + I_j^{(5)}.
\end{aligned}$$

For $I_j^{(1)}$, recalling $\mathscr{L}_t = \mathcal{L}^\nu_{\sigma(t)}$ and by Lemma 3.1, there is a $c>0$ such that

$$I_{-1}^{(1)} \leqslant 0, \quad I_j^{(1)} \leqslant -c2^{\alpha j}\|\Pi_j u\|_q^q, \quad j = 0,1,2,\cdots.$$

For $I_j^{(2)}$, using Lemma 2.3 with

$$f = b^i, \quad g = \partial_i u \quad \text{for } i = 1,\cdots,d,$$

and

$$\beta_1 = \beta, \quad \beta_2 = \eta-\beta, \quad q_1 = \infty \quad \text{and} \quad q_2 = 1,$$

by Hölder's inequality and recalling $b = b_1 + b_2$, we have for all $j = -1,0,1,\cdots$,

$$\begin{aligned}
I_j^{(2)} &\leqslant \|[\Pi_j, b\cdot\nabla]u\|_q\|\Pi_j u\|_q^{q-1}\\
&\lesssim 2^{-\eta j}\Big(\|b_1\|_{B^\beta_{p,\infty}}\|u\|_{B^{1-\beta+\eta}_{r,1}} + \|b_2\|_{B^\beta_{\infty,\infty}}\|u\|_{B^{1-\beta+\eta}_{q,1}}\Big)\|\Pi_j u\|_q^{q-1},
\end{aligned}$$

where

$$1/r = 1/q - 1/p.$$

For $I_j^{(3)}$, note that

$$
\begin{aligned}
I_j^{(3)} &= \int_{\mathbb{R}^d}((b-S_jb)\cdot\nabla)\Pi_j u\,|\Pi_j u|^{q-2}\Pi_j u\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^d}(S_jb\cdot\nabla)\Pi_j u\,|\Pi_j u|^{q-2}\Pi_j u\mathrm{d}x =: I_j^{(31)}+I_j^{(32)}.
\end{aligned}
$$

By Hölder's inequality and Bernstein's inequality (2.9), we have

$$
\begin{aligned}
I_j^{(31)} &\leqslant \sum_{k\geqslant j}\|(\Pi_k b\cdot\nabla)\Pi_j u\|_q\|\Pi_j u\|_q^{q-1}\\
&\leqslant \sum_{k\geqslant j}\Big(\|\Pi_k b_1\|_p\|\nabla\Pi_j u\|_r+\|\Pi_k b_2\|_\infty\|\nabla\Pi_j u\|_q\Big)\|\Pi_j u\|_q^{q-1}\\
&\lesssim 2^{(1+d/p)j}\|\Pi_j u\|_q^q\sum_{k\geqslant j}\Big(\|\Pi_k b_1\|_p+\|\Pi_k b_2\|_\infty\Big)\\
&\lesssim 2^{(1+d/p-\beta)j}\|\Pi_j u\|_q^q\Big(\|b_1\|_{B^\beta_{p,\infty}}+\|b_2\|_{B^\beta_{\infty,\infty}}\Big).
\end{aligned}
$$

For $I_j^{(32)}$, we have by the divergence theorem and (2.9) again,

$$
\begin{aligned}
I_j^{(32)} &= \frac{1}{q}\int_{\mathbb{R}^d}(S_jb\cdot\nabla)|\Pi_j u|^q\mathrm{d}x = -\frac{1}{q}\int_{\mathbb{R}^d}S_j\mathrm{div}b\,|\Pi_j u|^q\mathrm{d}x\\
&\leqslant \frac{1}{q}\|S_j\mathrm{div}b\|_\infty\|\Pi_j u\|_q^q \leqslant \frac{1}{q}\sum_{k\leqslant j}\|\Pi_k\mathrm{div}b\|_\infty\|\Pi_j u\|_q^q\\
&\lesssim \sum_{k\leqslant j}2^{k(1+d/p)}\Big(\|\Pi_k b_1\|_p+\|\Pi_k b_2\|_\infty\Big)\|\Pi_j u\|_q^q\\
&\lesssim 2^{(1+d/p-\beta)j}\Big(\|b_1\|_{B^\beta_{p,\infty}}+\|b_2\|_{B^\beta_{\infty,\infty}}\Big)\|\Pi_j u\|_q^q.
\end{aligned}
$$

Combining the above estimates, we obtain

$$
\begin{aligned}
\partial_t\|\Pi_j u\|_q^q/q &\leqslant -c2^{\alpha j}1_{j\geqslant 0}\|\Pi_j u\|_q^q-\lambda\|\Pi_j u\|_q^q\\
&\quad+C2^{-\eta j}\Big(\|u\|_{B^{1-\beta+\eta}_{r,1}}+\|u\|_{B^{1-\beta+\eta}_{q,1}}\Big)\|\Pi_j u\|_q^{q-1}\\
&\quad+C2^{(1-\beta+d/p)j}\|\Pi_j u\|_q^q+C\|\Pi_j u\|_q^{q-1}\|\Pi_j f\|_q\\
&\leqslant -\Big(c2^{\alpha j}1_{j\geqslant 0}+\lambda-C2^{(1-\beta+d/p)j}\Big)\|\Pi_j u\|_q^q\\
&\quad+C\Big(2^{-\eta j}\big(\|u\|_{B^{1-\beta+\eta}_{r,1}}+\|u\|_{B^{1-\beta+\eta}_{q,1}}\big)+\|\Pi_j f\|_q\Big)\|\Pi_j u\|_q^{q-1}.
\end{aligned}
$$

Since $1-\beta+d/p<\alpha$, by dividing both sides by $\|\Pi_j u\|_q^{q-1}$ and using Young's inequality, we get for some $c_0,\lambda_0>0$ and all $j\geqslant -1$,

$$
\partial_t\|\Pi_j u\|_q \leqslant -(c_02^{\alpha j}+\lambda-\lambda_0)\|\Pi_j u\|_q+C2^{-\eta j}\big(\|u\|_{B^{1-\beta+\eta}_{r,1}}+\|u\|_{B^{1-\beta+\eta}_{q,1}}\big)+C\|\Pi_j f\|_q,
$$

which implies by Gronwall's inequality that for all $j\geqslant -1$,

$$
\begin{aligned}
\|\Pi_j u(t)\|_q &\lesssim \int_0^t \mathrm{e}^{-(c_02^{\alpha j}+\lambda-\lambda_0)(t-s)}\Big(2^{-\eta j}\big(\|u\|_{B^{1-\beta+\eta}_{r,1}}+\|u\|_{B^{1-\beta+\eta}_{q,1}}\big)+\|\Pi_j f\|_q\Big)\mathrm{d}s\\
&\leqslant 2^{-\eta j}\int_0^t \mathrm{e}^{-(c_02^{\alpha j}+\lambda-\lambda_0)(t-s)}\Big(\|u\|_{B^{1-\beta+\eta}_{r,1}}+\|u\|_{B^{1-\beta+\eta}_{q,1}}+\|f\|_{B^\eta_{q,\infty}}\Big)\mathrm{d}s \qquad (3.6)\\
&\leqslant 2^{-\eta j}2^{\lambda_0 t}\int_0^t \mathrm{e}^{-c_02^{\alpha j}s}\mathrm{d}s\Big(\|u\|_{L^\infty_t(B^{1-\beta+\eta}_{r,1})}+\|u\|_{L^\infty_t(B^{1-\beta+\eta}_{q,1})}+\|f\|_{L^\infty_t(B^\eta_{q,\infty})}\Big).
\end{aligned}
$$

Hence,

$$\begin{aligned}\|u(t)\|_{B^{\alpha+\eta}_{q,\infty}} &= \sup_{j\geqslant -1}\Big(2^{(\alpha+\eta)j}\|\Pi_j u(t)\|_q\Big)\\ &\lesssim \|u\|_{L^\infty_t(B^{1-\beta+\eta}_{r,1})}+\|u\|_{L^\infty_t(B^{1-\beta+\eta}_{q,1})}+\|f\|_{L^\infty_t(B^\eta_{q,\infty})},\end{aligned}\tag{3.7}$$

where we have used that $2^{\alpha j}\int_0^t \mathrm{e}^{-c_0 2^{\alpha j}s}\mathrm{d}s=(1-\mathrm{e}^{-c_0 2^{\alpha j}})/c_0\leqslant 1/c_0$.

Let $\theta\in(0,\alpha+\beta-d/p-1)$. By embedding relation (2.4) and interpolation theorem, we have for all $\varepsilon\in(0,1)$,

$$\|u\|_{B^{1-\beta+\eta}_{r,1}}\leqslant C\|u\|_{B^{\alpha+\eta-\theta}_{q,\infty}}\leqslant C\|u\|^{1-\frac{\theta}{\alpha}}_{B^{\alpha+\eta}_{q,\infty}}\|u(s)\|^{\frac{\theta}{\alpha}}_{B^\eta_{q,\infty}}\leqslant \varepsilon\|u\|_{B^{\alpha+\eta}_{q,\infty}}+C_\varepsilon\|u\|_{B^\eta_{q,\infty}},$$

and similarly,

$$\|u\|_{B^{1-\beta+\eta}_{q,1}}\leqslant \varepsilon\|u\|_{B^{\alpha+\eta}_{q,\infty}}+C_\varepsilon\|u\|_{B^\eta_{q,\infty}}.$$

Substituting these into (3.7) and letting ε be small enough, we get

$$\|u\|_{L^\infty_t(B^{\alpha+\eta}_{q,\infty})}\lesssim \|u\|_{L^\infty_t(B^\eta_{q,\infty})}+\|f\|_{L^\infty_t(B^\eta_{q,\infty})},\tag{3.8}$$

and also,

$$\|u\|_{L^\infty_t(B^{1-\beta+\eta}_{r,1})}+\|u\|_{L^\infty_t(B^{1-\beta+\eta}_{q,1})}\lesssim \|u\|_{L^\infty_t(B^\eta_{q,\infty})}+\|f\|_{L^\infty_t(B^\eta_{q,\infty})}.\tag{3.9}$$

Now, multiplying both sides of (3.6) by $2^{\eta j}$ and then taking supremum over j, we obtain

$$\begin{aligned}\|u(t)\|_{B^\eta_{q,\infty}} &\lesssim \int_0^t \mathrm{e}^{-(\lambda-\lambda_0)(t-s)}\Big(\|u\|_{B^{1-\beta+\eta}_{r,1}}+\|u\|_{B^{1-\beta+\eta}_{q,1}}+\|f\|_{B^\eta_{q,\infty}}\Big)\mathrm{d}s\\ &\lesssim \int_0^t \|u\|_{L^\infty_s(B^\eta_{q,\infty})}\mathrm{d}s+\Big(1\wedge|\lambda-\lambda_0|^{-1}\Big)\|f\|_{L^\infty_t(B^\eta_{q,\infty})}.\end{aligned}$$

Thus by Gronwall's inequality we get

$$\|u\|_{L^\infty_T(B^\eta_{q,\infty})}\leqslant C\Big(1\wedge|\lambda-\lambda_0|^{-1}\Big)\|f\|_{L^\infty_T(B^\eta_{q,\infty})},\tag{3.10}$$

where $C=C(\eta,q,\Theta)$, which together with (3.8) yields (3.4). Combining (3.4) with (3.10) and using the interpolation theorem again, we obtain (3.5). ☐

Remark 3.4. If we take $\eta=0$ and $q>d/\alpha$ in (3.5), then by embedding (2.4),

$$\|u\|_{L^\infty_T(L^\infty)}\leqslant c_\lambda\|f\|_{L^\infty_T(B^0_{q,\infty})}\leqslant c_\lambda\|f\|_{L^\infty_T(L^q)}.$$

Such type maximal estimate is useful for deriving Krylov's estimate, which is crucial in the study of SDEs with rough drifts (cf. [30]).

By the above a priori estimate, we have the following existence and uniqueness of classical solutions to PDE (3.1).

Theorem 3.5. *Let $\beta\in(0,1)$ and $\alpha\in(0,2)$ with $\alpha+\beta>1$. Let $T>0$ and $p\in(\frac{d}{\alpha+\beta-1}\vee\frac{d}{\beta}\vee 2,\infty)$. Suppose that $(\mathbf{H}^{\beta,p}_b)$ holds and for some $c_0>0$,*

$$\mathrm{Re}(\psi^\nu_{\sigma(t)}(\xi))\leqslant -c_0|\xi|^\alpha,\quad \xi\in\mathbb{R}^d,\quad t\in[0,T].$$

For any $f\in L^\infty_T(B^\beta_{p,\infty})$ and $\lambda\geqslant 0$, there exists a unique classical solution $u\in L^\infty_T(B^{\alpha+\beta}_{p,\infty})$ to the nonlocal parabolic equation (3.1) in the sense that for all $(t,x)\in[0,T]\times\mathbb{R}^d$,

$$u(t,x)=\int_0^t(\mathscr{L}_s-\lambda)u(s,x)\mathrm{d}s+\int_0^t(b\cdot\nabla u)(s,x)\mathrm{d}s+\int_0^t f(s,x)\mathrm{d}s.\tag{3.11}$$

Moreover, there is a constant $C = C(\Theta) > 0$ independent of $\lambda \geqslant 0$,

$$\|u\|_{L^\infty_T(B^{\alpha+\beta}_{p,\infty})} \leqslant C\|f\|_{L^\infty_T(B^\beta_{p,\infty})}, \tag{3.12}$$

and for any $\gamma \in (0, \alpha+\beta)$,

$$\|u\|_{L^\infty_T(B^\gamma_{p,\infty})} \leqslant c_\lambda\|f\|_{L^\infty_T(B^\beta_{p,\infty})}, \tag{3.13}$$

where $c_\lambda = c(\lambda, \gamma, \Theta) \to 0$ as $\lambda \to \infty$.

Proof. Let ρ be a non-negative smooth function with compact support in $\mathbb{R}^d$ and $\int_{\mathbb{R}^d} \rho(x)\mathrm{d}x = 1$. Define $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(\varepsilon^{-1}x)$, $b_\varepsilon := \rho_\varepsilon * b$, $f_\varepsilon := \rho_\varepsilon * f$. Let u_ε be the smooth solution of PDE (3.1) corresponding to b_ε and f_ε. That is, for all $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$u_\varepsilon(t,x) = \int_0^t (\mathscr{L}_s - \lambda)u_\varepsilon(s,x)\mathrm{d}s + \int_0^t (b_\varepsilon \cdot \nabla u_\varepsilon)(s,x)\mathrm{d}s + \int_0^t f_\varepsilon(s,x)\mathrm{d}s. \tag{3.14}$$

By the a priori estimate (3.12) and (2.4), for any $\delta \in (0, \beta - d/p)$, we have

$$\sup_{0<\varepsilon\leqslant 1} \|u_\varepsilon\|_{L^\infty_T(B^{\alpha+\delta}_{\infty,\infty})} \lesssim \sup_{0<\varepsilon\leqslant 1} \|u_\varepsilon\|_{L^\infty_T(B^{\alpha+\beta}_{p,\infty})} \leqslant C\|f\|_{L^\infty_T(B^\beta_{p,\infty})}. \tag{3.15}$$

Since $\alpha + \beta > 1$ and $p > \frac{d}{\alpha+\beta-1}$, we can choose $\delta < \beta - \frac{d}{p}$ so that $\alpha + \delta > 1$. Note that for every $(s,x) \in [0,T] \times \mathbb{R}^d$,

$$\begin{aligned} |\mathscr{L}_s u_\varepsilon(s,x)| &\leqslant \int_{0<|z|\leqslant 1} |u_\varepsilon(s, x+\sigma(s)z) - u_\varepsilon(x) - \sigma(s)z \cdot \nabla u_\varepsilon(s,x)|\,\nu(\mathrm{d}z) \\ &\quad + \int_{|z|>1} |u_\varepsilon(s, x+\sigma(s)z) - u_\varepsilon(x)|\,\nu(\mathrm{d}z) \\ &\leqslant c_1\|u_\varepsilon\|_{L^\infty_T(B^{\alpha+\delta}_{\infty,\infty})} \leqslant c_2\|f\|_{L^\infty_T(B^\beta_{p,\infty})}, \end{aligned}$$

where the positive constants c_1 and c_2 are independent of $\varepsilon > 0$. Thus, by (3.14)-(3.15), we have for all $0 \leqslant t \leqslant t' \leqslant T$,

$$\lim_{|t-t'|\to 0} \sup_{0<\varepsilon\leqslant 1} \|u_\varepsilon(t) - u_\varepsilon(t')\|_\infty = 0.$$

Now, by Ascolli-Arzela's theorem, there is a decreasing sequence $\varepsilon_k \to 0$ and a continuous function u so that for any $R > 0$,

$$\lim_{k\to\infty} \|u_{\varepsilon_k} - u\|_{L^\infty([0,T];C^1(B_R))} \to 0,$$

and for any $\delta \in (0, \beta - d/p)$,

$$\|u\|_{L^\infty_T(B^{\alpha+\delta}_{\infty,\infty})} \overset{(2.4)}{\lesssim} \|u\|_{L^\infty_T(B^{\alpha+\beta}_{p,\infty})} \lesssim \|f\|_{L^\infty_T(B^\beta_{p,\infty})}.$$

By taking $\varepsilon \to 0$ along the sequence ε_k in (3.14), one concludes that u is a classical solution of PDE (3.1) and (3.11) holds. □

3.2. **Variable diffusion matrix case.** In this subsection we consider the variable diffusion coefficient case, and introduce the following assumptions on $\sigma(t,x)$:

$(\mathbf{H}^\theta_\varepsilon)$ There are $\theta, \varepsilon \in (0,1)$ and $\Lambda \geqslant 1$ such that

$$\|\sigma(t,x) - \sigma(t,y)\| \leqslant \Lambda|x-y|^\theta \quad \text{and} \quad \sigma(t,x) = \sigma(t,0) \ \text{ for } |x| \geqslant \varepsilon, \tag{3.16}$$

$$\Lambda^{-1}|\xi|^2 \leqslant |\sigma(t,0)\xi|^2 \leqslant \Lambda|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^d. \tag{3.17}$$

Notice that (3.16) means that σ only varies near 0 and this implies that

$$\|\sigma(t,x+y)-\sigma(t,x)\|\leqslant C(|y|\wedge\varepsilon)^\theta. \tag{3.18}$$

About the Lévy measure ν, we assume

($\mathbf{H}^\alpha_\nu$) There are $\nu_1,\nu_2\in\mathbb{L}^{(\alpha)}_{non}$ so that

$$\nu_1(A)\leqslant\nu(A)\leqslant\nu_2(A)\quad\text{for every } A\in\mathscr{B}(B_1).$$

In particular, for any $\gamma>\alpha>\gamma'$ and $\delta>0$,

$$\int_{|z|\leqslant\delta}|z|^\gamma\nu(\mathrm{d}z)\leqslant C\delta^{\gamma-\alpha},\ \int_{\delta<|z|\leqslant 1}|z|^{\gamma'}\nu(\mathrm{d}z)\leqslant C\delta^{\gamma'-\alpha}. \tag{3.19}$$

Since the Lévy measure ν is not necessarily absolutely continuous with respect to the Lebesgue measure, it seems hard to show that for any $f\in B^{\alpha+\gamma}_{p,\infty}$,

$$x\mapsto\int_{|z|>1}f(x+\sigma(t,x)z)\nu(\mathrm{d}z)\in B^\gamma_{p,\infty},$$

which is very essential if one wants to use the perturbation argument. Thus we have to first remove the large jump part and consider the following operator

$$\begin{aligned}\widetilde{\mathscr{L}}_tf(x)&:=\widetilde{\mathcal{L}^\nu_{\sigma(t,x)}}f(x)\\ &:=\int_{|z|\leqslant1}\Big(f(x+\sigma(t,x)z)-f(x)-\mathbf{1}_{\alpha\in[1,2)}\sigma(t,x)z\cdot\nabla f(x)\Big)\nu(\mathrm{d}z).\end{aligned} \tag{3.20}$$

The following theorem is the main result of this subsection. Although this analytic result needs a special assumption on the oscillation of $\sigma(t,\cdot)$, it is enough for us to get our Theorem 1.1 .

Theorem 3.6. *Let $\beta\in(0,1)$ and $\alpha\in(0,2)$ with $\alpha+\beta>1$. Let $T>0$ and $p\in(\frac{d}{\alpha+\beta-1}\vee\frac{d}{\alpha\wedge\beta}\vee 2,\infty)$, $\theta\in(\beta,1]$. Suppose that ($\mathbf{H}^\alpha_\nu$) and ($\mathbf{H}^{\beta,p}_b$) hold. Then there are $\varepsilon_0\in(0,1)$ and $\lambda_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ and $\lambda\in(\lambda_0,\infty)$, under ($\mathbf{H}^\theta_\varepsilon$), for any $f\in L^\infty_T(B^\beta_{p,\infty})$, there is a unique classical solution $u\in L^\infty_T(B^{\alpha+\beta}_{p,\infty})$ to the following PDE*

$$\partial_tu=(\widetilde{\mathscr{L}}_t-\lambda)u+b\cdot\nabla u+f,\quad u(0)=0, \tag{3.21}$$

that is, for all $(t,x)\in[0,T]\times\mathbb{R}^d$,

$$u(t,x)=\int_0^t(\widetilde{\mathscr{L}}_s-\lambda)u(s,x)\mathrm{d}s+\int_0^t(b\cdot\nabla u)(s,x)\mathrm{d}s+\int_0^tf(s,x)\mathrm{d}s.$$

Moreover, there is a $C=C(\varepsilon_0,\lambda_0,\Lambda,\nu_1,\nu_2,T,d,p,\alpha,\beta,\theta,\kappa)>0$ such that

$$\|u\|_{L^\infty_T(B^{\alpha+\beta}_{p,\infty})}\leqslant C\|f\|_{L^\infty_T(B^\beta_{p,\infty})}, \tag{3.22}$$

and for any $\gamma\in(0,\alpha+\beta)$,

$$\|u\|_{L^\infty_T(B^\gamma_{p,\infty})}\leqslant c_\lambda\|f\|_{L^\infty_T(B^\beta_{p,\infty})}, \tag{3.23}$$

where $c_\lambda=c(\lambda,\gamma,\varepsilon_0,\Lambda,\nu_1,\nu_2,T,d,p,\alpha,\beta,\theta,\kappa)\to0$ as $\lambda\to\infty$.

In order to get the above result, we need the following commutator estimate.

Lemma 3.7. *Under ($\mathbf{H}^\theta_\varepsilon$) and ($\mathbf{H}^\alpha_\nu$), for any $p>1$, we have*

$$\big\|[\Delta^{s/2},\widetilde{\mathscr{L}}_t]u\big\|_p\leqslant C\begin{cases}\varepsilon^{\theta\delta-s+d/p}\|u\|_{B^\delta_{\infty,\infty}}, & \alpha\in(0,1),\delta\in(\alpha,1],s\in(0,\theta\delta);\\ \varepsilon^{\theta-s+d/p}\|\nabla u\|_{B^{\delta-1}_{\infty,\infty}}, & \alpha\in[1,2),\delta\in(\alpha,2),s\in(0,\theta),\end{cases}$$

where $[\Delta^{s/2}, \widetilde{\mathscr{L}_t}]u := \Delta^{s/2}\widetilde{\mathscr{L}_t}u - \widetilde{\mathscr{L}_t}\Delta^{s/2}u$, *and the constant* $C > 0$ *is independent of* ε.

Proof. We only prove it for $\alpha \in [1, 2)$ since the case $\alpha \in (0, 1)$ is similar. For simplicity of notation, we drop the variable t in $\sigma(t, x)$ and write

$$\begin{aligned}\Gamma_u^\sigma(x, y, z) := u(x + y + \sigma(x + y)z) - u(x + y + \sigma(x)z)\\ - (\sigma(x + y) - \sigma(x))z \cdot \nabla u(x + y),\end{aligned}$$

and

$$\|[\Delta^{s/2}, \widetilde{\mathscr{L}_t}]u\|_p^p = \left(\int_{|x|\leqslant 2\varepsilon} + \int_{|x|>2\varepsilon}\right) \left|[\Delta^{s/2}, \widetilde{\mathscr{L}_t}]u(x)\right|^p \mathrm{d}x =: \mathscr{J}_1 + \mathscr{J}_2.$$

Let $\delta \in (\alpha, 2)$. By (3.18) and (2.12), we have

$$\begin{aligned}|\Gamma_u^\sigma(x, y, z)| &\lesssim (|y| \wedge \varepsilon)^\theta |z| \int_0^1 |\nabla u(x + y + (1 - r)\sigma(x + y)z + r\sigma(x)z)\\ &\qquad - \nabla u(x + y)|\mathrm{d}r\\ &\lesssim (|y| \wedge \varepsilon)^\theta |z|^\delta \|\nabla u\|_{B^{\delta-1}_{\infty,\infty}},\end{aligned}$$

and by definition,

$$[\Delta^{s/2}, \widetilde{\mathscr{L}_t}]u(x) = \int_{|z|\leqslant 1} \nu(\mathrm{d}z) \int_{\mathbb{R}^d} \frac{\Gamma_u^\sigma(x, y, z)}{|y|^{d+s}} \mathrm{d}y.$$

Thus, for $\mathscr{J}_1$, by (3.19) we have

$$\begin{aligned}\mathscr{J}_1 &\lesssim \|\nabla u\|^p_{B^{\delta-1}_{\infty,\infty}} \int_{|x|\leqslant 2\varepsilon} \left|\int_{|z|\leqslant 1} \nu(\mathrm{d}z) \int_{\mathbb{R}^d} \frac{|z|^\delta (|y| \wedge \varepsilon)^\theta}{|y|^{d+s}} \mathrm{d}y\right|^p \mathrm{d}x\\ &\lesssim \varepsilon^d \|\nabla u\|^p_{B^{\delta-1}_{\infty,\infty}} \left|\int_{|y|\leqslant\varepsilon} \frac{|y|^\theta \mathrm{d}y}{|y|^{d+s}} + \int_{|y|>\varepsilon} \frac{\varepsilon^\theta \mathrm{d}y}{|y|^{d+s}}\right|^p \lesssim \|\nabla u\|^p_{B^{\delta-1}_{\infty,\infty}} \varepsilon^{(\theta-s)p+d}.\end{aligned}$$

For $\mathscr{J}_2$, since $\Gamma_u^\sigma(x, y, z) = 0$ for $|x|, |x + y| > \varepsilon$ by (3.16), we have

$$\begin{aligned}\mathscr{J}_2 &= \int_{|x|>2\varepsilon} \left|\int_{|z|\leqslant 1} \nu(\mathrm{d}z) \int_{|x+y|\leqslant\varepsilon} \frac{\Gamma_u^\sigma(x, y, z)}{|y|^{d+s}} \mathrm{d}y\right|^p \mathrm{d}x\\ &\lesssim \|\nabla u\|^p_{B^{\delta-1}_{\infty,\infty}} \int_{|x|>2\varepsilon} \left|\int_{|z|\leqslant 1} \nu(\mathrm{d}z) \int_{|x+y|\leqslant\varepsilon} \frac{|z|^\delta (|y| \wedge \varepsilon)^\theta}{|y|^{d+s}} \mathrm{d}y\right|^p \mathrm{d}x\\ &\lesssim \|\nabla u\|^p_{B^{\delta-1}_{\infty,\infty}} \varepsilon^{\theta p} \int_{|x|>2\varepsilon} \left|\int_{|x+y|\leqslant\varepsilon} \frac{1}{|y|^{d+s}} \mathrm{d}y\right|^p \mathrm{d}x\\ &\lesssim \|\nabla u\|^p_{B^{\delta-1}_{\infty,\infty}} \varepsilon^{\theta p+dp} \int_{|x|>2\varepsilon} \frac{1}{(|x| - \varepsilon)^{p(d+s)}} \mathrm{d}x \lesssim \|\nabla u\|^p_{B^{\delta-1}_{\infty,\infty}} \varepsilon^{(\theta-s)p+d}.\end{aligned}$$

Combining the above calculations, we obtain the desired estimate. □

Lemma 3.8. *Suppose that* $(\mathbf{H}^\theta_\varepsilon)$ *and* $(\mathbf{H}^\alpha_\nu)$ *holds with* $\theta, \varepsilon \in (0, 1)$ *and* $\alpha \in (0, 2)$. *For any* $p \in (\frac{d}{\alpha\wedge 1}, \infty)$, *we have*

$$\left\|(\widetilde{\mathcal{L}^\nu_{\sigma(t,\cdot)}} - \widetilde{\mathcal{L}^\nu_{\sigma(t,0)}})f\right\|_{B^\beta_{p,\infty}} \leqslant c_\varepsilon \begin{cases} \|f\|_{B^{\alpha+\beta}_{p,\infty}}, & \alpha \in (0, 1),\ \beta \in (0, \frac{(p\alpha-d)\theta}{p(1-\theta)} \wedge \theta);\\ \|f\|_{B^{\alpha+\beta}_{p,\infty}}, & \alpha \in [1, 2),\ \beta \in (0, \theta),\end{cases}$$

where c_ε *is a positive constant so that* $\lim_{\varepsilon\to 0} c_\varepsilon = 0$.

Proof. We only give the proof of the estimate for $\alpha\in(0,1)$. The case $\alpha\in[1,2)$ is similar. For simplicity of notation, we drop the time variable t and write

$$\mathcal{T}_\sigma f(x):=\widetilde{\mathcal{L}^\nu_{\sigma(x)}}f(x)-\widetilde{\mathcal{L}^\nu_{\sigma(0)}}f(x)=\int_{|z|\leqslant 1}\Big(f(x+\sigma(x)z)-f(x+\sigma(0)z)\Big)\nu(\mathrm{d}z).$$

For $\gamma\in(d/p,1]$, $\delta>0$ and $y_0\in\mathbb{R}^d$, by [10, Lemma 2.2], we have

$$\Big\|\sup_{|y-y_0|\leqslant\delta}|f(\cdot+y)-f(\cdot+y_0)|\Big\|_p\lesssim\delta^\gamma\|f\|_{H^\gamma_p},$$

which implies under condition $(\mathbf{H}^\theta_\varepsilon)$ that

$$\begin{aligned}&\|f(\cdot+\sigma(\cdot)z)-f(\cdot+\sigma(0)z)\|_p\\&\leqslant\Big\|\sup_{|y-\sigma(0)z|\leqslant\Lambda\varepsilon^\theta|z|}|f(\cdot+y)-f(\cdot+\sigma(0)z)|\Big\|_p\lesssim\varepsilon^{\gamma\theta}|z|^\gamma\|f\|_{H^\gamma_p}.\end{aligned}$$

Since $p>\frac{d}{\alpha}$, we can take γ and γ' so that $\frac{d}{p}<\gamma'<\alpha<\gamma\leqslant 1$. By the above estimates,

$$\|\mathcal{T}_\sigma f\|_p\lesssim\int_{|z|\leqslant 1}(\varepsilon^{\gamma\theta}|z|^\gamma\|f\|_{H^\gamma_p})\wedge(\varepsilon^{\gamma'\theta}|z|^{\gamma'}\|f\|_{H^{\gamma'}_p})\nu(\mathrm{d}z).\tag{3.24}$$

For each $i\geqslant -1$, by (2.8),

$$\|\Pi_i\mathcal{T}_\sigma f\|_p\leqslant\sum_{j>i}\|\Pi_i\mathcal{T}_\sigma\Pi_j f\|_p+\sum_{j\leqslant i}\|\Pi_i\mathcal{T}_\sigma\Pi_j f\|_p=:\mathcal{J}_1+\mathcal{J}_2.\tag{3.25}$$

By (3.24) and the Bernstein's inequality (2.10),

$$\begin{aligned}\|\mathcal{T}_\sigma\Pi_j f\|_p&\lesssim\int_{|z|\leqslant 1}(\varepsilon^{\gamma\theta}|z|^\gamma\|\Pi_j f\|_{H^\gamma_p})\wedge(\varepsilon^{\gamma'\theta}|z|^{\gamma'}\|\Pi_j f\|_{H^{\gamma'}_p})\nu(\mathrm{d}z)\\&\lesssim\int_{|z|\leqslant 1}(\varepsilon^{\gamma\theta}|z|^\gamma 2^{\gamma j})\wedge(\varepsilon^{\gamma'\theta}|z|^{\gamma'}2^{\gamma' j})\nu(\mathrm{d}z)\|\Pi_j f\|_p.\end{aligned}\tag{3.26}$$

Note that by (3.19),

$$\begin{aligned}&\int_{|z|\leqslant 1}(\varepsilon^{\gamma\theta}|z|^\gamma 2^{\gamma j})\wedge(\varepsilon^{\gamma'\theta}|z|^{\gamma'}2^{\gamma' j})\nu(\mathrm{d}z)\\&\quad\leqslant\varepsilon^{\gamma\theta}2^{\gamma j}\int_{|z|\leqslant 2^{-j}}|z|^\gamma\nu(\mathrm{d}z)+\varepsilon^{\gamma'\theta}2^{\gamma' j}\int_{2^{-j}<|z|\leqslant 1}|z|^{\gamma'}\nu(\mathrm{d}z)\\&\quad\lesssim\varepsilon^{\gamma\theta}2^{\alpha j}+\varepsilon^{\gamma'\theta}2^{\alpha j}\lesssim\varepsilon^{\gamma'\theta}2^{\alpha j}.\end{aligned}$$

Substituting this into (3.26), we obtain

$$\|\mathcal{T}_\sigma\Pi_j f\|_p\lesssim\varepsilon^{\gamma'\theta}2^{\alpha j}\|\Pi_j f\|_p.\tag{3.27}$$

Thus,

$$\mathcal{J}_1\lesssim\varepsilon^{\gamma'\theta}\sum_{j>i}2^{\alpha j}\|\Pi_j f\|_p\lesssim\varepsilon^{\gamma'\theta}\sum_{j>i}2^{-\beta j}\|f\|_{B^{\alpha+\beta}_{p,\infty}}=\frac{\varepsilon^{\gamma'\theta}2^{-\beta i}}{1-2^{-\beta}}\|f\|_{B^{\alpha+\beta}_{p,\infty}}.$$

For $\mathcal{J}_2$, for any $\beta\in(0,\frac{(p\alpha-d)\theta}{p(1-\theta)}\wedge\theta)$, one can choose $\delta\in(\alpha,\frac{(p\alpha-d)}{p(1-\theta)}\wedge 1]$ such that $\beta<\delta\theta$. Since $\delta<\frac{(p\alpha-d)}{p(1-\theta)}$, we get $\delta+\frac{d}{p}-\alpha<\delta\theta$. By this, we can fix $s\in(\beta,\delta\theta)$

such that $s > \delta + \frac{d}{p} - \alpha$. Using Bernstein's inequality and Lemma 3.7 and noting that $\Delta^{s/2}\Pi_j = \Pi_j\Delta^{s/2}$, we have

$$\begin{aligned}
\mathcal{J}_2 &= \sum_{j\leqslant i} \|\Pi_i \Delta^{-s/2}\Delta^{s/2}\mathcal{T}_\sigma \Pi_j f\|_p \lesssim 2^{-si}\sum_{j\leqslant i}\|\Delta^{s/2}\mathcal{T}_\sigma\Pi_j f\|_p \\
&\leqslant 2^{-si}\sum_{j\leqslant i}\Big(\|[\Delta^{s/2},\mathcal{T}_\sigma]\,\Pi_j f\|_p + \|\mathcal{T}_\sigma\Delta^{s/2}\Pi_j f\|_p\Big) \\
&= 2^{-si}\sum_{j\leqslant i}\Big(\|[\Delta^{s/2},\widetilde{\mathscr{L}_t}]\,\Pi_j f\|_p + \|\mathcal{T}_\sigma\Delta^{s/2}\Pi_j f\|_p\Big) \\
&\overset{(3.27)}{\lesssim} 2^{-si}\sum_{j\leqslant i}\Big(\varepsilon^{\theta\delta - s + d/p}\|\Pi_j f\|_{C^\delta} + \varepsilon^{\gamma'\theta}2^{\alpha j}\|\Delta^{s/2}\Pi_j f\|_p\Big) \\
&\lesssim 2^{-si}\sum_{j\leqslant i}\Big(\varepsilon^{\theta\delta - s+d/p}2^{(\delta+d/p)j}\|\Pi_j f\|_p + \varepsilon^{\gamma'\theta}2^{(s+\alpha)j}\|\Pi_j f\|_p\Big) \\
&\leqslant c_\varepsilon 2^{-si}\sum_{j\leqslant i}\Big(2^{(\delta+d/p)j} + 2^{(\alpha+s)j}\Big)\|\Pi_j f\|_p \\
&\leqslant c_\varepsilon\left(2^{-si}\sum_{j\leqslant i}2^{(\delta+\frac{d}{p}-\alpha-\beta)j} + 2^{-si}\sum_{j\leqslant i}2^{(s-\beta)j}\right)\|f\|_{B^{\alpha+\beta}_{p,\infty}} \\
&\leqslant c_\varepsilon 2^{-\beta i}\|f\|_{B^{\alpha+\beta}_{p,\infty}},
\end{aligned}$$

where $\lim_{\varepsilon\to 0} c_\varepsilon = 0$. Thus we have by (3.25) that

$$\|\mathcal{T}_\sigma f\|_{B^\beta_{p,\infty}} = \sup_{i\geqslant -1}\left(2^{i\beta}\|\Pi_i\mathcal{T}_\sigma f\|_p\right) \lesssim \left(\frac{\varepsilon^{\gamma'\theta}}{1-2^{-\beta}} + c_\varepsilon\right)\|f\|_{B^{\alpha+\beta}_{p,\infty}}.$$

This gives the desired estimate. □

We are in a position to give

Proof of Theorem 3.6. Since we are considering the truncated operator $\widetilde{\mathscr{L}_t}$, without loss of generality we may assume $\nu|_{B_1^c} = \nu_1$. Thus, by the assumptions,

$$\begin{aligned}
-\mathrm{Re}(\psi_{\sigma(t,0)}(\xi)) &= \int_{\mathbb{R}^d}(1-\cos(\sigma(t,0)z\cdot\xi))\nu(\mathrm{d}z) \\
&\geqslant \int_{\mathbb{R}^d}(1-\cos(\sigma(t,0)z\cdot\xi))\nu_1(\mathrm{d}z) \geqslant c_0|\xi|^\alpha,
\end{aligned}$$

where $c_0 = c_0(\Lambda,\nu_1,d,\alpha) > 0$. Now we use Picard's iteration to show the existence. Let $u_1 \equiv 0$. For $n\in\mathbb{N}$, by induction and Theorem 3.5, the following PDE admits a unique classical solution $u_{n+1}\in L^\infty_T(B^{\alpha+\beta}_{p,\infty})$:

$$\begin{aligned}
&\partial_t u_{n+1} + \lambda u_{n+1} - \mathcal{L}^\nu_{\sigma(t,0)}u_{n+1} - \widetilde{b}\cdot\nabla u_{n+1} \\
&\quad = f + (\widetilde{\mathcal{L}^\nu_{\sigma(t,\cdot)}} - \widetilde{\mathcal{L}^\nu_{\sigma(t,0)}})u_n - \int_{|z|>1}\Big(u_n(\cdot+\sigma(t,0)z) - u_n(\cdot)\Big)\nu(\mathrm{d}z),
\end{aligned} \tag{3.28}$$

where

$$\widetilde{b} := b - \mathbf{1}_{\alpha\in(0,1)}\int_{|z|\leqslant 1}\sigma(t,0)z\nu(\mathrm{d}z).$$

Moreover, by the assumption $\theta > \beta > 1 + \frac{d}{p} - \alpha$, $\frac{(p\alpha - d)\theta}{p(1-\theta)} \geqslant \theta$. So, by (3.12), (3.13) and Lemma 3.8,

$$\begin{aligned} &c_\lambda^{-1}\|u_{n+1}\|_{L^\infty_T(B^\beta_{p,\infty})} + \|u_{n+1}\|_{L^\infty_T(B^{\alpha+\beta}_{p,\infty})} \\ &\leqslant C_1\|f\|_{L^\infty_T(B^\beta_{p,\infty})} + \|(\widetilde{\mathcal{L}^\nu_{\sigma(t,\cdot)}} - \widetilde{\mathcal{L}^\nu_{\sigma(t,0)}})u_n\|_{L^\infty_T(B^\beta_{p,\infty})} + C_2\|u_n\|_{L^\infty_T(B^\beta_{p,\infty})} \\ &\leqslant C_1\|f\|_{L^\infty_T(B^\beta_{p,\infty})} + c_\varepsilon\|u_n\|_{L^\infty_T(B^{\alpha+\beta}_{p,\infty})} + C_2\|u_n\|_{L^\infty_T(B^\beta_{p,\infty})}, \end{aligned}$$

where $C_1, C_2 \geqslant 1$ is independent of ε, λ and n. Here

$$\lim_{\varepsilon\downarrow 0} c_\varepsilon = 0, \quad \lim_{\lambda\uparrow\infty} c_\lambda = 0.$$

In particular, for any $m \in \mathbb{N}$,

$$\begin{aligned} &c_\lambda^{-1}\sup_{n\leqslant m}\|u_n\|_{L^\infty_T(B^\beta_{p,\infty})} + \sup_{n\leqslant m}\|u_n\|_{L^\infty_T(B^{\alpha+\beta}_{p,\infty})} \\ &\leqslant C_1\|f\|_{L^\infty_T(B^\beta_{p,\infty})} + c_\varepsilon\sup_{n\leqslant m}\|u_n\|_{L^\infty_T(B^{\alpha+\beta}_{p,\infty})} + C_2\sup_{n\leqslant m}\|u_n\|_{L^\infty_T(B^\beta_{p,\infty})}. \end{aligned}$$

Choosing ε_0 small and λ_0 large enough so that $c_{\varepsilon_0} = \frac{1}{2}$ and $c_{\lambda_0} = \frac{1}{2C_2}$, we get for all $\varepsilon \in (0, \varepsilon_0)$ and $\lambda \in (\lambda_0, \infty)$,

$$\tfrac{1}{2}c_\lambda^{-1}\sup_{n\leqslant m}\|u_n\|_{L^\infty_T(B^\beta_{p,\infty})} + \tfrac{1}{2}\sup_{n\leqslant m}\|u_n\|_{L^\infty_T(B^{\alpha+\beta}_{p,\infty})} \leqslant C_1\|f\|_{L^\infty_T(B^\beta_{p,\infty})}.$$

Letting $m \to \infty$, we obtain the following uniform estimate:

$$c_\lambda^{-1}\sup_{n\in\mathbb{N}}\|u_n\|_{L^\infty_T(B^\beta_{p,\infty})} + \sup_{n\in\mathbb{N}}\|u_n\|_{L^\infty_T(B^{\alpha+\beta}_{p,\infty})} \leqslant 2C_1\|f\|_{L^\infty_T(B^\beta_{p,\infty})}. \tag{3.29}$$

Similarly, for any $n, k \in \mathbb{N}$, we have

$$\begin{aligned} &c_\lambda^{-1}\|u_{n+1} - u_{k+1}\|_{L^\infty_T(B^\beta_{p,\infty})} + \|u_{n+1} - u_{k+1}\|_{L^\infty_T(B^{\alpha+\beta}_{p,\infty})} \\ &\leqslant c_\varepsilon\|u_n - u_k\|_{L^\infty_T(B^{\alpha+\beta}_{p,\infty})} + C_2\|u_n - u_k\|_{L^\infty_T(B^\beta_{p,\infty})}. \end{aligned}$$

As above, for all $\varepsilon \in (0, \varepsilon_0)$ and $\lambda \in (\lambda_0, \infty)$, we deduce that

$$\limsup_{n,k\to\infty}\Big(\|u_n - u_k\|_{L^\infty_T(B^{\alpha+\beta}_{p,\infty})} + \|u_n - u_k\|_{L^\infty_T(B^\beta_{p,\infty})}\Big) = 0. \tag{3.30}$$

Finally, by (3.29) and taking limits in (3.28), we obtain the existence of a classical solution. By (3.29) and interpolation theorem, we also have (3.22) and (3.23). As for the uniqueness, it follows by the same calculation as that for (3.30). This completes the proof of the theorem. □

4. Strong well-posedness of SDE (1.1)

In this section, we give a proof for the main result of this paper, Theorem 1.1. Define

$$\bar{Z}_t := \sum_{0<s\leqslant t}(Z_s - Z_{s-})\mathbf{1}_{\{|Z_s - Z_{s-}|>1\}} \quad \text{and} \quad \widetilde{Z}_t := Z_t - \bar{Z}_t. \tag{4.1}$$

It is well known (see, e.g., [5]) that both $\widetilde{Z}$ and $\bar{Z}$ are pure jump Lévy processes with Lévy measures $1_{\{|z|\leqslant 1\}}\nu(dz)$ and $1_{\{|z|>1\}}\nu(dz)$, respectively, and they are independent to each other. We call $\widetilde{Z}$ a truncated stable-like process as it only has jumps of size no larger than 1. The Lévy process $\bar{Z}$ has finite Lévy measure and hence is a compound Poisson process. SDE (1.1) can be written as

$$X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_{s-})\mathrm{d}\widetilde{Z}_s + \int_0^t \sigma(s, X_{s-})\mathrm{d}\bar{Z}_s. \tag{4.2}$$

To solve SDE (1.1), by standard interlacing technique, it suffices to solve the following SDE

$$X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_{s-})\mathrm{d}\widetilde{Z}_s. \tag{4.3}$$

Its generator is given by

$$\widetilde{\mathscr{A}_t} = \widetilde{\mathscr{L}_t} + \widetilde{b_t} \cdot \nabla,$$

where $\widetilde{\mathscr{L}_t}$ is defined by (3.20) and

$$\widetilde{b}(t, x) := b(t, x) - \sigma(t, x)\ell \tag{4.4}$$

with $\ell := \mathbf{1}_{\alpha\in(0,1)} \int_{|z|\leqslant 1} z\nu(\mathrm{d}z)$.

In the following we shall fix a stochastic basis $(\Omega, \mathscr{F}, \mathbb{P}; (\mathscr{F}_t)_{t\geqslant 0})$ so that all the processes are defined on it.

Theorem 4.1. *Let $b(t, x, \omega)$ and $\sigma(t, x, \omega)$ be two $\mathscr{B}(\mathbb{R}_+)\times\mathscr{B}(\mathbb{R}^d)\times\mathscr{F}_0$-measurable functions. Let $\beta \in (1 - \frac{\alpha}{2}, 1)$ and $p \in (\frac{d}{\alpha/2+\beta-1} \vee \frac{d}{\alpha\wedge\beta} \vee 2, \infty)$. Suppose that*

$$\sup_{\omega\in\Omega} \|b_\cdot(\cdot, \omega)\|_{L^\infty_T(B^\beta_{p,\infty})} < \infty, \quad T > 0,$$

and $\sigma_\cdot(\cdot, \omega)$ satisfies $(\mathbf{H}^1_\varepsilon)$ with common bound Λ for almost every ω, where ε is a small constant as in Theorem 3.6. For any $\mathscr{F}_0$-measurable random variable X_0, there is a unique $\mathscr{F}_t$-adapted strong solution X_t so that

$$X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_{s-})\mathrm{d}\widetilde{Z}_s.$$

Proof. Let $N(\mathrm{d}t, \mathrm{d}z)$ be the Poisson random measure associated with Z, that is,

$$N((0, t] \times E) = \sum_{s\leqslant t} \mathbf{1}_E(Z_s - Z_{s-}), \quad E \in \mathscr{B}(\mathbb{R}^d \setminus \{0\}),$$

whose intensity measure is given by $\mathrm{d}t\nu(\mathrm{d}z)$. Let $\tilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - \mathrm{d}t\nu(\mathrm{d}z)$ be the compensated Poisson random martingale measure. By Lévy-Itô's decomposition, we know

$$\widetilde{Z}_t = \int_0^t\int_{\{z\in\mathbb{R}^d:0<|z|\leqslant 1\}} z\widetilde{N}(ds, dz) \quad \text{and} \quad \bar{Z}_t = \int_0^t\int_{\{z\in\mathbb{R}^d:|z|>1\}} zN(ds, dz).$$

Let $T > 0$. Consider the following backward nonlocal parabolic system with random coefficients:

$$\partial_t \mathbf{u}_t + (\widetilde{\mathscr{L}_t} - \lambda)\mathbf{u}_t + \widetilde{b} \cdot \nabla \mathbf{u}_t + b = 0, \quad \mathbf{u}_T = 0, \tag{4.5}$$

where $\widetilde{\mathscr{L}_t}$ is defined by (3.20) and $\widetilde{b_t}$ is defined by (4.4). By the assumptions and Theorem 3.6, for some $\lambda_0 > 0$, and for each ω and $\lambda > \lambda_0$, there is a unique solution $\mathbf{u}_\cdot(\cdot, \omega) \in L^\infty_T(B^{\alpha+\beta}_{p,\infty})$ to the above equation with

$$\sup_{\omega\in\Omega} \|\mathbf{u}_\cdot(\cdot, \omega)\|_{L^\infty_T(B^{\alpha+\beta}_{p,\infty})} \leqslant C,$$

and for any $\gamma \in (0, \alpha + \beta)$,

$$\sup_{\omega\in\Omega} \|\mathbf{u}_\cdot(\cdot, \omega)\|_{L^\infty_T(B^\gamma_{p,\infty})} \leqslant c_\lambda, \tag{4.6}$$

where $c_\lambda \to 0$ as $\lambda \to \infty$. Thanks to $\alpha/2 + \beta > 1$ and $p > \frac{d}{\alpha/2+\beta-1}$, by Sobolev's embedding (2.4), one can choose $\lambda > \lambda_0$ large enough so that

$$\sup_{\omega\in\Omega} \|\nabla \mathbf{u}_\cdot(\cdot, \omega)\|_\infty \leqslant 1/2. \tag{4.7}$$

Since $\mathbf{u}$ is $\mathscr{F}_0$-measurable, by Itô's formula (see, e.g., [14]), we have

$$\mathbf{u}_t(X_t) = \mathbf{u}_0(X_0) + \int_0^t [\partial_s \mathbf{u}_s + \widetilde{\mathscr{L}_s}\mathbf{u}_s + \widetilde{b}_s \cdot \nabla \mathbf{u}_s](X_s)\mathrm{d}s$$
$$+ \int_0^t\!\!\int_{0<|z|\leqslant 1} [\mathbf{u}_s(X_{s-} + \sigma(s, X_{s-})z) - \mathbf{u}_s(X_{s-})]\tilde{N}(\mathrm{d}s, \mathrm{d}z).$$

Let $\Phi_t(x, \omega) = x + \mathbf{u}_t(x, \omega)$. Then by (4.7), $x \mapsto \Phi_t(x, \omega)$ is a C^1-diffeomorphism and, by (4.5),

$$\begin{aligned} Y_t := \Phi_t(X_t) &= \Phi_0(X_0) + \int_0^t \Big(\lambda \mathbf{u}_s(X_s) - \sigma(s, X_s)\ell\Big)\mathrm{d}s \\ &\quad + \int_0^t\!\!\int_{0<|z|\leqslant 1} [\Phi_s(X_{s-} + \sigma(s, X_{s-})z) - \Phi_s(X_{s-})]\tilde{N}(\mathrm{d}s, \mathrm{d}z) \\ &= \Phi_0(X_0) + \int_0^t a(s, Y_s)\mathrm{d}s + \int_0^t\!\!\int_{0<|z|\leqslant 1} g(s, Y_{s-}, z)\tilde{N}(\mathrm{d}s, \mathrm{d}z), \end{aligned} \tag{4.8}$$

where

$$a(t, y) := \lambda \mathbf{u}_t(\Phi_t^{-1}(y)) - \sigma(t, \Phi_t^{-1}(y))\ell, \quad g(t, y, z) := \Phi_t\left(\Phi_t^{-1}(y) + \sigma(t, \Phi_t^{-1}(y))z\right) - y.$$

Fix $\eta \in (\alpha/2, \alpha + \beta - 1 - d/p)$. Noting that

$$|[f(x+z) - f(x)] - [f(y+z) - f(y)]| \leqslant \|\nabla f\|_{B^\eta_{\infty,\infty}} |x - y|\, |z|^\eta,$$

we have by (4.6)-(4.7) that for all $x, y \in \mathbb{R}^d$ and $|z| \leqslant 1$,

$$\begin{aligned} |g(t,x,z) - g(t,y,z)| &\leqslant \Big|\Big(\Phi_t\left(\Phi_t^{-1}(x) + \sigma(t, \Phi_t^{-1}(x))z\right) - \Phi_t(\Phi^{-1}(x))\Big) \\ &\quad - \Big(\Phi_t\left(\Phi_t^{-1}(y) + \sigma(t, \Phi_t^{-1}(x))z\right) - \Phi_t\left(\Phi_t^{-1}(y)\right)\Big)\Big| \\ &\quad + \Big|\Phi_t\left(\Phi_t^{-1}(y) + \sigma(t, \Phi_t^{-1}(x))z\right) - \Phi_t\left(\Phi_t^{-1}(y) + \sigma(t, \Phi_t^{-1}(y))z\right)\Big| \\ &\leqslant \|\nabla \Phi_t\|_{B^\eta_{\infty,\infty}} |\Phi_t^{-1}(x) - \Phi_t^{-1}(y)|\, |\sigma(t, \Phi_t^{-1}(x))z|^\eta \\ &\quad + \|\nabla \Phi_t\|_\infty \left|\left(\sigma(t, \Phi_t^{-1}(x)) - \sigma(t, \Phi_t^{-1}(y)\right) z\right| \\ &\lesssim |x - y|\, |z|^\eta + |x - y|\, |z| \\ &\lesssim |x - y|\, |z|^\eta. \end{aligned}$$

Moreover, we also have

$$|a(t, x) - a(t, y)| \lesssim |x - y|.$$

Since the coefficients of SDE (4.8) are Lipschitz continuous, by the classical result, SDE (4.8) admits a unique solution (cf. [14]). In particular, one can check that $X_t = \Phi_t^{-1}(Y_t)$ satisfies the original equation (4.3). The proof is complete. □

We also need the following technical lemma in order to patch up the solution.

Lemma 4.2. *Let X_t be a $\mathbb{R}^d$-valued right continuous process. Let τ be an $\mathscr{F}_t$-stopping time. Suppose that for each $t \geqslant 0$, $X_{t+\tau}$ is $\mathscr{F}_{t+\tau}$-measurable. Then for each $t \geqslant 0$, $\mathbf{1}_{\{\tau \leqslant t\}} X_t$ is $\mathscr{F}_t$-measurable.*

Proof. Since X_t is right continuous, we have

$$\mathbf{1}_{\{\tau \leqslant t\}} X_t = \lim_{n\to\infty} \mathbf{1}_{\{\tau \leqslant t\}} X_{t+\tau-[2^n\tau]2^{-n}} = \lim_{n\to\infty} \sum_{j=0}^{[2^n t]} \mathbf{1}_{\{\tau \leqslant t\}} X_{t+\tau-j2^{-n}} \mathbf{1}_{\{j \leqslant 2^n\tau < j+1\}}.$$

On the other hand, since by assumption $X_{t+\tau-j2^{-n}}=X_{\tau+(t-j2^{-n})}$ is $\mathscr{F}_{\tau+(t-j2^{-n})}$-measurable and $\tau+t-j2^{-n}$ is a stopping time when $t\geqslant j2^{-n}$, we have for each $n\geqslant 1$ and $1\leqslant j\leqslant [2^n t]$,

$$\begin{aligned}\mathbf{1}_{\{\tau\leqslant t\}}X_{t+\tau-j2^{-n}}\mathbf{1}_{\{j\leqslant 2^n\tau<j+1\}}&=X_{t+\tau-j2^{-n}}\mathbf{1}_{\{2^{-n}j\leqslant\tau<(j+1)2^{-n}\}}\mathbf{1}_{\{\tau\leqslant t\}}\\&=X_{\tau+t-j2^{-n}}\mathbf{1}_{\{\tau+t-j2^{-n}<t+2^{-n}\}}1_{\{j2^{-n}\leqslant\tau\leqslant t\}}.\end{aligned}$$

Noticing $X_{\tau+t-j2^{-n}}\mathbf{1}_{\{\tau+t-j2^{-n}<t+2^{-n}\}}\in\mathscr{F}_{t+2^{-n}}$ and $\mathbf{1}_{\{j2^{-n}\leqslant\tau\leqslant t\}}\in\mathscr{F}_t$, we get

$$\mathbf{1}_{\{\tau\leqslant t\}}X_t\in\cap_{n\geqslant 1}\mathscr{F}_{t+2^{-n}}=\mathscr{F}_t.$$

The proof is complete. □

Now we can give

Proof of Theorem 1.1 . By the discussion at the beginning of this section, we only need to prove the global well-posedness of (4.3). By Remark 1.3, we can further assume that b has support contained in ball B_R. Let $p\geqslant 1$. By definition (2.6), we have

$$\begin{aligned}\|\Pi_j b_t\|_p^p&=\int_{\mathbb{R}^d}\left|\int_{B_R}h_j(x-y)b(t,y)\mathrm{d}y\right|^p\mathrm{d}x\\&\leqslant\|\Pi_j b_t\|_\infty^p|B_{2R}|+\|b\|_\infty^p\int_{B_{2R}^c}\left(\int_{B_R}|h_j(x-y)|\mathrm{d}y\right)^p\mathrm{d}x.\end{aligned}$$

Noting that $h_j(x)=2^{jd}h_0(2^jx)$ by (2.5), we have

$$\begin{aligned}\int_{B_{2R}^c}\left(\int_{B_R}|h_j(x-y)|\mathrm{d}y\right)^p\mathrm{d}x&\leqslant\|h_0\|_1^{p-1}\int_{B_{2R}^c}\int_{B_R}|h_j(x-y)|\mathrm{d}y\mathrm{d}x\\&\leqslant C(h_0)\int_{B_{2R}^c}\int_{B_R}2^{jd}(2^j|x-y|)^{-2d}\mathrm{d}y\mathrm{d}x\leqslant C(h_0,d,R)2^{-jd},\end{aligned}$$

where the second inequality is due to the polynomial decay property of Schwartz function h_0. Hence,

$$\|b_t\|_{B^\beta_{p,\infty}}=\sup_{j\geqslant -1}2^{j\beta}\|\Pi_j b_t\|_p\leqslant C\sup_{j\geqslant -1}2^{j\beta}(\|\Pi_j b_t\|_\infty+2^{-jd}\|b_t\|_\infty)\leqslant C\|b_t\|_{C^\beta}.$$

Below we use induction to construct a sequence of finite stopping times $(\tau_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}\tau_n=\infty$ a.s. and such that SDE (4.3) is strong well-posed up to each τ_n. Let $\tau_0\equiv 0$. For $n\in\mathbb{N}$, suppose that we have constructed stopping time τ_n and the existence and uniqueness of strong solutions up to time τ_n. That is, there is a unique strong solution X_t satisfying

$$X_t=X_0+\int_0^t b(s,X_s)\mathrm{d}s+\int_0^t\sigma(s,X_{s-})\mathrm{d}\widetilde{Z}_s,\quad\text{for } t\in[0,\tau_n).$$

Now define

$$X_{\tau_n}:=X_{\tau_n-}+\sigma(\tau_n,X_{\tau_n-})(\widetilde{Z}_{\tau_n}-\widetilde{Z}_{\tau_n-}),\quad\mathscr{F}'_t:=\mathscr{F}_{t+\tau_n},\quad t\geqslant 0.$$

Clearly, $X_{\tau_n}\in\mathscr{F}'_0$. We introduce $\mathscr{F}'_0$-measurable random $\mathbb{R}^d$-valued function b' and $(d\times d)$-matrix valued function σ' as follows:

$$b'(t,x,\omega)=b(t+\tau_n,x+X_{\tau_n}(\omega))$$

and

$$\sigma'(t,x,\omega):=\begin{cases}\sigma(t+\tau_n(\omega),x+X_{\tau_n}(\omega)), & |x|\leqslant\varepsilon/2,\\ \frac{2(\varepsilon-|x|)}{\varepsilon}\sigma\big(t+\tau_n(\omega),\frac{\varepsilon x}{2|x|}+X_{\tau_n}(\omega)\big) & \\ \quad+\frac{2(|x|-\varepsilon/2)}{\varepsilon}\sigma\big(t+\tau_n(\omega),X_{\tau_n}(\omega)\big), & \varepsilon/2<|x|\leqslant\varepsilon,\\ \sigma(t+\tau_n(\omega),X_{\tau_n}(\omega)), & |x|>\varepsilon.\end{cases}$$

It is easy to see that σ' satisfies $(\mathbf{H}^1_\varepsilon)$. Thus, by Theorem 4.1, the following SDE admits a unique strong solution

$$X'_t=\int_0^t b'(s,X'_s)\mathrm{d}s+\int_0^t\sigma'(s,X'_{s-})\mathrm{d}\widetilde{Z}_{\tau_n+s},\quad t\geqslant 0. \tag{4.9}$$

Define $\mathscr{F}'_t$-stopping time

$$\tau':=\inf\{t>0:|X'_t|\geqslant\varepsilon/2\}, \tag{4.10}$$

and for $t\geqslant 0$,

$$\widetilde{X}_t:=X_t\mathbf{1}_{t<\tau_n}+(X'_{t-\tau_n}+X_{\tau_n})\mathbf{1}_{t\geqslant\tau_n}. \tag{4.11}$$

Since $t\mapsto\widetilde{X}_t$ is right continuous and $\widetilde{X}_{t+\tau_n}=X'_t+X_{\tau_n}\in\mathscr{F}_{t+\tau_n}$, by Lemma 4.2,

$$\mathbf{1}_{\{\tau_n\leqslant t\}}\widetilde{X}_t\text{ is }\mathscr{F}_t\text{-measurable.}$$

Thus, for each $t\geqslant 0$, by the change of variable,

$$\int_{\tau_n}^{t+\tau_n}\sigma(s,\widetilde{X}_{s-})\mathrm{d}\widetilde{Z}_s=\int_0^t\sigma(s+\tau_n,\widetilde{X}_{(s+\tau_n)-})\mathrm{d}\widetilde{Z}_{\tau_n+s}. \tag{4.12}$$

Now, by definition and (4.12), we have for $t\in[0,\tau')$,

$$\begin{aligned}\widetilde{X}_{t+\tau_n}&=X_{\tau_n}+\int_0^t b(s+\tau_n,\widetilde{X}_{s+\tau_n})\mathrm{d}s+\int_0^t\sigma(s+\tau_n,\widetilde{X}_{(s+\tau_n)-})\mathrm{d}\widetilde{Z}_{\tau_n+s}\\&=X_{\tau_n}+\int_{\tau_n}^{t+\tau_n}b(s,\widetilde{X}_s)\mathrm{d}s+\int_{\tau_n}^{t+\tau_n}\sigma(s,\widetilde{X}_{s-})\mathrm{d}\widetilde{Z}_s\\&=x+\int_0^{t+\tau_n}b(s,\widetilde{X}_s)\mathrm{d}s+\int_0^{t+\tau_n}\sigma(s,\widetilde{X}_{s-})\mathrm{d}\widetilde{Z}_s.\end{aligned} \tag{4.13}$$

Define $\tau_{n+1}:=\tau'+\tau_n$. Observe that for each $s>0$,

$$\{\tau_{n+1}<s\}=\cup_{t\in\mathbb{Q},t<s}(\{\tau'<t\}\cap\{\tau_n<s-t\})\in\mathscr{F}_s.$$

This means that τ_{n+1} is an $\mathscr{F}_t$-stopping time. By (4.13) and induction hypothesis, we obtain that $\widetilde{X}_t$ uniquely solves

$$\widetilde{X}_t=x+\int_0^t b(s,\widetilde{X}_s)\mathrm{d}s+\int_0^t\sigma(s,\widetilde{X}_{s-})\mathrm{d}\widetilde{Z}_s,\quad t\in[0,\tau_{n+1}). \tag{4.14}$$

Finally, we show that $\zeta:=\lim_{n\to\infty}\tau_n$ is infinite $\mathbb{P}$-a.s.. Define

$$Y_t:=x+\int_0^t H_s\mathrm{d}s+\int_0^t K_s\mathrm{d}\widetilde{Z}_s,$$

where

$$H_s:=\begin{cases}b(s,X_s) & \text{for } s<\zeta,\\ 0 & \text{for } s\geqslant\zeta\end{cases}\quad\text{and}\quad K_s=\begin{cases}\sigma(s,X_{s-}) & \text{for } s<\zeta,\\ 0_{d\times d} & \text{for } s\geqslant\zeta\end{cases}$$

Here $0_{d\times d}$ denotes the zero matrix. Clearly, $Y_t = X_t$ for $t < \zeta$. Note that $t \mapsto \int_0^t K_s \mathrm{d}\widetilde{Z}_s$ is a square-integrable martingale. Hence $\mathbb{P}$-a.s. Y_t has finite left-limits in $t \in (0, \infty)$. Since

$$|Y_{\tau_n} - Y_{\tau_{n-1}}| = |X_{\tau_n} - X_{\tau_{n-1}}| \geqslant \varepsilon/2 \quad \text{for every } n \geqslant 1,$$

it follows that $\zeta = \lim_{n\to\infty} \tau_n = \infty$ $\mathbb{P}$-a.s.. This completes the proof of the theorem. □

5. Weak well-posedness of SDE (1.1)

Let $\mathbb{D}$ be the space of all $\mathbb{R}^d$-valued càdlàg functions on $\mathbb{R}_+$, which is endowed with the Skorokhod topology so that $\mathbb{D}$ becomes a Polish space. Denote by $\mathcal{P}(\mathbb{D})$ the space of all probability measures on $\mathbb{D}$. Let ω_t be the canonical process on $\mathbb{D}$. For $t \geqslant s \geqslant 0$, let $\mathcal{B}_t^s$ denote the natural filtration generated by $\{\omega_r; r \in [s,t]\}$ and define

$$\mathscr{B}_t^s := \cap_{r\geqslant s} \cap_{P\in\mathcal{P}(\mathbb{D})} (\bar{\mathcal{B}}_r^s)^P, \quad \mathscr{B}_t := \mathscr{B}_t^0, \quad \mathscr{B}^s := \mathscr{B}_\infty^s,$$

where $(\bar{\mathcal{B}}_r^s)^P$ stands for the completion of $\mathcal{B}_r^s$ with respect to P.

We first introduce the notion of martingale solutions to SDE (1.1). Recall that the generator of SDE (1.1) is given by $\mathscr{A}_t := \mathscr{L}_t + b \cdot \nabla$.

Definition 5.1 (Martingale solutions)**.** *For $(s, y) \in \mathbb{R}_+ \times \mathbb{R}^d$, a probability measure $\mathbb{P} \in \mathcal{P}(\mathbb{D})$ is called a martingale solution of $\mathscr{A}_t$ starting from y at time s if*

(i) $\mathbb{P}(\omega_s = y) = 1$.

(ii) For any $f \in C_b^2(\mathbb{R}^d)$, M_t^f is a $\mathscr{B}_t$-martingale under $\mathbb{P}$, where

$$M_t^f := f(\omega_t) - f(\omega_s) - \int_s^t \mathscr{A}_r f(\omega_r)\mathrm{d}r, \quad t \geqslant s. \tag{5.1}$$

The set of all the above martingale solutions is denoted by $\mathcal{M}_y^s(\mathscr{A}_t)$.

5.1. Martingale problems for SDEs driven by truncated stable processes.

In this subsection we show the well-posedness of the martingale problem associated with the truncated operator $\widetilde{\mathscr{A}_t}$,

$$\widetilde{\mathscr{A}_t} = \widetilde{\mathscr{L}_t} + \widetilde{b} \cdot \nabla,$$

where $\widetilde{\mathscr{L}_t}$ is defined by (3.20) and $\widetilde{b}_t$ is defined by (4.4). We also write $\widetilde{\mathscr{A}_t}^{\sigma,b}$ for $\widetilde{\mathscr{A}_t}$ when we want to emphasize its dependence on $\sigma(t,x)$ and $b(t,x)$.

The following general localization result can be proven along the same lines as in [25, Theorem 6.6.1].

Lemma 5.2. *Let $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be bounded measurable functions. Suppose that for each $(s, y) \in \mathbb{R}_+ \times \mathbb{R}^d$, there is an open set $\mathscr{U}$ of (s, y) and bounded measurable σ' and b' such that*

(i) *$\sigma = \sigma'$ and $b = b'$ on $\mathscr{U}$;*

(ii) *there is a unique element in $\mathcal{M}_{y'}^{s'}(\widetilde{\mathscr{A}_t}^{\sigma',b'})$ for each $(s', y') \in \mathbb{R}_+ \times \mathbb{R}^d$.*

Then there is a unique element in $\mathcal{M}_y^s(\widetilde{\mathscr{A}_t})$ for each $(s, y) \in \mathbb{R}_+ \times \mathbb{R}^d$.

We can use the above localization lemma to establish the following.

Theorem 5.3. *Suppose that $b(t,x)$ and $\sigma(t,x)$ satisfy (1.6)-(1.7) with $\alpha \in (0, 2)$, and $\beta, \theta \in ((1-\alpha)^+, 1)$. Then for each $(s, y) \in \mathbb{R}_+ \times \mathbb{R}^d$, there is a unique element in $\mathcal{M}_y^s(\widetilde{\mathscr{A}_t})$.*

Proof. Without loss of generality we assume $\beta<\theta$. We fix $\varepsilon\in(0,1)$ being small as in Theorem 3.6. For fixed $(s,y)\in\mathbb{R}_+\times\mathbb{R}^d$, we define

$$b^{s,y}(t,x):=b(t+s,x+y),$$

and

$$\sigma^{s,y}(t,x):=\begin{cases}\sigma(t+s,x+y), & |x|\leqslant\varepsilon/2,\\ \frac{2(\varepsilon-|x|)}{\varepsilon}\sigma\big(t+s,\frac{\varepsilon x}{2|x|}+y\big)+\frac{2(|x|-\varepsilon/2)}{\varepsilon}\sigma(t+s,y), & \varepsilon/2<|x|\leqslant\varepsilon,\\ \sigma(t+s,y), & |x|>\varepsilon.\end{cases}$$

By Lemma 5.2, it suffices to prove that there is a unique element in $\mathcal{M}^{s'}_{y'}(\widetilde{\mathscr{A}}^{s,y}_t)$ for each $(s',y')\in\mathbb{R}_+\times\mathbb{R}^d$, where $\widetilde{\mathscr{A}}^{s,y}_t$ is the operator associated with $(b^{s,y},\sigma^{s,y})$. Without loss of generality, we assume $s'=0$. Since the coefficients are bounded and continuous in x, the existence of martingale solutions is well-known (for example, see [15, p.536, Theorem 2.31]). Let us show the uniqueness. Let $\mathbb{P}_1,\mathbb{P}_2\in\mathcal{M}^0_{y'}(\widetilde{\mathscr{A}}^{s,y}_t)$ be two martingale solutions associated with $\widetilde{\mathscr{A}}^{s,y}_t$ with starting point y' at time 0. Let $T>0$ and $f\in C^\infty_0(\mathbb{R}^{d+1})$. By the assumptions and Theorem 3.6, for $p>\frac{d}{\alpha+\beta-1}\vee\frac{d}{\alpha\wedge\beta}\vee 2$ and λ large enough, there is a unique classical solution $u\in L^\infty_T(B^{\alpha+\beta}_{p,\infty})$ solving the following backward nonlocal parabolic equation:

$$\partial_t u_t+\Big(\widetilde{\mathscr{A}}^{s,y}_t-\lambda\Big)u_t+f=0\quad\text{with}\quad u_T=0.$$

Let $u^\lambda_t(x):=\mathrm{e}^{\lambda(T-t)}u_t(x)$. Then

$$\partial_t u^\lambda_t+\widetilde{\mathscr{A}}^{s,y}_t u^\lambda_t+\mathrm{e}^{\lambda(T-t)}f=0\quad\text{with}\quad u^\lambda_T=0. \tag{5.2}$$

By the definition of martingale solutions, we have

$$u^\lambda_t(X_t)-u^\lambda_0(y')-\int_0^t(\partial_r u^\lambda_r+\widetilde{\mathscr{A}}^{s,y}_r u^\lambda_r)(X_r)\mathrm{d}r$$

are $\mathscr{B}_t$-martingales under $\mathbb{P}_i, i=1,2$. In particular, by (5.2), we obtain

$$\mathbb{E}^{\mathbb{P}_1}\int_0^T\mathrm{e}^{\lambda(T-r)}f(r,X_r)\mathrm{d}r=u^\lambda_0(y')=\mathbb{E}^{\mathbb{P}_2}\int_0^T\mathrm{e}^{\lambda(T-r)}f(r,X_r)\mathrm{d}r.$$

From this, we derive that $\mathbb{P}_1$ and $\mathbb{P}_2$ have the same one-dimensional marginal distributions. By a standard induction method, we deduce that $\mathbb{P}_1=\mathbb{P}_2$ (see [25, p.147, Theorem 6.2.3]). The proof is complete. □

Theorem 5.4. *Suppose that $b(t,x)$ and $\sigma(t,x)$ satisfy (1.6)-(1.7) with $\alpha\in(0,2)$, and $\beta,\theta\in((1-\alpha)^+,1)$. Then for each $x\in\mathbb{R}^d$, there is a unique weak solution to (4.3). More precisely, there are a stochastic basis $(\Omega,\mathscr{F},\mathbf{P};\{\mathscr{F}_t\}_{t\geqslant0})$ and $(X,\widetilde{Z})$ two càdlàg $\mathscr{F}_t$-adapted processes defined on it such that*

(i) *$\widetilde{Z}$ is a pure jump $\{\mathscr{F}_t\}_{t\geqslant0}$-Lévy process with Lévy measure $\mathbf{1}_{\{|z|\leqslant1\}}\nu(dz)$ in the sense that $\widetilde{Z}_t$ is $\mathscr{F}_t$-measurable for each $t\geqslant0$ and for each $t,s\geqslant0$, $\widetilde{Z}_{t+s}-\widetilde{Z}_t$ is independent of $\mathscr{F}_t$;*

(ii) *$(X,\widetilde{Z})$ satisfies a.s. that*

$$X_t=x+\int_0^t b(s,X_s)\mathrm{d}s+\int_0^t\sigma(s,X_{s-})\mathrm{d}\widetilde{Z}_s\quad\textit{for all } t\geqslant0.$$

Proof. Since the coefficients are bounded and continuous in x, the existence of a weak solution to SDE (4.3) is standard by a weak convergence method. The uniqueness follows by Theorem 5.3 since $\mathbf{P}\circ X^{-1}\in\mathcal{M}_x^0(\mathscr{A}_t)$ for each weak solution X of (4.3). □

5.2. **Weak well-posedness for SDEs (1.1).** In this section, we will establish the following existence and uniqueness of weak solution for SDE (1.1). The following is the detailed version of the weak well-posedness part of Theorem 1.1.

Theorem 5.5. *Under conditions (1.5), (1.6) and (1.7), for each $x\in\mathbb{R}^d$, there is a unique weak solution to SDE (1.1). More precisely, there are a stochastic basis $(\Omega,\mathscr{F},\mathbf{P};\{\mathscr{F}_t\}_{t\geqslant0})$ and (X,Z) two càdlàg $\mathscr{F}_t$-adapted processes defined on it such that*

(i) *Z is a pure jump $\{\mathscr{F}_t\}_{t\geqslant0}$-Lévy process with Lévy measure ν in the sense that Z_t is $\mathscr{F}_t$-measurable for each $t\geqslant0$, and for each $t,s\geqslant0$, $Z_{t+s}-Z_t$ is independent of $\mathscr{F}_t$;*

(ii) *(X,Z) satisfies that for all $t\geqslant0$,*

$$X_t=x+\int_0^t b(s,X_s)\mathrm{d}s+\int_0^t\sigma(s,X_{s-})\mathrm{d}Z_s,\ \ a.s.$$

Before we present the proof of this theorem, let us first explain the difficulty, our main idea and the strategy to prove the theorem. The key is to show that if (X,Z) is a weak solution to (1.1), the distribution of X is unique. Let $\widetilde{Z}$ and $\bar{Z}$ be defined as in (4.1). The processes $\widetilde{Z}$ and $\bar{Z}$ are pure jump $\{\mathscr{F}_t\}_{t\geqslant0}$-Lévy processes with Lévy measures $\mathbf{1}_{\{|z|\leqslant1\}}\nu(dz)$ and $\mathbf{1}_{\{|z|>1\}}\nu(dz)$, respectively, and they are mutually independent. Define

$$\tau:=\inf\{t>0:|Z_t-Z_{t-}|>1\}=\inf\{t>0:|\sigma^{-1}(X_{t-})(X_t-X_{t-})|>1\},$$

which is an $\{\mathscr{F}_t\}_{t\geqslant0}$-stopping time. On the other hand,

$$\tau=\inf\{t>0:\bar{Z}_t\neq0\}.$$

So τ is exponentially distributed with parameter $\lambda_0:=\nu(\{|z|>1\})$ and is independent of the truncated Lévy process $\widetilde{Z}$. Note that

$$X_{t\wedge\tau}=X_0+\int_0^{t\wedge\tau}b(X_s)\mathrm{d}s+\int_0^{t\wedge\tau}\sigma(s,X_{s-})\mathrm{d}\widetilde{Z}_s+\mathbf{1}_{\{t\geqslant\tau\}}\sigma_\tau(X_{\tau-})(Z_\tau-Z_{\tau-}).$$

If we define

$$Y_t^\tau:=X_t\mathbf{1}_{t<\tau}+X_{\tau-}\mathbf{1}_{t\geqslant\tau},$$

then

$$Y_{t\wedge\tau}^\tau=X_0+\int_0^{t\wedge\tau}b(Y_s^\tau)\mathrm{d}s+\int_0^{t\wedge\tau}\sigma(s,Y_{s-}^\tau)\mathrm{d}\widetilde{Z}_s,$$

solves SDE (4.3) driven by truncated stable process $\widetilde{Z}$ on $[0,\tau]$. However, we can not get the uniqueness in law of Y^τ from Theorem 5.4 as we can not identify directly $\{Y_t^\tau;t<\tau\}$ *in distribution* with the unique weak solution of (4.3) killed at an independent exponentially distributed time with parameter λ_0. In other words, one does not know a priori whether the *local uniqueness* in distribution holds for SDE (4.3). Instead, we will extend the process of Y^τ beyond time τ by running a weak solution of (4.3) with initial value $X_{\tau-}$ that is independent of $\{(X_t,Z_t);t<\tau\}$ conditioned on $X_{\tau-}$. The extended solution is a weak solution of (4.3) on $[0,\infty)$ and so its law is unique by Theorem 5.4. This would imply that the law of $\{X_t; t \in [0,\tau)\}$ is unique. Consequently, the law of $\{X_t; t \in [0,\tau]\}$ is unique as $X_\tau = X_{\tau-} + \sigma(X_{\tau-})(Z_\tau - Z_{\tau-})$ and $Z_\tau - Z_{\tau-}$ is independent of $X_{\tau-}$. Iterating this would give the uniqueness of the distribution of $\{X_t; t \in [0,\infty)\}$. In the remaining of this subsection, we will make rigorous of this idea and in fact establish the uniqueness of solutions to the martingale problem $\mathcal{M}^s_y(\mathscr{A}_t)$. This localizing procedure is very similar to that of [24, Sections 3]. For reader's convenience, we spell out the details below.

Denote by X the canonical process on $\mathbb{D}$ taking values in $\mathbb{R}^d$; that is, $X_t(\omega) = \omega(t)$ for $\omega \in \mathbb{D}$. For each $s > 0$, define X^{s-} by

$$X^{s-}_t := X_t \mathbf{1}_{\{t<s\}} + X_{s-}\mathbf{1}_{\{t \geqslant s\}}, \quad t \geqslant 0. \tag{5.3}$$

For a $\{\mathscr{B}_t\}$-stopping time τ, define

$$\mathscr{B}_{\tau-} := \sigma\{A \cap \{t < \tau\} : A \in \mathscr{B}_t, t \geqslant 0\}.$$

Clearly, $\tau, X^{\tau-}_t, X_{\tau-} \in \mathscr{B}_{\tau-}$. For $t \geqslant 0$, denote by θ_t the time shift operator on $\mathbb{D}$; that is, $\theta_t\omega(s) = \theta(t+s)$ for $\omega \in \mathbb{D}$.

The following two lemmas are analogy of [25, Lemma 6.1.1 and Theorem 6.1.2].

Lemma 5.6. *Fix $s > 0$ and $\eta \in \mathbb{D}$. For any probability measure Q on $\mathbb{D}$ with $Q(\omega \in \mathbb{D} : \omega(s) = \eta(s-)) = 1$, there is a unique probability measure $\delta_\eta \otimes_{s-} Q$ on $\mathbb{D}$ such that*

$$(\delta_\eta \otimes_{s-} Q)(\{\omega \in \mathbb{D} : \omega(t) = \eta(t) \text{ for all } t \in [0,s)\}) = 1$$

and

$$\delta_\eta \otimes_{s-} Q = Q \quad \text{on } \mathscr{B}^s.$$

For $\omega \in \mathbb{D}$ and $t \geqslant 0$, we also denote $\omega(t)$ by ω_t.

Lemma 5.7. *Let τ be a finite $\{\mathscr{B}_t\}$-stopping time on $\mathbb{D}$. Suppose that $Q : \eta \mapsto Q_\eta$ is a map from $\mathbb{D}$ to $\mathcal{P}(\mathbb{D})$ such that for each $A \in \mathscr{B}_\infty$, $\eta \mapsto Q_\eta(A)$ is $\mathscr{B}_{\tau-}$-measurable and*

$$Q_\eta\left(\{\omega \in \mathbb{D} : \omega(\tau(\eta)) = \eta(\tau(\eta)-)\}\right) = 1 \quad \text{for every } \eta \in \mathbb{D}.$$

Let $P \in \mathcal{P}(\mathbb{D})$. Then

- (i) *There exists a unique $P \otimes_{\tau-} Q \in \mathcal{P}(\mathbb{D})$ such that $P \otimes_{\tau-} Q = P$ on $\mathscr{B}_{\tau-}$ and $\{\delta_\eta \otimes_{\tau(\eta)-} Q_\eta(\cdot)\}_{\eta\in\mathbb{D}}$ is a regular conditional probability distribution of $P \otimes_{\tau-} Q$ given $\mathscr{B}_{\tau-}$.*
- (ii) *If $M : \mathbb{R}_+ \times \mathbb{D} \to \mathbb{R}$ is progressive measurable and right continuous such that $M^{\tau-}_t$ is a P-martingale and $M_t - M^{\tau(\eta)-}_t$ is a Q_η-martingale for each $\eta \in \mathbb{D}$, then M_t is a $P \otimes_{\tau-} Q$-martingale.*

Proof. (i) For $0 = t_0 < t_1 < \cdots < t_n < \infty$ and $\Gamma_i \in \mathscr{B}(\mathbb{R}^d)$, let

$$A_k := \cap_{i=1}^k \{\omega : \omega_{t_i} \in \Gamma_i\}, \ A^k = \cap_{i=k}^n \{\omega : \omega_{t_i} \in \Gamma_i\}, \ k = 1, \cdots, n.$$

Observe that

$$\eta \mapsto (\delta_\eta \otimes_{\tau(\eta)-} Q_\eta)(A_n) = \sum_{k=1}^n \mathbf{1}_{\{t_{k-1} < \tau(\eta) \leqslant t_k\}} \mathbf{1}_{A_{k-1}}(\eta) Q_\eta(A^k) \in \mathscr{B}_{\tau-}.$$

We conclude by a monotone class argument that for any $A \in \mathscr{B}_\infty$, the mapping $\eta \mapsto (\delta_\eta \otimes_{\tau(\eta)-} Q_\eta)(A)$ is $\mathscr{B}_{\tau-}$-measurable. Now we define

$$\mathbb{P}(A) := \int_{\mathbb{D}} (\delta_\eta \otimes_{\tau(\eta)-} Q_\eta)(A) P(\mathrm{d}\eta), \quad A \in \mathscr{B}_\infty.$$

It is easy to see that $\mathbb{P}$ has the desired properties.

(ii) For simplicity we write $\mathbb{E}$ for $\mathbb{E}^{\mathbb{P}}$. Let $0 \leqslant s < t$ and $A \in \mathscr{B}_s$. By definition, we have

$$\mathbb{E}[M_t \mathbf{1}_A] = \mathbb{E}\Big[\mathbb{E}(M_t \mathbf{1}_A | \mathscr{B}_{\tau-})\Big] = \mathbb{E}\Big[\mathbb{E}^{Q_\cdot}(M_t \mathbf{1}_A)\Big].$$

Note that for each $\eta \in \mathbb{D}$,

$$\begin{aligned}\mathbb{E}^{Q_\eta}(M_t \mathbf{1}_A) &= \mathbb{E}^{Q_\eta}\left[(M_t - M_t^{\tau(\eta)-})\mathbf{1}_A\right] + \mathbb{E}^{Q_\eta}\left[M_t^{\tau(\eta)-}\mathbf{1}_A\right] \\ &= \mathbb{E}^{Q_\eta}\left[(M_s - M_s^{\tau(\eta)-})\mathbf{1}_A\right] + \mathbb{E}^{Q_\eta}\left[M_t^{\tau(\eta)-}\mathbf{1}_A\right] \\ &= \mathbb{E}^{Q_\eta}\left[M_s \mathbf{1}_{A\cap\{\tau(\eta)\leqslant s\}}\right] + \mathbb{E}^{Q_\eta}\left[M_t^{\tau(\eta)-}\mathbf{1}_{A\cap\{\tau(\eta)>s\}}\right].\end{aligned}$$

It follows from this and the fact that $Q_\eta(\{\omega : \tau(\omega) = \tau(\eta)\}) = 1$ for each $\eta \in \mathbb{D}$ and $M^{\tau-}$ is a P-martingale that

$$\begin{aligned}\mathbb{E}[M_t \mathbf{1}_A] &= \mathbb{E}\left[\mathbb{E}^{Q_\eta}\left[(M_s \mathbf{1}_{A\cap\{\tau(\eta)\leqslant s\}}\right]\right] + \mathbb{E}\left[\mathbb{E}^{Q_\eta}(M_t^{\tau(\eta)-}\mathbf{1}_{A\cap\{\tau(\eta)>s\}})\right] \\ &= \mathbb{E}\left[M_s \mathbf{1}_{A\cap\{\tau\leqslant s\}}\right] + \mathbb{E}\left[M_t^{\tau-}\mathbf{1}_{A\cap\{\tau>s\}}\right] \\ &= \mathbb{E}\left[M_s \mathbf{1}_{A\cap\{\tau\leqslant s\}}\right] + \mathbb{E}\left[M_s^{\tau-}\mathbf{1}_{A\cap\{\tau>s\}}\right] = \mathbb{E}\left[M_s \mathbf{1}_A\right].\end{aligned}$$

The proof is complete. □

The following Lévy system formula can be proved as in [3, 9].

Lemma 5.8. *Let $(s, y) \in \mathbb{R}_+ \times \mathbb{R}^d$, $\mathbb{P} \in \mathcal{M}_y^s(\mathscr{A}_t)$ and $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ be a measurable function with $F(t, x, x) = 0$. For any $\{\mathscr{B}_t\}$-stopping times $\tau_2 \geqslant \tau_1 \geqslant s$ and any non-negative $\{\mathscr{B}_t\}$-predictable process H_t,*

$$\mathbb{E}\Big[\sum_{\tau_1 < t \leqslant \tau_2} H_t(\omega) F(t, \omega_{t-}, \omega_t)\Big] = \mathbb{E}\left[\int_{\tau_1}^{\tau_2}\!\!\int_{\mathbb{R}^d} H_t(\omega) F(t, \omega_t, \omega_t + \sigma(t, \omega_t) z)\nu(\mathrm{d}z)\mathrm{d}t\right].$$

We have the following key lemma.

Lemma 5.9. *Let $(s, y) \in \mathbb{R}_+ \times \mathbb{R}^d$. Define*

$$\tau(\omega) := \inf\Big\{t > s : |\sigma^{-1}(t, \omega_{t-})(\omega_t - \omega_{t-})| > 1\Big\},$$

which is a $\{\mathscr{B}_t\}$-stopping time, and for $f \in C_b^2(\mathbb{R}^d)$,

$$\widetilde{M}_t^f := f(\omega_t) - f(\omega_s) - \int_s^t \widetilde{\mathscr{A}_r} f(\omega_r)\mathrm{d}r.$$

Suppose $\mathbb{P} \in \mathcal{M}_y^s(\mathscr{A}_t)$. Then $(\widetilde{M}^f)_t^{\tau-}$ is a $\mathbb{P}$-martingale with respect to the filtration $\{\mathscr{B}_t\}$. Moreover, if we let $Q_\eta := \widetilde{\mathbb{P}}_{\tau(\eta), \eta_{\tau(\eta)}-}$ for $\eta \in \mathbb{D}$, where $\widetilde{\mathbb{P}}_{s,y}$ is the unique element in $\mathcal{M}_y^s(\widetilde{\mathscr{A}_t})$, then $\mathbb{P} \otimes_{\tau-} Q \in \mathcal{M}_y^s(\widetilde{\mathscr{A}_t})$.

Proof. (i) For $f \in C_b^2(\mathbb{R}^d)$, by definition of (5.3),

$$(\widetilde{M}^f)_t^{\tau-} = f(\omega_{t\wedge\tau}) - [f(\omega_\tau) - f(\omega_{\tau-})]\mathbf{1}_{\{\tau\leqslant t\}} - \int_s^{t\wedge\tau} \widetilde{\mathscr{A}_r} f(\omega_r)\mathrm{d}r.$$

Thus for any $t \geqslant t' \geqslant s$,

$$\begin{aligned}(\widetilde{M}^f)_t^{\tau-} - (\widetilde{M}^f)_{t'}^{\tau-} = f(\omega_{t\wedge\tau}) - f(\omega_{t'\wedge\tau}) - (f(\omega_\tau) - f(\omega_{\tau-}))\,\mathbf{1}_{\{t'<\tau\leqslant t\}} \\ - \int_{t'\wedge\tau}^{t\wedge\tau} \widetilde{\mathscr{A}_r} f(\omega_r)\mathrm{d}r. \qquad (5.4)\end{aligned}$$

For $E \in \mathscr{B}_{t'}$, since $\mathbb{P} \in \mathcal{M}^s_y(\mathscr{A}_t)$, we have with $\mathbb{E} := \mathbb{E}^{\mathbb{P}}$,

$$\mathbb{E}\Big[\mathbf{1}_E(f(\omega_{t\wedge\tau}) - f(\omega_{t'\wedge\tau}))\Big] = \mathbb{E}\left[\mathbf{1}_E \int_{t'\wedge\tau}^{t\wedge\tau} \mathscr{A}_r f(\omega_r)\mathrm{d}r\right]. \tag{5.5}$$

On the other hand, if we let $F(t,x,y) := (f(y) - f(x))\mathbf{1}_{\{|\sigma^{-1}(t,x)(y-x)|>1\}}$, then

$$(f(\omega_\tau) - f(\omega_{\tau-}))\mathbf{1}_{\{t'<\tau\leqslant t\}} = \sum_{t'\wedge\tau<r\leqslant t\wedge\tau} F(r,\omega_{r-},\omega_r).$$

Since $E \cap \{\tau > t'\} \in \mathscr{B}_{t'\wedge\tau}$, we have by Lemma 5.8 that

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_E(f(\omega_\tau) - f(\omega_{\tau-}))\mathbf{1}_{\{t'<\tau\leqslant t\}}\right] &= \mathbb{E}\left[\mathbf{1}_{E\cap\{\tau>t'\}} \sum_{t'\wedge\tau<r\leqslant t\wedge\tau} F(r,\omega_{r-},\omega_r)\right] \\
&= \mathbb{E}\left[\mathbf{1}_{E\cap\{\tau>t'\}} \int_{t'\wedge\tau}^{t\wedge\tau}\int_{\mathbb{R}^d} F(r,\omega_r,\omega_r+\sigma_r(\omega_r)z)\nu(\mathrm{d}z)\mathrm{d}r\right] \\
&= \mathbb{E}\left[\mathbf{1}_E \int_{t'\wedge\tau}^{t\wedge\tau}\int_{\mathbb{R}^d} (f(\omega_r+\sigma_r(\omega_r)z) - f(\omega_r))\mathbf{1}_{\{|z|>1\}}\nu(\mathrm{d}z)\mathrm{d}r\right].
\end{aligned} \tag{5.6}$$

We thus obtain from (5.4)-(5.6) that for any $t \geqslant t' \geqslant s$ and for any $E \in \mathscr{B}_{t'}$,

$$\mathbb{E}\left[\mathbf{1}_E\left((\widetilde{M}^f)^{\tau-}_t - (\widetilde{M}^f)^{\tau-}_{t'}\right)\right] = 0.$$

This establishes that $(\widetilde{M}^f)^{\tau-}_t$ is a $\mathbb{P}$-martingale with respect to the filtration $\{\mathscr{B}_t\}$.

(ii) Note that $(s,y) \mapsto \widetilde{\mathbb{P}}_{s,y}(A)$ is $\mathscr{B}(\mathbb{R}_+)\times\mathscr{B}(\mathbb{R}^d)$-measurable and $\eta \mapsto (\tau(\eta), \eta_{\tau(\eta)-})$ is $\mathscr{B}_{\tau-}$-measurable. One can also verify by definition that $M^f - (M^f)^{\tau(\eta)-}$ is a $\{\mathscr{B}_t\}$-martingale under each Q_η. Thus $\mathbb{P} \otimes_{\tau-} Q \in \mathcal{M}^s_y(\widetilde{\mathscr{A}_t})$ by Lemma 5.7. □

Theorem 5.10. *The uniqueness of $\mathcal{M}^s_y(\widetilde{\mathscr{A}_t})$ for each $(s,y) \in \mathbb{R}_+ \times \mathbb{R}^d$ implies the uniqueness of $\mathcal{M}^s_y(\mathscr{A}_t)$ for each $(s,y) \in \mathbb{R}_+ \times \mathbb{R}^d$.*

Proof. Without loss of generality, we assume $(s,y) = (0,0)$ and $\mathbb{P}^{(1)}, \mathbb{P}^{(2)} \in \mathcal{M}^0_0(\mathscr{A}_t)$. Let $\tau_0 \equiv 0$ and for each $n \in \mathbb{N}$, define stopping time τ_n by

$$\tau_n = \inf\Big\{t > \tau_{n-1} : |\sigma^{-1}(t,\omega_{t-})(\omega_t - \omega_{t-})| > 1\Big\} \quad \text{with } \inf\emptyset := \infty.$$

For each $(s_0,y_0) \in \mathbb{R}_+\times\mathbb{R}^d$, let $\widetilde{\mathbb{P}}_{s_0,y_0} \in \mathcal{M}^{s_0}_{y_0}(\widetilde{\mathscr{A}_t})$ be the unique martingale solution of SDE (4.3). For $\eta \in \mathbb{D}$, let $Q_\eta := \widetilde{\mathbb{P}}_{\tau_1(\eta),\omega_{\tau_1(\eta)-}}$. By Lemma 5.9,

$$\mathbb{P}^{(i)} \otimes_{\tau_1-} Q \in \mathcal{M}^0_0(\widetilde{\mathscr{A}_t}), \quad i = 1,2.$$

It follows that $\mathbb{P}^{(i)} \otimes_{\tau_1-} Q = \widetilde{\mathbb{P}}_{0,0}$ by the uniqueness of $\mathcal{M}^0_0(\widetilde{\mathscr{A}_t})$, where $i = 1,2$. In particular,

$$\mathbb{P}^{(1)} = \mathbb{P}^{(2)} = \widetilde{\mathbb{P}}_{0,0} \quad \text{on } \mathscr{B}_{\tau_1-}. \tag{5.7}$$

Next we show that

$$\mathbb{P}^{(1)} = \mathbb{P}^{(2)} \quad \text{on } \mathscr{B}_{\tau_1}. \tag{5.8}$$

It suffices to show that for any $n \geqslant 1$, $0 = s_0 < s_1 < \cdots < s_n$ and $\Gamma_j \in \mathscr{B}(\mathbb{R}^d)$ for $0 \leqslant j \leqslant n$,

$$\mathbb{P}^{(1)}\left(\cap_{j=0}^n\{\omega_{s_j\wedge\tau_1} \in \Gamma_j\}\right) = \mathbb{P}^{(2)}\left(\cap_{j=0}^n\{\omega_{s_j\wedge\tau_1} \in \Gamma_j\}\right). \tag{5.9}$$

We have by (5.7)

$$
\begin{aligned}
&\mathbb{P}^{(i)}\left(\cap_{j=0}^{n}\{\omega_{s_j\wedge\tau_1}\in\Gamma_j\}\right)\\
&=\mathbb{P}^{(i)}\left(\cap_{j=0}^{n}\{\omega_{s_j\wedge\tau_1}\in\Gamma_j\};\ \tau_1>s_n\right)+\mathbb{P}^{(i)}\left(\cap_{j=0}^{n}\{\omega_{s_j\wedge\tau_1}\in\Gamma_j\};\ \tau_1\leqslant s_n\right)\\
&=\mathbb{P}^{(i)}\left(\cap_{j=0}^{n}\{\omega^{\tau_1-}_{s_j\wedge\tau_1}\in\Gamma_j\}\right)+\sum_{k=1}^{n}\Big(\mathbb{P}^{(i)}\left(\cap_{j=0}^{n}\{\omega_{s_j\wedge\tau_1}\in\Gamma_j\};\ s_{k-1}<\tau_1\leqslant s_k\right)\\
&\quad-\mathbb{P}^{(i)}\left(\cap_{j=0}^{n}\{\omega^{\tau_1-}_{s_j\wedge\tau_1}\in\Gamma_j\};\ s_{k-1}<\tau_1\leqslant s_k\right)\Big)\\
&=:\widetilde{\mathbb{P}}_{0,0}\left(\cap_{j=0}^{n}\{\omega^{\tau_1-}_{s_j\wedge\tau_1}\in\Gamma_j\}\right)+\sum_{k=1}^{n}J^{(k)}.
\end{aligned}
\tag{5.10}
$$

By the the Markov property of $\mathbb{P}^{(i)}$ on $\mathscr{B}_{\tau_1-}$ and the Lévy system formula of Lemma 5.8, for each $1\leqslant k\leqslant n$,

$$
\begin{aligned}
J^{(k)}&=\mathbb{E}^{\mathbb{P}^{(i)}}\Big[\sum_{s_{k-1}<r\leqslant s_k\wedge\tau_1}\left(\mathbf{1}_{\cap_{i=k}^{n}\Gamma_i}(\omega_r)-\mathbf{1}_{\cap_{i=k}^{n}\Gamma_i}(\omega_{r-})\right)\mathbf{1}_{\{|\sigma_r^{-1}(\omega_{r-})(\omega_r-\omega_{r-})|>1\}};\\
&\qquad\qquad \cap_{j=0}^{k-1}\{\omega_{s_j}\in\Gamma_j\}\cap\{s_{k-1}<\tau_1\leqslant s_k\}\Big]\\
&=\mathbb{E}^{\mathbb{P}^{(i)}}\Big[\int_{s_{k-1}}^{s_k\wedge\tau_1}\int_{\{|z|>1\}}\left(\mathbf{1}_{\cap_{i=k}^{n}\Gamma_i}(\omega_r+\sigma_r(z))-\mathbf{1}_{\cap_{i=k}^{n}\Gamma_i}(\omega_r)\right)\nu(\mathrm{d}z)\mathrm{d}r;\\
&\qquad\qquad \cap_{j=0}^{k-1}\{\omega_{s_j}\in\Gamma_j\}\cap\{\tau_1>s_{k-1}\}\Big]\\
&=\mathbb{E}^{\widetilde{\mathbb{P}}_{0,0}}\Big[\int_{s_{k-1}}^{s_k\wedge\tau_1}\int_{\{|z|>1\}}\left(\mathbf{1}_{\cap_{i=k}^{n}\Gamma_i}(\omega_r+\sigma_r(z))-\mathbf{1}_{\cap_{i=k}^{n}\Gamma_i}(\omega_r)\right)\nu(\mathrm{d}z)\mathrm{d}r;\\
&\qquad\qquad \cap_{j=0}^{k-1}\{\omega_{s_j}\in\Gamma_j\}\cap\{\tau_1>s_{k-1}\}\Big].
\end{aligned}
$$

This together with (5.10) establishes (5.9) and thus (5.8).

Finally, let $\{\mathbb{P}^{(i)}_\omega\}_{\omega\in\mathbb{D}}$ be the regular conditional probability distribution of $\mathbb{P}^{(i)}$ with respect to $\mathscr{B}_{\tau_1}$. By [25, Theorem 6.1.3], there is a common $\mathbb{P}^{(i)}$-null set $\mathcal{N}\in\mathscr{B}_{\tau_1}$ so that for all $\omega\in\mathbb{D}\setminus\mathcal{N}$,

$$
\delta_\omega\otimes_{\tau_1(\omega)}\mathbb{P}^{(i)}_{\cdot}\in\mathcal{M}^{\omega_{\tau_1(\omega)}}_{\tau_1(\omega)}(\mathscr{A}_t).
$$

Repeating the above proof, we can derive that $\mathbb{P}^{(1)}=\mathbb{P}^{(2)}$ on $\mathscr{B}_{\tau_2}$. By the induction and the fact that $\lim_{n\to\infty}\tau_n=\infty$, we have $\mathbb{P}^{(1)}=\mathbb{P}^{(2)}$ on $\mathscr{B}_\infty$. □

We now give the proof for the weak well-posdeness part of Theorem 1.1, that is, Theorem 5.5.

Proof of Theorem 5.5. Since the coefficients are bounded and continuous in x, the existence of a weak solution to SDE (1.1) is standard by a weak convergence argument. The uniqueness follows by Theorem 5.10 since $\mathbf{P}\circ X^{-1}\in\mathcal{M}^0_x(\mathscr{A}_t)$ for each weak solution X of (1.1). □

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