Isometric C^1 embedding of flat tori into $\mathbb{R}^3$

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Abstract

Smooth embeddings of the torus into Euclidean space cannot preserve its flat metric. But one can obtain a C^1 embedding which is isometric, by repeatedly applying the process of Convex Integration on an initial embedding. This paper provides an exposition on this result, describing the ideas behind the explicit construction for isometric C^1 embeddings of flat tori into $\mathbb{R}^3$.

1 Introduction

The "donut-shaped" embedding of the flat torus $\mathbb{T}^2$ into $\mathbb{R}^3$ is one of the fundamental introductory visuals of topology.But this embedding is geometrically lacking, as it does not preserve distances. In fact, it is impossible to create an isometric embedding of the torus into $\mathbb{R}^n$ which is also smooth. Such an embedding must have a region of positive curvature, and would violate the metric on the torus which must be flat everywhere.

In 1954, John Nash surprised the mathematical community by proving that if you relax the smooth (or C^∞) requirement to be simply C^1, then there does exist an embedding of the torus into $\mathbb{R}^3$ [4]. Later, Gromov developed Complex Integration Theory which linked Nash's theorem to other results in differential topology. This theory was used in [1] to convert the Nash-Kuiper process for C^1 embedding of the sphere into an explicit algorithm. The paper further uses the algorithm to generate the first images of an isometric embedding of the torus, revealing that the structure of the embedded object lies in-between a fractal and a smooth surface. The authors of the paper have also prepared a great exposition on these results (see [5]) which focus on the history and visuals without going into detail on the mathematics.

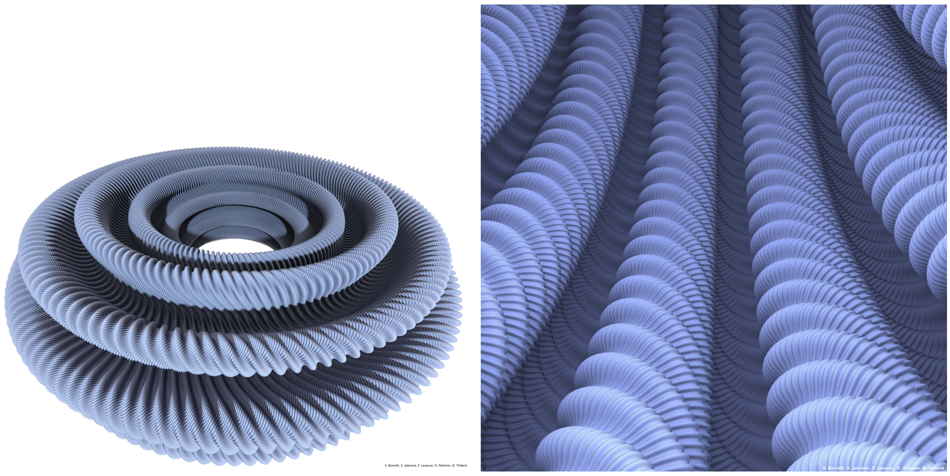

Figure 1: Images of the isometric embedding of the torus. Note the fractal-like nature when you zoom in. Figure from [5].

This paper follows [1] to provide a different kind of exposition, focusing on the mathematical ideas underlying the isometric embedding. We discuss the key ideas in the construction, skipping over rigorous proofs and technical details. Section 2 introduces the ideas behind Convex Integration, by illustrating the one-dimensional case of approximating a curve under constraints on the derivative. Section 3 starts generalizing to the torus, showing how convex integration can be applied in two dimensions and how it might constrain the metric of the approximation. We use several assumptions to introduce the theory simply. Section 4 provides the elements needed to complete the proof.

This paper assumes familiarity with the basic definitions of differential geometry. See [3] for an introduction to the topic. Many results in this paper are stated without proof, and for the rest only a sketch of the proof is provided. Our exposition follows [1], and motivated readers can check the corresponding lemmas and theorems there for proofs. We also omit the more technical aspects of certain definitions and notation. For example, the constants determining the bounds in certain lemmas are dependant on various parameters, which we do not track very closely.

2 Introduction to Convex Integration

Convex Integration essentially provides an approximation for a given function subject to certain constraints. In this section, we describe the one-dimensional case for Convex Integration, to provide intuition for the two-

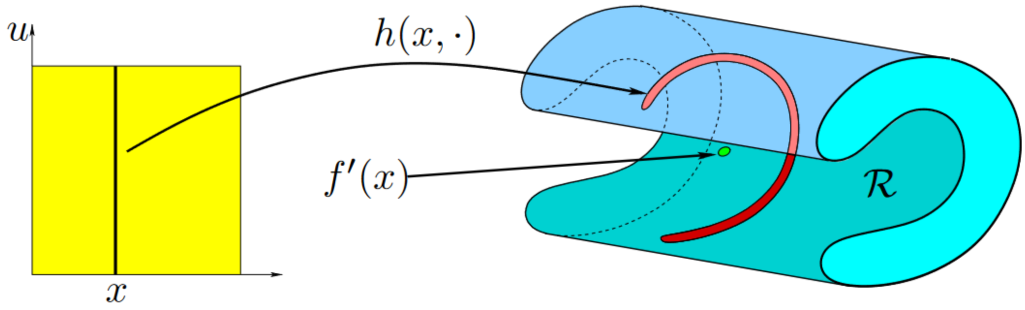

Figure 2: The loop $h(x, u)$ for a fixed value of x, encircling the point $f'(x)$. Figure 2.1 in [1].

dimensional case used for the torus. In the one-dimensional case, we are approximating a curve $f : I := [0, 1] \to \mathbb{R}^n$. To describe the constraints on the approximation, we define differential relations and their solutions:

Definition 2.1. *Consider a subset $\mathcal{R}_x \subset \mathbb{R}^n$ of vectors for all points $x \in I$. The union $\mathcal{R} := \cup_{x \in I} \mathcal{R}_x$ is called a* differential relation*. A curve $F : I \to \mathbb{R}^n$ is a* solution *of $\mathcal{R}$ if $f'(x) \in \mathcal{R}_x$ for all $x \in I$.*

Differential relations thus impose a condition on the derivative of a curve. For a curve f and differential relation $\mathcal{R}$, Convex Integration allows us to construct a solution of $\mathcal{R}$ that is C^0 close to f (if certain conditions are met). This happens in two steps: first we define a family of loops $h(x, u)$ in $\mathcal{R}_x$ such that f' is the average of h. Then we obtain the approximation F by integrating h over a winding path in (x, u)-space, so that $F'(x)$ is always in $\mathcal{R}_x$, but in aggregate F turns out to be close to f. These steps are elaborated below:

For a given x, we take $h(x, u)$ to be a function $\mathbb{R}/\mathbb{Z} \to \mathcal{R}_x$, such that

$$f'(x) = \int_0^1 h(x, u)du.$$

It turns out that a necessary and sufficient condition for such h to exist is for f to be *strictly short*, i.e. for $f'(x)$ to be interior to the convex hull of $\mathcal{R}_x$ for all $x \in I$. We assume that this condition holds for the cases we work with; it is not hard to satisfy for the $\mathcal{R}_x$ we work with for the isometric embedding problem.

In the case of our problem, the differential relation $\mathcal{R}$ constrains the norm of the derivative. Thus its shape is spherical and the loops h can be constructed from f and $\mathcal{R}$ as a circular loop whose speed is regulated by the Bessel function. See Section 2.2 of [1] for details of the construction and references to further theory.

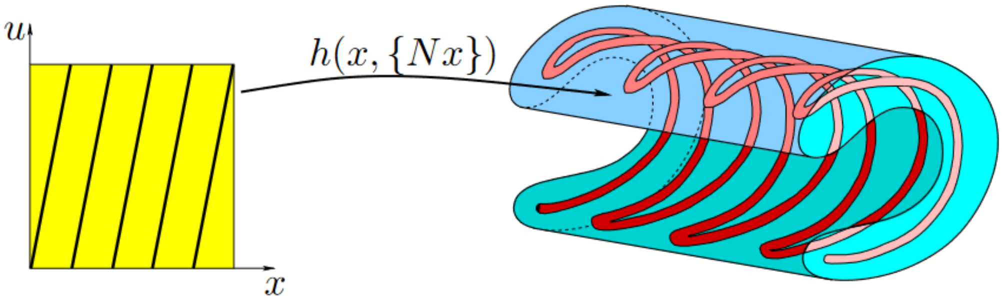

Figure 3: The path $x \mapsto (x, \{Nx\})$ winds N times around the cylinder and traces a path in $\mathcal{R}$ (pictured in blue). Figure 2.2 in [1].

Then, for a given positive integer N, we define

$$F(t) := f(0) + \int_0^t h(x, \{Nx\})dx$$

where $\{Nx\}$ is the fractional part of Nx. Intuitively, F is integrating h along a periodic curve with period $1/N$. As N increases, each individual period gets closer to a single loop $h(x, \cdot)$ whose integral is $f'(x)$. Summing over the N periods, F is roughly equal to a Riemann sum of f' and hence close to f. This can be formally stated as:

Lemma 2.2. *Given a curve f, differential relation $\mathcal{R}$ and positive integer N, define h and F as above. Then F is a solution of $\mathcal{R}$ and*

$$\|F - f\|_\infty \leq \frac{K}{N}.^{1}$$

In other words, given a curve f we can find F which is C^0 close to f and satisfying the relation $\mathcal{R}$.

3 Convex Integration over the torus

Convex Integration gives us a method to approximate curves. How do we generalize this to an embedding of a surface, and how does it help preserve

[1] For any function g, its C^0 norm is defined as $\|g\|_\infty := \sup_{p \in D} \|g(p)\|$, where D is the domain of g

metrics? For the first question, we treat our embedding as a family of parallel curves. Convex Integration guarantees that there is an approximate map F which is C^0 close to f. But by carefully choosing the curves representing the embedding, we can also ensure that the directional derivatives of the maps are close. This allows us to control the metric, making F an almost isometric embedding.

To understand the context of *almost* isometric, we define the isometric default:

Definition 3.1. *Let f be an embedding of Riemannian manifold (M, g) into $\mathbb{R}^n$. We call the difference between g and the pullback of the Euclidean metric the* isometric default $:= g - f^* \langle \cdot, \cdot \rangle_{\mathbb{R}^n}$.

In this section, we will focus on the simpler case where the isometric default is something called a primitive metric. We show that the convex integration process in this case yields an approximation whose isometric default can be made arbitrarily small.[2] The primitive metric depends on the parameter ρ, which is some positive function $\rho : \mathbb{T}^2 \to \mathbb{R}_+$. ρ appears in several of the bounds in this section.

3.1 The Naive Choice of Parallel Curves

We produce coordinates for $\mathbb{T}$ to construct our function F explicitly. Assume (for this section) that we can pick $V \in \ker \ell$ with co-prime integer coordinates. Then the path starting at an arbitrary origin O and going along V is a simple closed curve on $\mathbb{T}^2$. We can therefore cut along the path and obtain a cylinder Cyl. Take U to be orthogonal to V with $\|U\| \, \|V\| = 1$. Then the rectangle determined by the taking vectors U and V from the origin O in $\mathbb{R}^2$ is a fundamental domain of $\mathbb{T}^2$ under the action of $\mathbb{Z}^2$. Thus we can represent points on the cylinder Cyl on this rectangle as $O + tV + sU$ for $t \in \mathbb{R}/\mathbb{Z}$ and $s \in I$.

The most obvious method of producing F by convex integration might be to consider Cyl as a collection of curves $\varphi_t : I \to Cyl$ with $\varphi_t(s) = O + tV + sU$. To satisfy the isometry condition along the curve φ_t, we want to constrain the norm of the derivative along the curve:

$$\left\| \frac{\partial (F \circ \varphi_t)}{\partial s} \right\| = \sqrt{\mu(U, U)}$$

[2] In the general case, we shall show following Nash that the actual difference of the metrics is a sum of primitive metrics, and thus can be resolved by successive applications of the Convex Integration process.

This is the condition analogous to $\mathcal{R}_x$ from section 2, from which we can define an F by convex integration. This condition means that $F^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} (U, U) = \mu(U, U)$. Also, the fact that $V \in \ker \ell$ means that $F^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} (V, V) = \mu(V, V) + O(1/N)$. We have the metrics agreeing for two pairings, but the third pairing leads to issues. If we compute the pullback metric for the pair (U, V), it turns out to be fundamentally different from $\mu(U, V)$. Therefore convex integration along the curves φ_t does not product an almost isometry.

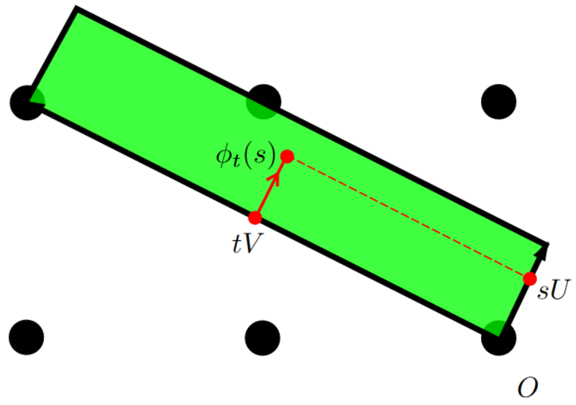

Figure 4: The naive curve φ_t does not yield an isometry if convex integrated. Figure 2.5 in [1].

3.2 Shifting U to produce directional derivative bounds

To get the actual isometry F, we replace U by $W = U + \zeta V$, with $\zeta = -\mu(U, V)/\mu(V, V)$. This is chosen so that $\mu(W, V) = 0$. We therefore replace φ_t with the integral curve $\varphi(t, \cdot) : I \to Cyl$, which starts at $O + tV$ at $s = 0$ and then follows W (thus $\partial\varphi/\partial s = W$). In U, V coordinates, we can write this as

$$\varphi(t, s) = O + sU + \psi(t, s)V$$

for some function $\psi : \mathbb{R}/\mathbb{Z} \times I \to \mathbb{R}$ with $\psi(t, 0) = t$.

Now, we apply the convex integration process to each curve $f \circ \varphi(t, \cdot)$. We now want to constrain the norm of the derivative to be $\sqrt{\mu(W, W)}$. This leads to defining a function $h(t, s, u)$ (analogous to $h(x, u)$ from section 2) such that

$$(W \cdot f)(\varphi(t, s)) = \frac{\partial(f \circ \varphi)}{\partial s}(t, s) = \int_0^1 h(t, s, u)du.$$

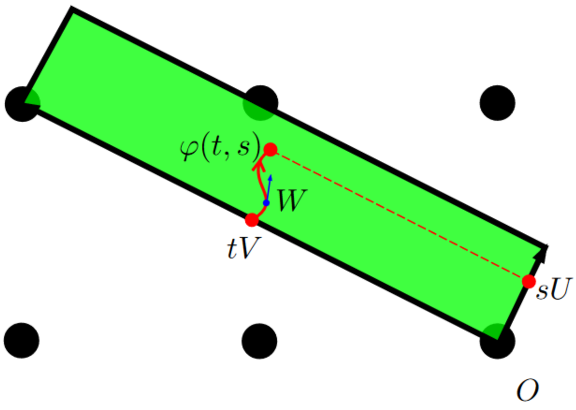

Figure 5: The curve φ following the nonconstant vector W yields the isometry through convex integration. Figure 2.6 in [1].

Then we obtain the map $F : Cyl \to \mathbb{R}^3$ satisfying

$$F \circ \varphi(t,s) := f(O + tV) + \int_0^s h(t, u, \{Nu\})du.$$

Applying Theorem 2.2, we see that F and f are C^0-close as $F \circ \varphi$ and $f \circ \varphi$ are close and φ is a diffeomorphism. In fact, we can get something stronger: the derivative of F with respect to t is also obtained from the corresponding derivative of f by a convex integration process. Thus we have

Lemma 3.2. *If F is produced by convex integration from f and h for a given N, we have*

$$\left\| \frac{\partial(F \circ \varphi)}{\partial t} - \frac{\partial(f \circ \varphi)}{\partial t} \right\|_\infty \leq \frac{K}{N}$$

where K depends only on the C^2 norm of h.

This essentially bounds $\|V \cdot F - V \cdot f\|$, as $\partial\varphi/\partial t = (\partial\psi/\partial t)V$ and thus

$$\|V \cdot F - V \cdot f\| = \left| \frac{\partial \psi}{\partial t} \right|^{-1} \left\| \frac{\partial(F \circ \varphi)}{\partial t} - \frac{\partial(f \circ \varphi)}{\partial t} \right\|.$$

Turns out, we can get a similar bound on $\|W \cdot F - W \cdot f\|$ in terms of U and the metric difference ρ (where $\mu = f^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} + \rho \ell \otimes \ell$):

Lemma 3.3.

$$\|W \cdot F - W \cdot f\| \leq \sqrt{7} \|U\| \|\rho\|_\infty^{1/2}.$$

Now, (U, V) is an orthogonal basis with respect to the Euclidean metric, which means

$$\|dF - df\| \leq \frac{\|U \cdot F - U \cdot f\|}{\|U\|} + \frac{\|V \cdot F - V \cdot f\|}{\|V\|}.$$

Since $W = U + \zeta V$ we can bound the first term with the inequality

$$\|U \cdot F - U \cdot f\| \leq \|W \cdot F - W \cdot f\| + |\zeta| \, \|V \cdot F - V \cdot f\| .$$

This shows that the differential maps are close:

Lemma 3.4.

$$\|dF - df\|_\infty \leq \sqrt{7} \, \|\rho\|_\infty^{1/2} + K/N.$$

We can also use Theorem 3.2 to bound the difference of the pullbacks of the metrics under φ. This shows that the metrics are close:

Lemma 3.5.

$$\|\mu - F^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3}\|_\infty \leq \frac{K}{N} \left\|d\varphi^{-1}\right\|_\infty^2 .$$

3.3 Extending the map from the cylinder to the torus

So far, we have constructed an almost isometric map F on the cylinder Cyl that is C^0-close to the map induced on Cyl by f. But F will not generally coincide on the two boundaries of Cyl, which means we cannot simply quotient it into a map on $\mathbb{T}^2$. To do that, first we make a modification to define a new map $\bar{F}$ such that

$$\bar{F} \circ \varphi(t, s) = F \circ \varphi(t, s) - w(s)(F \circ \varphi(t, 1) - f \circ \varphi(t, 1))$$

where $w : I \to I$ is a smooth S-shaped function. It turns out that if f and w are smooth, then $\bar{F}$ quotiented by the boundary ∂Cyl is a smooth map on $\mathbb{T}^2$. Using all the previous results, we can then show that $\bar{F}$ is the desired almost isometric map:

Theorem 3.6 (One Step Theorem)**.** *Let $f : \mathbb{T}^2 \to \mathbb{R}^3$ be a smooth embedding such that the pullback metric of f differs from the Euclidean metric by the primitive metric $\rho\ell \otimes \ell$. Define $\bar{F}$ as above. Then we have*

1. $\left\|\bar{F} - f\right\|_\infty \leq K_1/N$ *and* $\left\|\bar{F} - f\right\|_\infty \leq 2\sqrt{7} \, \|U\| \, \|\rho\|_\infty^{1/2}$,

2. $\left\|d\bar{F} - df\right\|_\infty \leq K_2/N + \sqrt{7}\left\|\rho\right\|_\infty^{1/2}$,
3. $\left\|V \cdot \bar{F} - V \cdot f\right\| \leq K_3/N$,
4. $\left\|W \cdot \bar{F} - W \cdot f\right\| \leq \sqrt{7}\left\|U\right\|(1 + \left\|w'\right\|_\infty)\left\|\rho\right\|_\infty^{1/2}$, *and*
5. $\left\|\mu - \bar{F}^* \langle\cdot,\cdot\rangle_{\mathbb{R}^3}\right\|_\infty \leq K_4/N$.

In the end, this theorem will represent each step of Convex Integration we apply. Thus it is called the One Step theorem, as it produces our desired approximations in one step.

4 Isometric embeddings of the square flat torus

The previous section we saw how to build an almost isometry when the isometric default $g - f^* \langle\cdot,\cdot\rangle_{\mathbb{R}^3}$ is a primitive metric $\rho\ell \otimes \ell$. This section covers the general case where this default is any metric.

For the case of the torus, it turns out that we can always decompose the isometric default of an embedding f into three components. These components are of the form $\ell_i \otimes \ell_i$, obtained from the following linear forms:

$$\ell_1 := dx, \quad \ell_2 := \frac{1}{\sqrt{5}}(dx + 2dy), \quad \ell_3 := \frac{1}{\sqrt{5}}(dx - 2dy).$$

Thus the isometric default lies in the open cone

$$\mathcal{C} := \{\rho_1\ell_1 \otimes \ell_1 + \rho_2\ell_2 \otimes \ell_2 + \rho_3\ell_3 \otimes \ell_3 | \rho_1, \rho_2, \rho_3 > 0\}$$

This might seem like an impressive fact, but a standard donut-shaped parametrization of the torus (with a suitable choice of minor radius and major radius) satisfies the condition of lying in this cone. But since the isometric default is uniformly captured by three primitive metrics, three successive convex integrations is enough to build an almost isometry which is close to f.

First, we define a crucial piece of notation. Note that the almost isometry $\bar{f}$ obtained from Theorem 3.6 depends only on the initial embedding f, the metric μ and the parameter N. Thus we notate this function as

$$IC(f, \mu, N) := \bar{f}.^3$$

[3]The term IC is from the original paper and probably stands for intégration convexe, which is convex integration in french.

For the final result, we shall apply the convex integration thrice, addressing a different component of the cone $\mathcal{C}$ at each step. First, suppose $D_1 = g - f^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3}$. Then set $\mu_1 := f^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} + \rho_1(D_1)\ell_1 \otimes \ell_1$, and $f_1 := IC(f, \mu_1, N_1)$. Then, for N_1 large enough, the new isometric default $D_2 := g - f_1^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ has ρ_1 component almost 0, and ρ_2, ρ_3 components unchanged.

Then we set $\mu_2 := f_1^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} + \rho_2(D_2)\ell_2 \otimes \ell_2$ and built the almost isometry $f_2 := IC(f_1, \mu_2, N_2)$. Then, for N_2 large enough, the isometric default $D_3 := g - f_2^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ has $\rho_1, \rho_2 \approx 0$ and $\rho_3(D_3) \approx \rho_3(D_2)$. Then finally we set $\mu_3 := f_2^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} + \rho_3(D_3)\ell_3 \otimes \ell_3$ and $f_3 := IC(f_2, \mu_3, N_3)$ which is an almost isometry for the metric g (for N_3 large enough). We denote this last immersion as

$$IC(f, g, N_1, N_2, N_3).$$

Now if N_1, N_2, N_3 all tend to infinity, the C^0 proximity property in the One Step Theorem (Theorem 3.6) suggests that the limit immersion is the initial immersion f, which is not an isometry. Also finite values of N_i only provide approximations of an isometry. Therefore we must repeat the process indefinitely to produce the desired isometry.

The key idea is to consider a sequence g_k of metrics that converges to the flat metric $dx \otimes dx + dy \otimes dy$. Then we define the recursive sequence $\mathcal{F}_k$ by

$$\mathcal{F}_k := IC(\mathcal{F}_{k-1}, g_k, N_{k,1}, N_{k,2}, N_{k,3}),$$

which should converge to an isometry. To iterate this process, we start with $\mathcal{F}_0 = f$ an initial embedding, and must ensure that the isometric default at each step $D_k := g_k - \mathcal{F}_{k-1}^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ lies in the cone $\mathcal{C}$. To ensure this, we establish the following result, which allows us to stage an almost isometric approximation with respect to a new metric $\bar{g}$ from one with respect to g:

Theorem 4.1 (Stage Theorem)**.** *Let g and $\bar{g}$ be two Riemannian metrics on $\mathbb{T}^2$ and let $f : \mathbb{T}^2 \to \mathbb{R}^3$ be an immersion such that*

1. $\bar{g} - g \in C^\infty(\mathbb{T}^2, \mathcal{C})$,[4]
2. $g - f^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} \in C^\infty(\mathbb{T}^2, \mathcal{C})$.

Then there exist integers N_1, N_2 and N_3 such that the immersion

$$\bar{f} := IC(f, g, N_1, N_2, N_3)$$

satisfies

[4]Note that $g \in C^\infty(\mathbb{T}^2, \mathcal{C})$ essentially means that g lies in the cone $\mathcal{C}$ at every point of $\mathbb{T}^2$.

1. $\bar{f}(0,0) = f(0,0)$,

2. $\bar{g} - \bar{f}^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} \in C^\infty(\mathbb{T}^2, \mathcal{C})$,

3. $\left\| g - \bar{f}^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} \right\|_\infty \leq \| g - \bar{g} \|_\infty$,

4. $\left\| d\bar{f} - df \right\|_\infty \leq 11 \left\| g - f^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} \right\|_\infty^{1/2}$.

The first two results ensure that this result can recursively build a sequence of immersions. The last two are used to prove that the limit is an isometry. The 11 in the final condition is an artifact of a constant chosen in the proof of the theorem. One can pick a bigger number for this bound, and obtain the result for smaller values of the N_i.

To actually build this sequence, we use the sequence g_k constructed by

$$\Delta := \langle \cdot, \cdot \rangle_{\mathbb{R}^2} - \mathcal{F}_0^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} ; \quad g_k := \mathcal{F}_0^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} + \delta_k \Delta$$

for an increasing sequence δ_k converging to 1. Here we are assuming $\mathcal{F}_0$ is our initial immersion, and that $\Delta \in C^\infty(\mathbb{T}^2, \mathcal{C})$. Then we recursively obtain $\mathcal{F}_k$ from $\mathcal{F}_{k-1}$ by applying the Stage Theorem (Theorem 4.1) with $g = g_{k-1}, \bar{g} = g_k$. Note that the first hypothesis condition of the theorem holds as $g_k - g_{k-1} = (\delta_k - \delta_{k-1})\Delta \in C^\infty(\mathbb{T}^2, \mathcal{C})$, and the second hypothesis condition holds inductively. Then, subject to one final condition, we can build an isometry:

Theorem 4.2. *Suppose the sequence δ_k satisfies $\sum \sqrt{\delta_k - \delta_{k-1}} < \infty$. Then teh sequence $\mathcal{F}_k$ is C^1-converging towards a C^1 isometric immersion $\mathcal{F}_\infty : \mathbb{T}^2 \to \mathbb{R}^3$.*

Proof. Using the last two results of the Stage Theorem, we obtain the bound

$$\| d\mathcal{F}_k - d\mathcal{F}_{k-1} \|_\infty \leq 22 \| g_k - g_{k-1} \|_\infty^{1/2} = 22 \sqrt{\delta_k - \delta_{k-1}} \, \| \Delta \|_\infty^{1/2} .$$

The series $\sum \sqrt{\delta_k - \delta_{k-1}}$ converges, and so the sequence $\mathcal{F}_k$ does C^1 converged to a C^1 map $\mathcal{F}_\infty$. Now, the third result of the stage theorem states that

$$\| g_k - \mathcal{F}_k^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} \|_\infty \leq \| g_{k+1} - g_k \|_\infty .$$

Taking the limit as $k \to \infty$, we obtain

$$\| \langle \cdot, \cdot \rangle_{\mathbb{R}^2} - \mathcal{F}_\infty^* \langle \cdot, \cdot \rangle_{\mathbb{R}^3} \|_\infty \leq 0$$

and thus $\mathcal{F}_\infty$ is a C^1 isometry. □ □

Therefore, for an appropriate choice of δ_k, we can build a sequence of metrics which converge to the flat metric, and apply the stage theorem recursively to apply several layers of convex integration to our initial immersion to obtain an isometry.

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