Convex Integration: Embedding the flat torus into $\mathbb{R}^3$

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THE SQUARE TORUS

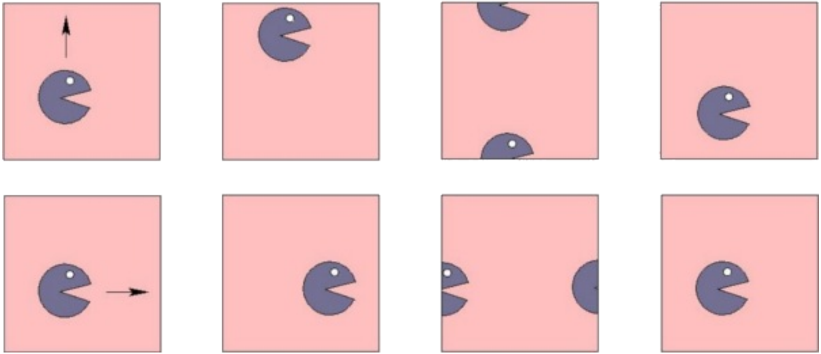

Figure: Cartoon of the square torus, from [5]

The Square Torus

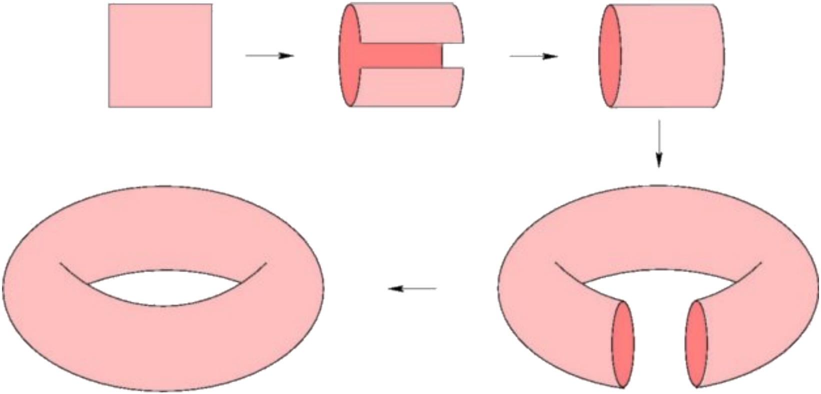

Figure: Embedding the torus into $\mathbb{R}^3$, from [5]

This embedding isn't isometric (has regions of positive curvature)

Nash-Kuiper Theorem

Theorem (Nash-Kuiper Theorem)

Suppose (M^m, g) is a Riemannian maniforld with short smooth embedding $f : M \to \mathbb{R}^n$. There is a sequence of C^1 isometric embeddings $M \to \mathbb{R}^n$ which converge uniformly to f.

Proof is nonconstructive. How do you visualize the C^1 embedding of the Torus?

Borrelli-Jabrane-Lazarus-Thibert's construction

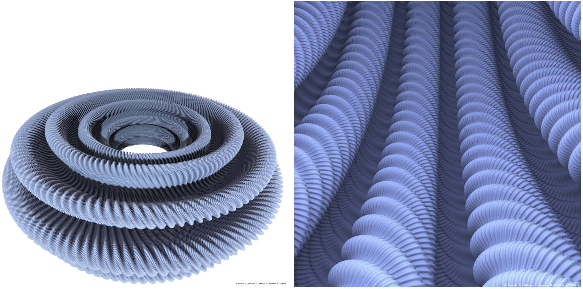

Figure: Images of the isometric embedding of the torus. Note the fractal-like nature when you zoom in. Figure from [5].

Differential Relations

Definition

Let $I = [0, 1]$. Consider a subset $\mathcal{R}_x \subset \mathbb{R}^n$ of vectors for all points $x \in I$. The union $\mathcal{R} := \cup_{x \in I} \mathcal{R}_x$ is called a *differential relation*. A curve $F : I \to \mathbb{R}^n$ is a *solution* of $\mathcal{R}$ if $F'(x) \in \mathcal{R}_x$ for all $x \in I$.

Problem: Find an approximation F to curve $f : I \to \mathbb{R}^n$ which is a solution of $\mathcal{R}$.

STEP 1: DEFINE LOOPS SATISFYING THE RELATION

For a given x, we take $h(x,u)$ to be a function $\mathbb{R}/\mathbb{Z} \to \mathcal{R}_x$, such that

$$f'(x) = \int_0^1 h(x,u)du.$$

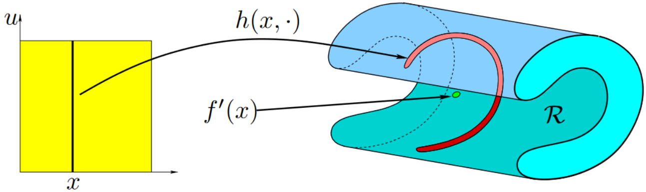

Figure: The loop $h(x,u)$ for a fixed value of x, encircling the point $f'(x)$. Figure 2.1 in [1].

Step 2: Wind around the loops

$$F(t) := f(0) + \int_0^t h(x, \{Nx\})dx$$

Intuitively, F is integrating h along a periodic curve with period $1/N$.

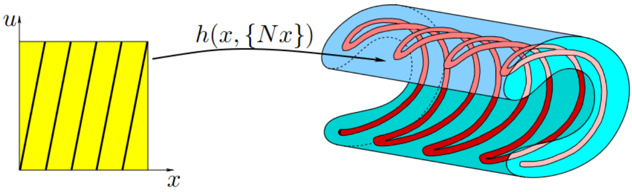

Figure: The path $x \mapsto (x, \{Nx\})$ winds N times around the cylinder and traces a path in $\mathcal{R}$ (pictured in blue). Figure 2.2 in [1].

Lemma

There exists constant K so that $\|F - f\|_\infty \leq \frac{K}{N}$.

2D Convex Integration: Approximating families of curves

Cut $\mathbb{T}^2$ along a vector V to obtain cylinder Cyl, which can be embedded into $\mathbb{R}^2$ as the rectangle $O + tV + sU$.

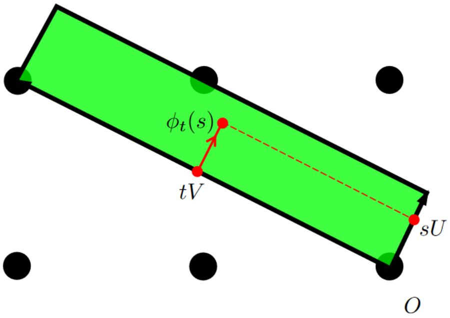

Figure: The naive curve ϕ_t does not yield an isometry if convex integrated. Figure 2.5 in [1].

Reducing the isometric default

Definition

Let f be an embedding of Riemannian manifold (M, g) into $\mathbb{R}^n$. We call the difference between g and the pullback of the Euclidean metric the *isometric default* $:= g - f^* \langle \cdot, \cdot \rangle_{\mathbb{R}^n}$.

Idea: To bound the isometric default, bound it's value for pairs (δ_s, δ_s), (δ_t, δ_t) and (δ_s, δ_t).

Torus Embedding Construction

- Simple case: Isometric default is primitive metric $\rho\ell \otimes \ell$.
  - Find approximation F on the cylinder.
  - Modify F to $\bar{F}$ which works on the torus (when glued along the edge)
- Turns out the natural embedding of the torus into $\mathbb{R}^3$ has isometric default that can be decomposed into three primitive metrics.
  - Applying convex integration thrice can approximate metric better.
  - Apply this stage to a sequence of metrics g_k on $\mathbb{T}$ whose limit is an isometry.

The curves for Torus Convex Integration

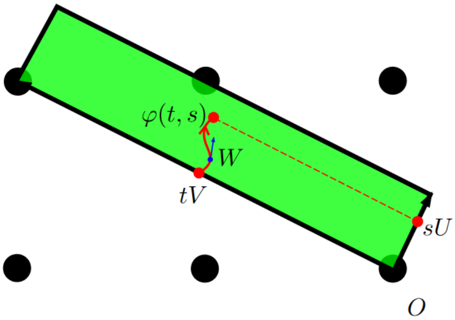

Figure: The curve ϕ following the nonconstant vector W yields the isometry through convex integration. Figure 2.6 in [1].

Replace U by $W = U + \zeta V$, so that $g(W, V) = 0$. Apply convex integration to each curve $f \circ \phi(t, \cdot)$, constraining norm of derivative to be $\sqrt{g(W, W)}$.

References

[1] Vincent Borrelli, Saıd Jabrane, Francis Lazarus, and Boris Thibert. "Isometric embeddings of the square flat torus in ambient space". In: *Ensaios Matemáticos* 24 (2013), pp. 1–91.

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[5] Hevea Project. "The folder : flat tori finally visualized !" In: (). URL: `http://hevea-project.fr/ENPageToreDossierDePresse.html`.