2018 Special Camp - Combinatorial Games Pset

Zawadx

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1 Terms

Combinatorial games satisfy the following conditions:

1. There are two players competing against each other. They alternate moves.
2. There is a fixed set of possible positions for the game.
3. (Usually enforced, to break draws) The game eventually ends in a finite number of moves.
4. The game ends when the player to play has no moves left.

A **position** is simply a listing of the possible moves for both players. A **move** specifies what the next position will be if the player chooses that move.

An **imparital** game is one where each position lists the same moves for both players. Otherwise it's a **partizan** game.

Under **normal play**, the last player to move wins. Under **misere play**, the last player to move loses.

In an impartial game, a **P-position** is one where the **P**revious player is going to win, and an **N-position** is one where the Next player is going to win. If you're like me and dislike this notation, you can instead use the game numbers 0 and $*$ instead of **P** and **N**, which also applies to general games.

A **nimber** is the size of the nim pile which a given game is equivalent to. Nimbers are added by bitwise xor. The **Sprague-Grundy Theorem** states that all impartial games are equivalent to some nimber.

2 Problems

Unless stated otherwise, solve the given game.

Problem 1: The **Subtraction Game** is an impartial game, specified by a fixed set $S \subset \mathbb{N}$ and $N \in \mathbb{N}$. It starts with a pile of N stones, and on each move a player can remove $k \in S$ stones from the pile.

1. Solve the Subtraction game for $S = \{1, 2, 3\}$. Look familiar?
2. Solve the Subtraction game for $S = [n]$[1] under both normal and misere play.
3. Solve the Subtraction game for $S = \{1, 3, 5, 7\}, S = \{1, 3, 6\}$ and for S being all nonnegative powers of two, all fibonacci numbers and all primes.
4. Prove that the position results for the Subtraction game is periodic given finite S. What is this period?
5. The **Thirty-One Game** is like the Subtraction Game for $S = [6]$ and $N = 31$, but each number can only be chosen once. If the first player follows the strategy for the subtraction game, find a strategy for the second player to beat it. Also, solve the Thirty-One Game.

Problem 2: (Empty and Divide) Initially there are two boxes with m chips in one and n chips in the other. A move consists of emptying one box and dividing the chips in the other among the two boxes so that each box has at least one chip in it. The game ends when there is one chip in each box.

Problem 3: (Chomp!) We start with an $m \times n$ chocolate bar. A move consists of removing some cell, and also removing any cell that is above it or to its right (Imagine taking rectangular bites from the top right). The last player to take a bite loses.

1. Consider the position with a three row chocolate bar, having $1, 5, 8$ cells in the rows respectively. Prove that this is a winning position for the first player by finding the unique winning move.
2. It is known that the first player can win from *all rectangular starting positions*. The proof isn't hard, but is an ingenious "existance" proof. See if you can find it.

Problem 4: (Generalized Subtraction) We can generalize the game in Problem 1 by having the subtraction set depend on the opponent's last move.

1. The first move can remove any number of stones, but can't empty the whole pile. In all following moves, you may not remove more stones than removed in the last move.
2. **(Fibonacci Nim)** Just like the above, except each move can be at most twice the size of the previous. The analysis here is harder than before, and the proof uses **Zeckendorf's Theorem**.[2]

[1] $[n]$ is short for the set $\{1, 2, \ldots, n\}$

[2] After solving this, try it out at `http://ceadserv1.nku.edu/longa//classes/2002fall/mat115/days/day09/newfib.html`

Problem 5: (SOS Game[3]) The board is a row of n squares. A move consists of picking an empty square and writing S or O on it. The first player to complete an SOS on consecutive squares wins. The game is a draw if SOS never appears.

1. If $n = 4$ and the first player writes S on the first square, prove that the second player can win.

2. Solve the game for $n = 7$, $n = 14$ and $n = 2000$.[4]

Problem 6: Nim is basically the subtraction game with $S = \mathbb{N}$, but with multiple piles. It is used to build up the entire theory of impartial games. Here's why:

1. **(Nimble)** is played on a row of squares which extends infinitely to the right. Finitely many coins are placed on the squares, possibly multiple on one row. A move consists of taking one coin and moving it to any square to the left. Show that this game is just Nim in disguise.

2. **(Turning Turtles)** Finitely many coins are laid in a row, with heads or tails up. A move involves turning over a coin from heads to tails, and possibly turning over a coin to the left of it (from heads to tails or vice versa). The game ends when all the coins show the same side. Show that this game is just Nim in disguise.

3. A position in **Northcott's Game** is a checkerboard with one black and one white checker on each row. One player moves the white checkers and one player moves the black checkers. A checker moves along its row, but cannot jump onto or over the other checker. The last to move wins. Even though this game seems to be partizan, show that it's just Nim in disguise.

Problem 7: (Staircase Nim) A staircase of n steps contains coins (possibly multiple) on some of the steps. A move consists of taking some positive number of coins from a single step and moving it to the previous step. And coins at the bottom of the staircase are out of the game.

Problem 8: (Moore's Nim) We define Nim_k to be like nim, but in a move you can remove stones from upto k piles. In each move at least one stone must be removed. Nim_1 is just ordinary nim. **Moore's Theorem** states that, $(x_1, x_2, \ldots, x_n)$ is a P-position in nim_k iff the base-$(k+1)$ bitwise sum of the binary expressions of the x_i is zero.

1. Prove Moore's Theorem.

2. What would be optimal play in a misere version of Nim_k?

Problem 9: The Sprague-Grundy function of a game f(x) outputs the nimber associated with a game of size x. The nimber associated with a game is the minimum nimber excluded from the set of nimbers associated with the positions that you can move to from that game. Find the Sprague-Grundy function for:

1. The subtraction game with $S = [n]$.

2. One pile Nim, where you must remove at least half the stones or at most half the stones

[3]No, not the space battle from Haruhi

[4]the last is USAMO 1999 P5

3. **(Dim$^+$)** One pile Nim, where you can only remove c stones from the pile of size m if $c|m$. Also consider the misere version.

4. **(Impatient Subtraction)** Suppose in a subtraction game with set S, we also let the player remove the whole pile on any move. Prove that the function for this game, $g^+(x) = g(x-1) + 1$ where $g(x)$ is the function for the normal subtraction game.

Problem 10: (Wythoff's Game) We place a Queen on some square of a chessboard. The only moves allowed is moving the Queen towards the left or the bottom, and the game ends if the Queen ends up in the bottom-left square. Find the Sprague-Grundy function over the chessboard, and use it to solve the game.

Problem 11: Add the following move to Nim: you may remove the same positive number of coins from three piles. Analyze this game. If $n = 3$, this becomes the three dimensional Wythoff's Game.

Problem 12: (Two-dimensional Nim) We divide the 1st quadrant into squares to form our game board, and place finitely many chips on the squares. A move consists of moving a chip to any square to the left on the same row, or any square on a row below (not necessarily to the left). A square can contain multiple chips. If all the chips are on the bottom row, then this is just Nimble.

1. Find the Sprague-Grundy function over the game board.

2. Consider the position with chips placed on $(0,4)$, $(1,2)$, $(2,1)$, and $(4,0)$ (Each square is identified by the lattice point to its bottom-left). Is it a P-position or N-position? If it's an N-position, what's the winning move? How many moves will the game last?

3. Suppose you are allowed to add finitely many chips to the left of or to any row below the chip removed. Is the game still finite?

Problem 13: Consider a game with n piles, with a move being either removing one or two chips from a pile, or removing one chip and splitting the remaining chips into two piles.

1. Find the Sprague-Grundy Function.

2. Consider the game with one pile of 15 chips. Find an optimal first move.

Problem 14: A game is played on a finite graph. A move consists of removing a vertex and all edges adjacent to that vertex. However, a disjoint vertex can't be removed in this way i.e. at least one edge must be removed. Investigate this game by finding the Sprague Grundy values for:

1. S_n, the star with $n+1$ vertices and n edges that share a common vertex.

2. L_n, a line with $n+1$ vertices and n edges (for small values).

3. C_n, a cycle of length n.

4. A double star (two stars joined by a single edge).

5. Who wins the 3×3 square lattice? Who wins the Tic-Tac-Toe grid? Generalize!

Problem 15: We shall analyze some games which are similar to Turning Tables from Problem 6.2, with a row of coins. Moves consist of turning some coins, with the condition that the rightmost coin turned is always from head to tails (to avoid endless games).

1. Prove that, for all such games can be decomposed into games with one heads at the right. So, g(TTHTTTHTH) = g(TTH) + g(TTTH) + g(TH).
2. **(Twins)** We are allowed to flip exactly two coins, with the rightmost being form heads to tails. Prove that this is also equivalent to Nim.
3. **(Mock Turtles)** Suppose we are allowed to flip upto three coins. The move is: turn over one, two or three coins with the rightmost going from head to tails. Find the Sprague-Grundy function.
4. **(Ruler)** Any number of coins may be turned, as long as they are consecutive and the rightmost is from heads to tails. Find the Sprague-Grundy function.
5. **Grunt** You may turn over coins numbered $(0, n, x-n, x)$, with the rightmost going from heads to tails. Show that this is equivalent to **Grundy's Game**, where the only legal move is to split a single pile into two non-empty unequal piles.

Problem 16: We generalize Coin Turning games to two dimensions. The coins to be turned are in a rectangular array, starting from 0. The condition that the rightmost coin is turned from heads to tails is changed to: there is a coin at (x, y) turned from heads to tails with all other turned coins being in the rectangle $\{(a, b) : 0 \leq a \leq x, 0 \leq b \leq y\}$.

1. **(Acrostic Twins)** A move is to turn over two coins, both in the same column or the same row. Prove that the Sprague Grundy Function for this game $g(x, y)$ is just the nim-sum of x and y.
2. **Turning Corners** A move consists of turning over four distinct coins at the corner of a rectangle. Find the Sprague-Grundy function of this game for small values.
3. Prove that the function defined above satisfies all the criteria for Nim-multiplication: the associative and commutative laws, a multiplicative identity and inverses, and distributive law with nim-sums. This is the nim Field, which is in a sense the simplest possible field using the positive integers!
4. Turning Corners is an example of a **Tartan Game**, a class of games whose solutions may be found using nim multiplication. Each Tartan Game is expressed as the product of two one-dimensional coin turning games. What is Turning Corners a product of?
5. Prove the **Tartan Theorem**: If the Sprague-Grundy functions of two coin turning games G_1 and G_2 are g_1 and g_2 respectively, the Sprague-Grundy function of their product Tartan Game $G_1 \times G_2$ is the nim-product $g(x, y) = g_1(x) \times g_2(y)$.