Combinatorics Problem Set

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National Camp 2017

Easy[1]

0. It takes Edward Elric five minutes to chop a wood into two pieces. How long would it take him to chop it into 3 pieces? (Not kidding!)

1. Zahin climbs a mountain. He starts at 8AM and reaches the summit at noon. He spends the night on the summit. The next morning, he leaves the summit at 8AM and descends by the same route that he used the day before, reaching the bottom at noon. Prove that there is a time between 8AM and noon at which Zahin was at exactly the same spot on the mountain on both days. (Notice that we do not specify anything about the speed that Zahin travels. For example, he could race at 1000 miles per hour for the first few minutes, then sit still for hours, then travel backward, etc. Nor does Zahin have to travel at the same speeds going up as going down.)[2]

2. There are integers from 1 to 2017 written on a blackboard. On a step, Naruto writes the sum of the two numbers and removes the two numbers. At one point, there would be only one number on the blackboard. What is it?

Hint: Think about what remains unchanged[3]

3. Same as the above problem, except instead of writing the sum, for two numbers a and b one writes $ab+a+b$. What would be the final number?

Hint: Try it out with smaller numbers instead of 2017.[4]

4. Given 7 lines on the plane, prove that two of them form an angle less than $26°$.

Hint: If you have 2 angles adding up to $180°$, is there always one less than or equal to $90°$? What if there are 3 angles? 7 angles?[5]

5. There are n distinct integers written on a board. Luffy will get a piece of meat if he takes two numbers from the board, remove them and write their g.c.d. and l.c.m. instead and get a different set of numbers. Prove that Luffy can't get infinitely many pieces of meat.

6. Let P be a set of points in the plane so that each point in P is the midpoint of two other points in P. Show that P is an infinite set.

Hint: Infinity looks scary, huh? When faced with such a large number of things to consider, a good tactic is to assume that the elements of your problem are "in order" if possible. Focus on the "largest" and "smallest" elements, as they may be constrained in interesting ways.[6]

7. At a Chinese restaurant, food is placed on a circular table which can be rotated. n people sat at the same table and ordered n different dishes. When the waiters serve the dishes, no one gets their ordered dish in front of them. Prove that the table may be rotated so that at least two people get their ordered dish in front of them.

8. Find all ordered triples of non-negative odd integers (a, b, c) such that $a + b + c = k$ for some constant non-negative odd integer k.

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*Thanks to The Art and Craft of Problem Solving - Paul Zeitz, Problem Solving Strategies - Arthur Engel and Olympiad Combinatorics - Pranav Sriram for being wonderful books and sources for many of the problems here. They are great books for further reading.

[1] This does not mean that they are not "problems". You will not be able to solve them unless you pay enough attention, and try hard enough!

[2] Try drawing a picture

[3] The formal name for something that remains unchanged in a transformation is Invariant.

[4] Experimentation is perhaps the most useful thing to do for a combinatorics problem. Trying out various cases and configurations can let you fathom and eventually solve some of the toughest problems.

[5] This intuition is the Pigeonhole Principle: make sure that you understand and can apply it properly!

[6] This is the Extremal Principle, which is useful even in the highest levels of mathematics.

9. Prove using combinatorial argument[7]:

$$1.\ \binom{n}{r}+\binom{n}{r+1}=\binom{n+1}{r+1} \qquad 2.\ \binom{2n}{2}=2\binom{n}{2}+n^2 \qquad 3.\ \sum_{k=0}^{n}k\binom{n}{k}=\ n2^{n-1}$$

10. a) A line l is in the same plane as triangle ABC doesn't pass through any of the vertices. How many sides could it pass through?
b) A plane p is in the same space as a tetrahedron $ABCD$ and doesn't pass through any of the vertices. How many edges could it pass through? [8]

11. The two white corners of a standard chessboard are removed. Can the remaining chessboard be tiled with 2×1 dominoes?[9]

12. In Kaori Miyazono's bubble chamber, there are three kinds of particles, namely X,Y, and Z. There are 10, 11 and 111 of them respectively. When two different kinds of particles collide, they both turn into particles of the other kind. Can it happen that eventually only one type of particle is present?

Hint: Invariants again! Whenever you have a chaotic situation with a transformation in the problem, look for invariants (or sometimes Monovariants, which change in only one direction)

13. (a) Can you tile a 6×6 board with 1×4 tiles?
(b) Let k be a positive integer. For which positive integers m, n can a $m \times n$ rectangle can be tiled with $1 \times k$ tiles?
Hint: Color the board to match the tiles!

14. Around a circle, 5 ones and 4 zeros are arranged in any order. Then between any two equal digits, you write 0 and between different digits 1. Finally, the original digits are wiped out. Prove that even if this process is repeated indefinitely, you can never get 9 zeros.

15. You are throwing darts at a target in the shape of an equilateral triangle with side length 2 unit. a) Prove that if you hit 5 darts on the board, you can find 2 darts within 1 unit of each other. b) If you hit 17 darts, what will be the maximum possible value of the minimum distance between two of the darts on the board? What about when you hit 26 darts?

16. There are n people in a room. You know someone if and only if they also know you. Prove that there are two people who know the same number of people in the room.

17. In how many ways can a $2 \times n$ rectangle be tiled by 2×1 dominoes?[10]

18. In how many ways can a $3 \times n$ rectangle be tiled by 2×1 dominoes?

19. Consider a language whose alphabet has just one letter, but in which words of any length are allowed. Messages begin and end with words, and when you type a message, you hit the space bar once between words. How many different messages in this language can be typed using exactly n keystrokes?

20. Points in the plane are each colored with one of three colors: red, green, or blue. Prove that, for a given distance d, there always exist two points of the same color at the distance d from each other.

21. In a convex n-gon, no three diagonals are concurrent. How many diagonals are there? How many intersection points are there of two diagonals?

22. The plane is divided into regions by straight lines. Show that it is always possible to color the regions with two colors so that adjacent regions are never the same color (like a checkerboard)

23. Rooks are placed on the $n \times n$ chessboard satisfying the following condition: If a square is free, there are at least n rooks on its row and column together. Find the minimum possible number of rooks on the board.

[7] This is just fancy language for a story that rigorously describes in English how you count something. See The Art and Craft of Problem Solving, pg. 208

[8] Another important technique in learning and problem solving is using Google. If you don't know what a Tetrahedron is, please Google it and look at images.

[9] Think about the colours on the chessboard, and how those would apply on a domino.

[10] This is actually related with the Fibonacci sequence, where $f_{n+1} = f_n + f_{n-1}$. Try to think of the combinatorial meaning of the Fibonacci sequence.

24. Is there a coloring of the plane with three colors such that any straight line is bichromatic, i.e. only contains points of two colors?

25. A finite set S of points in the plane has the property that any line through two of them passes through a third. Show that all the points lie on a line.

26. A $2n-1 \times 2n-1$ board is going to be tiled by L triominoes, Z tetrominoes and Box tetrominoes. Prove that at least $4n-1$ L triominoes must be used.

27. You have weights of mass $1^2, 2^2, 3^2, 4^2, \ldots,$ with one weight of each size. Given any integer k, prove that you can place some of the weights on the two scales of a balance so that the difference in mass of the two scales equals k. (*HARD MODE:* The weights you place must be $1^2, 2^2, \ldots, n^2$ for some positive integer n.)

28. What is the smallest number of squares on an 8×8 chessboard which would have to be painted so that no 3×1 rectangle could be placed on the board without covering a painted square?

29. In how many ways can you take an odd number of objects from n objects?

30. The numbers from 1 to n^2 are placed on an $n \times n$ grid. Two cells are considered adjacent if they share a side or corner. Prove that there exists two adjacent cells whose numbers differ by at least n.

31. Prove by combinatorial argument:

$$a \geq b \quad \Rightarrow \quad \binom{a}{2} + \binom{b}{2} < \binom{a+1}{2} + \binom{b-1}{2}$$

and

$$\binom{a}{2} + \binom{b}{2} + ab = \binom{a+b}{2}$$

32. Prove that among the p_n parts of a plane cut by n lines with no two parallel, no three concurrent, we have at least $\frac{2n-2}{3}$ triangles.

33. Can a $4k+2 \times 4k+2$ board be covered by T-tetrominoes?

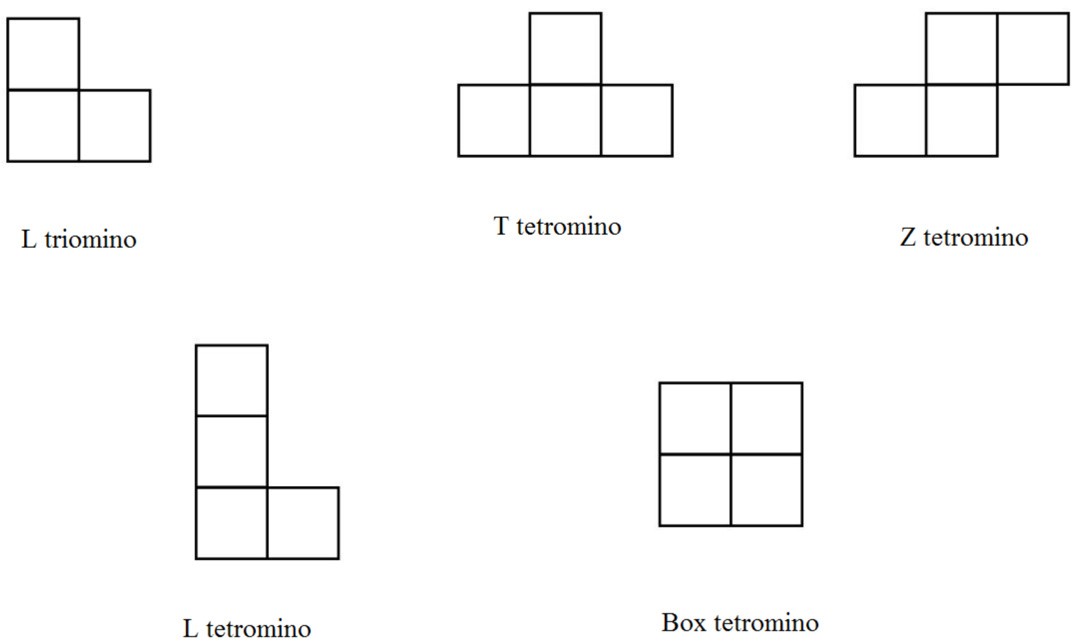

Figure 1: Triominoes and Tetrominoes mentioned in various tiling problems

Medium

1. Every road in Westeros is one-way. Every pair of cities is connected by exactly one one-way road. Show that there exists a city which can be reached from every city either directly or via at most one other city.[11]

2. a) Among six persons, there are always three who know each other or three who are complete strangers. Is this true for five persons? b) Each of 17 scientists corresponds with all the others about one of three topics, with each pair treating exactly one topic. Prove that there are three scientists, who correspond with each other about the same subject. Show that this isn't true for a group of 16 scientists. [12]

3. Every point of the plane is colored red or blue. Show that there exists a rectangle with vertices of the same color.

4. A square is removed from a $2^k \times 2^k$ board. Show that the remaining board can be completely tiled with L-triominoes.

5. In any convex polygon, there exists three consecutive vertices whose circumcircle covers the entire polygon.

6. Every road in Westeros is one-way. Every pair of cities is connected by exactly one one-way road. Show that there exists a path which traverses each city exactly once.

7. (IMO Shortlist 1989) An integer is written on each cell of an $m \times n$ chessboard. If two integers share a common side, Megumin can add k to both of them. Find a necessary and sufficient condition so that Megumin can make all the numbers 0 after a finite number of operations.

8. In a 7×7 grid, two squares are yellow and the rest are green. How many unique arrangements are there for the squares? (two arrangements are not unique if one can be obtained by rotating the other)

9. There are n identical cars on a circular track. Among all of them, they have just enough gas for one car to complete a lap. Show that there is a car which can complete a lap by collecting gas from the other cars on its way around.

10. In the Sikinian parliament, every member has at most three enemies among the other members. Prove that the parliament can be split into two houses, so that every member has at most one enemy in his house.

11. There are n boxes $B_1, B_2, \ldots, B_n$ from left to right, and there are n balls in these boxes. If there is at least 1 ball in B_1, we can move one to B_2. If there is at least 1 ball in B_n, we can move one to B_{n-1}. If there are at least 2 balls in B_k, $2 \le k \le n-1$ we can move one to B_{k-1}, and one to B_{k+1}. Prove that, for any arrangement of the n balls, we can achieve a final arrangement where each box has one ball in it.

12. Let D_n denote the number of derangements of a sequence of length n, i.e. a permutation where no element remains in its original spot. Prove through combinatorial argument that:

$$D_n = n! \sum_{r=0}^{n} (-1)^k \frac{1}{k!} \qquad \text{and,} \qquad n! = \sum_{r=0}^{n} \binom{n}{r} D_{n-r}$$

13. $2n$ points are given in the plane, no three collinear. Exactly n of these points are farms $F = \{F_1, F_2, \ldots, F_n\}$. The remaining n points are wells: $W = \{W_1, W_2, \ldots, W_n\}$. It is intended to build a straight line road from each farm to one well. Show that the wells can be assigned to the farms bijectively (i.e. such that one well is assigned to exactly one farm and vice versa), so that none of the roads intersect.

14. We call a rectangle good if the length of at least one of its sides is an integer. If a rectangle can be tiled with good rectangles, prove that the original rectangle must also be good.

15. Initially, some configuration S of cells of a given $n \times n$ chessboard are infected. Then, the infection spreads as follows: a cell becomes infected iff at least two of its neighbors (not including diagonal ones) are infected. If the entire board eventually becomes infected, prove that $|S| \ge n$ (that is, at least n of the cells were infected initially).

[11] The cities and the connections between them are conveniently expressed as the vertices and edges of a graph. This representation can allow you to bring easily use the vast range of notation and results of Graph Theory.

[12] These problems ask you to find the Ramsey Numbers R(3,3) and R(3,3,3). See Problem solving strategies, page 63

16. (IMO 2011/4) Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \cdots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed. Determine the number of ways in which this can be done.[13]

17. If you draw all the diagonals, how many regions do they divide the interior of the n-gon into? How many of these regions are triangles?

18. I invited n couples to my house. Everyone shook hands with some number of people, but no one shook hands with their partner. I asked everyone (including my wife) how many hands they shook, and everyone gave a different answer. How many hands did my wife shake?

19. (IMO Shortlist 2012 C1) Several positive integers are written in a row. Iteratively, Hiccup chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.

20. For a set S of integers, define $S + 1 = x + 1 : x \in S$. How many subsets S of $1, 2, \cdots, n$ satisfy $S \cup S + 1 = 1, 2, \cdots, n$?

21. (IMO Shortlist 1985) A set of 1985 points is distributed around the circumference of a circle and each of the points is marked with 1 or -1. A point is called "good" if the partial sums that can be formed by starting at that point and proceeding around the circle for any distance in either direction are all strictly positive. Show that if the number of points marked with -1 is less than 662, there must be at least one good point.

22. Find the number of subsets of $1, 2, \cdots, n$ which don't have two consecutive integers are its elements.

23. (IMO Shortlist 2015 N1) Determine all positive integers M such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

contains at least one integer term.

24. Laura prepared for the iMO (imaginary Math Olympiad) over a period of 72 days. She did at least one problem per day, but no more than 114 problems in total. Prove that there was a period of consecutive days when Laura had a streak of solving exactly 29 problems.

25. Let D_n denote the number of derangements of a sequence of length n, i.e. a permutation where no element remains in its original spot. Prove that $D_n = (n - 1)(D_{n-1} + D_{n-2})$. Use this to find a closed form for D_n.

26. In the plane, n lines are given ($n \geq 3$), no two of them parallel. Through every intersection of two lines there passes at least an additional line. Prove that all lines pass through one point.

27. Prove that a 4×11 rectangle cannot be tiled by L-shaped tetrominoes.

28. There are 12 spaces in a linear parking lot, 8 of which are occupied. An SUV pulls up, and requires 2 consecutive spaces to park. What is the probability that it will be able to?

29. We define a *selfish* set to be one which has its own cardinality as an element. How many subsets of $1, 2, \cdots, n$ is selfish? How many are *minimally* selfish i.e. have no proper subsets which are selfish?

30. (Zawad) In each square of a 4×4 board, a lightbulb is placed. Each lightbulb is either on or off. A move consists of selecting a lightbulb, and switching the states of that lightbulb and all lightbulbs sharing an edge with it. Is it possible to find a sequence of moves to switch all lightbulbs on, for all starting configurations?

31. Find an expression for the following sum using combinatorial argument:

$$\sum_{k=0}^{n} k^2 \binom{n}{k}, \quad \sum_{k=0}^{n} \binom{n}{k}^2$$

32. Prove that for any set of n positive integers, there will be a non-empty subset for which the sum of the elements will be divisible by n.

[13] Yes, I know it's an IMO problem. But there's no need to be scared. IMO problems are quite solveable if you approach it with proper mental endurance, and there's no better time to start doing them than right now!

33. Points on a straight line are colored in two colors. Prove that it is always possible to find three points of the same color with one being the midpoint of the other two.

34. A finite set of coins with distinct diameters are placed on the plane. Prove that there is a coin bordered by no more than 5 others.

35. Does the set $1, 2, \cdots, 3000$ contain a subset T of 2000 elements such that $x \in T \Rightarrow 2x \notin T$

36. A subset of $n + 1$ elements is taken from $1, 2, \cdots, 2n$.
a) Prove that there exists two elements in the subset, one of which is divisible by the other.
b) Prove that there exists two elements in the subset which are co-prime.

37. (USAMO 1998) A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Determine the minimum number of mouse clicks needed to make the chessboard all one color.

38. Let $A_1A_2 \cdots A_{2n}$ be a convex polygon with $2n$ sides. P is a point inside the polygon which does not lie on any of its diagonals. Prove that there exists a side of the polygon, the interior of which isn't intersected by any of the lines PA_1, PA_2, $\cdots$, PA_{2n}.

39. Seven dwarfs are sitting around a circular table. There is a cup in front of each. There is milk in some cups, altogether 3 liters. One of the dwarfs shares his milk uniformly with the other cups. Proceeding counter-clockwise, each of the other dwarfs, in turn, does the same. After the seventh dwarf has shared his milk, the initial content of each cup is restored. First guess the initial amount of milk in each cup, and then prove that these were indeed the amounts.

Hard

1. (IMO Shortlist 2013 C3) A crazy physicist discovered a new kind of particle which he called an imon, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be entangled, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.
1. If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.
2. At any moment, he may double the whole family of imons in his lab by creating a copy I_1 of each imon I. During this procedure, the two copies I_1 and J_1 become entangled if and only if the original imons I and J are entangled, and each copy I_1 becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.
Prove that the physicist may apply a sequence of such operations resulting in a family of imons, no two of which are entangled.

2. Consider 9 points in space, no four of which are coplanar. Each pair of points is joined by an edge, and each edge is coloured red or blue, or left uncoloured. Find the smallest value of n such that whenever exactly n edges are coloured, the set of coloured edges must contain a triangle all of whose edges are the same colour.

3. Barney scored 100 problems in his perfect week. Every problem used k plays. Among every 20 problems, there was a common play used on all of them. But no play was used on all problems. What is the minimum number of plays Barney could have used per problem ?

4. (IMO 2014/2) Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of n^2 unit squares. A configuration of n rooks on this board is peaceful if every row and every column contains exactly one rook. Find the greatest positive integer k such that, for each peaceful configuration of n rooks, there is a $k \times k$ square which does not contain a rook on any of its k^2 unit squares.

5. Let S be a set with 2002 elements, and let N be an integer with $0 \leq N \leq 2^{2002}$. Prove that it is possible to color every subset of S either black or white so that the following conditions hold:
(a) the union of any two white subsets is white;
(b) the union of any two black subsets is black;
(c) there are exactly N white subsets.

6. Students in a class form groups. Each group contains exactly three members and any two distinct groups have at most one member in common. Prove that if there are 46 students in the class, then there exists a set of at least 10 students in which no group is properly contained.

7. (IMO 2011/2) Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A windmill is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the pivot P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely. Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.

8. (APMO 2009/1) Consider the following operation on positive real numbers written on a blackboard: Choose a number r written on the blackboard, erase that number, and then write a pair of positive real numbers a and b satisfying the condition $2r^2 = ab$ on the board.
Assume that you start out with just one positive real number r on the blackboard, and apply this operation $k^2 - 1$ times to end up with k^2 positive real numbers, not necessarily distinct. Show that there exists a number on the board which does not exceed kr.

9. (IMO Shortlist 1996) A square $(n-1) \times (n-1)$ is divided into $(n-1)^2$ unit squares in the usual manner. Each of the n^2 vertices of these squares is to be coloured red or blue. Find the number of different colourings such that each unit square has exactly two red vertices. (Two colouring schemes are regarded as different if at least one vertex is coloured differently in the two schemes.)

10. (RMM 2017) Let n be an integer greater than 1 and let X be an n-element set. A non-empty collection of subsets $A_1, ..., A_k$ of X is tight if the union $A_1 \cup \cdots \cup A_k$ is a proper subset of X and no element of X lies in exactly one of the A_is. Find the largest cardinality of a collection of proper non-empty subsets of X, no non-empty subcollection of which is tight.
Note: A subset A of X is proper if $A \neq X$. Cardinality refers to the number of elements in a set. A collection is a set of sets, and the subset of a collection is a subcollection. The sets in a collection are assumed to be distinct. The whole collection is assumed to be a subcollection.

11. (USA TST 2011)In the nation of Onewaynia, certain pairs of cities are connected by roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges). Some roads have a traffic capacity of 1 unit and other roads have a traffic capacity of 2 units. However, on every road, traffic is only allowed to travel in one direction. It is known that for every city, the sum of the capacities of the roads connected to it is always odd. The transportation minister needs to assign a direction to every road. Prove that he can do it in such a way that for every city, the difference between the sum of the capacities of roads entering the city and the sum of the capacities of roads leaving the city is always exactly one.

12. Each face of a cube is coloured with a different colour. How many of the colourings are distinct (not obtainable by rotating another)?

13. One corner of a $(2n+1) \times (2n+1)$ chessboard is cut off. For which n can you cover the remaining squares by 2×1 dominoes, so that half of the dominoes are horizontal.

14. In a set S of $2n$ persons there are two with an even number of common friends.

15. (IMO 1994)Let m and n be two positive integers. Let $a_1, a_2, \ldots, a_m$ be m different numbers from the set $\{1, 2, \ldots, n\}$ such that for any two indices i and j with $1 \leq i \leq j \leq m$ and $a_i + a_j \leq n$, there exists an index k such that $a_i + a_j = a_k$. Show that

$$\frac{a_1 + a_2 + ... + a_m}{m} \geq \frac{n+1}{2}.$$

16. (APMO 2016) A positive integer is called fancy if it can be expressed in the form

$$2^{a_1} + 2^{a_2} + \cdots + 2^{a_{100}},$$

where $a_1, a_2, \cdots, a_{100}$ are non-negative integers that are not necessarily distinct. Find the smallest positive integer n such that no multiple of n is a fancy number.

17. (AUO 1973)Four non-coplanar points are given. How many parallelepiped boxes have these points as vertices?

18. (HMMT Guts 2012) Luna has an infinite supply of red, blue, orange, and green socks. She wants to arrange 2012 socks in a line such that no red sock is adjacent to a blue sock and no orange sock is adjacent to a green sock. How many ways can she do this?

19. The Fibonacci numbers f_n are defined by $f_1 = 1, f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$. Prove that for any positive integer k, there is a Fibonacci number ending with k zeroes.

20. Prove that there is no closed knight's tour (a sequence of moves starting and ending at the same square) on a $4 \times n$ chessboard.

21. (Canada 2004) Let T be the set of all positive integer divisors of 2004^{100}. What is the largest possible number of elements that a subset S of T can have if no element of S is an integer multiple any other element of S?

22. Let n be a positive integer. n people take part in a certain party. For any pair of the participants, either the two are acquainted with each other or they are not. What is the maximum possible number of the pairs for which the two are not acquainted but have a common acquaintance among the participants?

23. (USAMO 2008)Let $\mathcal{P}$ be a convex polygon with n sides, $n \geq 3$. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a triangulation of $\mathcal{P}$ into $n-2$ triangles. If $\mathcal{P}$ is regular and there is a triangulation of $\mathcal{P}$ consisting of only isosceles triangles, find all the possible values of n.

24. Let S be a set of 25 points arranged in a 5×5 unit square array. Show that among any 6 points in S, we can always select three of them so that the area of the triangle they form is at most 2 square units.

25. Find all sets of points (no 3 collinear) such that for any three points A, B, C you can always find a fourth point D such that the four points form a parallelogram in some order.

26. (IMO 1978) An international society has members from six different countries. The list of members contains 1978 names, numbered $1, 2, \ldots, 1978$. Prove that there is at least one member whose number is the sum of the numbers of two members from his country or twice as large as the number of one member from his own country.

27. (IMO 1997) An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a silver matrix if, for each $i = 1, 2, \ldots, n$, the i-th row and the i-th column together contain all elements of S. Show that:
(a) There is no silver matrix for $n = 1997$;
(b) Silver matrices exist for infinitely many values of n.

Constructions

In this section, you are only supposed to do the construction part of the problem and in many cases guess the optimal answer. Beware, some problems may be easier to solve if you find the bound first. You can try to solve the full problem if you wish, but be warned that it will probably be extremely difficult.

1. Let n, k be given positive integers satisfying $k \leq 2n-1$. On a table tennis tournament $2n$ players take part, they play a total of k rounds, each round is divided into n matches, each match two players play. The two players can match on many occasions in different rounds. Find the greatest positive integer $m = f(n, k)$ such that no matter how the tournament progresses, we always find m players each of pair of which didn't match each other.

2. Let $n \geq 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding n?

3. In a 999×999 square table some cells are white and the remaining ones are red. Let T be the number of triples (C_1, C_2, C_3) of cells, the first two in the same row and the last two in the same column, with C_1, C_3 white and C_2 red. Find the maximum value T can attain.

4. Let n be an positive integer. Find the smallest integer k with the following property; Given any real numbers $a_1, \cdots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \leq a_i \leq 1$ for $i = 1, 2, \cdots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

5. (IMO 2015/1) We say that a finite set $\mathcal{S}$ of points in the plane is balanced if, for any two different points A and B in $\mathcal{S}$, there is a point C in $\mathcal{S}$ such that $AC = BC$. We say that $\mathcal{S}$ is centre-free if for any three different points A, B and C in $\mathcal{S}$, there is no points P in $\mathcal{S}$ such that $PA = PB = PC$.
(a) Show that for all integers $n \geq 3$, there exists a balanced set consisting of n points.
(b) Determine all integers $n \geq 3$ for which there exists a balanced centre-free set consisting of n points.

6. (ISL 2015 C3) For a finite set A of positive integers, a partition of A into two disjoint nonempty subsets A_1 and A_2 is *good* if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

7. Elmo is drawing with colored chalk on a sidewalk outside. He first marks a set S of $n > 1$ collinear points. Then, for every unordered pair of points $\{X, Y\}$ in S, Elmo draws the circle with diameter XY so that each pair of circles which intersect at two distinct points are drawn in different colors. Count von Count then wishes to count the number of colors Elmo used. In terms of n, what is the minimum number of colors Elmo could have used?

8. Let n be a positive integer. The numbers $\{1, 2, ..., n^2\}$ are placed in an $n \times n$ grid, each exactly once. The grid is said to be Muirhead-able if the sum of the entries in each column is the same, but for every $1 \leq i, k \leq n-1$, the sum of the first k entries in column i is at least the sum of the first k entries in column $i+1$. For which n can one construct a Muirhead-able array such that the entries in each column are decreasing?

9. A $2^{2014}+1$ by $2^{2014}+1$ grid has some black squares filled. The filled black squares form one or more snakes on the plane, each of whose heads splits at some points but never comes back together. In other words, for every positive integer n greater than 2, there do not exist pairwise distinct black squares $s_1, s_2, \ldots, s_n$ such that s_i and s_{i+1} share an edge for $i = 1, 2, \ldots, n$ (here $s_{n+1} = s_1$).
What is the maximum possible number of filled black squares?

10. For any integer $n \geq 2$, let $N(n)$ be the maximal number of triples (a_i, b_i, c_i), $i = 1, \ldots, N(n)$, consisting of nonnegative integers a_i, b_i and c_i such that the following two conditions are satisfied: $a_i + b_i + c_i = n$ for all $i = 1, \ldots, N(n)$, If $i \neq j$ then $a_i \neq a_j$, $b_i \neq b_j$ and $c_i \neq c_j$ Determine $N(n)$ for all $n \geq 2$.

11. Let n be a positive integer. Determine the size of the largest subset of $\{-n, -n+1, \ldots, n-1, n\}$ which does not contain three elements a, b, c (not necessarily distinct) satisfying $a + b + c = 0$.

12. For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k+1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $\frac{N}{2}$.

13. A mathematics competition has n contestants and five problems. On each problem, each contestant is assigned a positive integer score which is at most seven. It turns out no pair of contestants got the same score on two different problems. Find the maximum possible value for n.

14. Let m, n be positive integers with $m > 1$. Sherlock partitions the integers $1, 2, \ldots, 2m$ into m pairs. Moriarty then chooses one integer from each pair and finds the sum of these chosen integers. Prove that Sherlock can select the pairs so that Moriarty cannot make his sum equal to n.

15. In a lottery, Rahad must select six distinct numbers from $\{1, 2, \cdots, 36\}$ to put on a ticket. The lottery committee will then draw six distinct numbers randomly from $\{1, 2, \cdots, 36\}$. Any ticket not containing any of these 6 numbers is a winning ticket. Show that there exists a set of nine tickets such that at least one of them will certainly be a winning ticket, whereas this statement is false if 9 is replaced by 8.

Algorithms

In this section, each problem asks you (implicitly or explicitly) to find an algorithm to accomplish an activity. Find the algorithm.

1. Given a sequence of numbers $a_1, a_2, ..., a_n$ sort them in increasing order. Find an algorithm that takes as few steps as possible (intuitively, as less "time" [14] as possible).[15]

2. In a graph G with n vertices, no vertex has degree greater than Δ. Show that one can color the vertices using at most $\Delta + 1$ colors, such that no two neighboring vertices are the same color.

3. In a $2 \times n$ array we have positive real numbers such that the sum of the numbers in each of the n columns is 1. Show that we can select one number in each column such that the sum of the selected numbers in each row is at most $\frac{(n+1)}{4}$.

4. A set of three nonnegative integers $\{x, y, z\}$ with $x < y < z$ satisfying $\{z - y, y - x\} = \{1776, 2001\}$ is called a historic set. Show that the set of all nonnegative integers can be written as a disjoint union of historic sets.

5. Peter has 3 accounts in a bank, each with an integral number of dollars. He is only allowed to transfer money from one account to another so that the amount of money in the latter is doubled. Prove that Peter can always transfer all his money into two accounts. Can he always transfer all his money into one account?

[14] Look up 'algorithm complexity' for a more rigorous idea
[15] If you want to know more about these, google sorting algorithms

6. n people are seated in a circle. A total of nk coins have been distributed among them, but not necessarily equally. A move is the transfer of a single coin between two adjacent people. Find an algorithm for making the minimum possible number of moves which result in everyone ending up with the same number of coins.

7. Complete problem 4 of Constructions.

8. Find an algorithm to find the convex hull of a set of points given their coordinates. Minimize the number of steps required.

9. Given a sequence of numbers $a_1, a_2, \dots a_k$ - devise an algorithm to find the maximum sum of a group of consecutive terms of the sequence. For example, in the sequence $-1, 2, 1, 0$ the maximum sum of a group of consecutive elements is $2 + 1 = 3$.

10. (Binary searching) You are given an increasing sequence $a_1, a_2 \dots a_k$ and a number L. Find whether L is in the sequence in as few steps as possible.

Final Bosses[16]

1. Let S be a set of n points in space. The segments joining these segments are of distinct length, and r of these segments are coloured red. Let m be the smallest integer for which $m \geq 2r/n$. Prove that there always exists a path of m red segments with their lengths sorted in ascending order.

2. (IMO Shortlist 2000 G7) Ten gangsters are standing on a flat surface, and the distances between them are all distinct. At twelve o'clock, when the church bells start chiming, each of them fatally shoots the one among the other nine gangsters who is the nearest. At least how many gangsters will be killed?

3. (IMO 2014 P6 easier) A set of lines in the plane is in general position if no two are parallel and no three pass through the same point. A set of line in general position cuts the plane into regions, some of which have finite area; we call these its finite regions. Prove that for all sufficiently large n, in any set of n lines in general position it is possible to colour at least $\sqrt{n/2}$ lines blue in such a way that none of its finite regions has a completely blue boundary.

4. Each square of a $2^n - 1 \times 2^n - 1$ square board contains either $+1$ or -1. Such an arrangement is deemed successful if each number is the product of its neighbours. Prove that, for $n > 1$, there exists only one successful arrangement, namely all 1.

Hint: Try to force symmetry.

5. An $(n, k)-$ tournament is a contest with n players held in k rounds such that:
(i) Each player plays in each round, and every two players meet at most once. (ii) If player A meets player B in round i, player C meets player D in round i, and player A meets player C in round j, then player B meets player D in round j.
Determine all pairs (n, k) for which there exists an $(n, k)-$ tournament.

6. (IMO 2016 P6) There are $n \geq 2$ line segments in the plane such that every two segments cross and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it facing the other endpoint. Then he will clap his hands $n - 1$ times. Every time he claps,each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.
(a) Prove that Geoff can always fulfill his wish if n is odd.
(b) Prove that Geoff can never fulfill his wish if n is even.

[16]They are not unbeatable, but they might need you to save and reload, try and try again for quite some time. But trust me, the reward will be worth it.