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Calculus of Variations

A constant rank theorem for special Lagrangian Equations

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Abstract
Constant rank theorems are obtained for saddle solutions to the special Lagrangian equation and the quadratic Hessian equation. The argument also leads to Liouville type results for the special Lagrangian equation with subcritical phase, matching the known rigidity results for semiconvex entire solutions to the quadratic Hessian equation.

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1 Introduction

In this note, we establish constant rank results for the special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i = \Theta, \tag{1.1}$$

and the quadratic Hessian equation on the positive branch

$$\sigma_2\left(D^2 u\right) = \sum_{1 \leq 1 < j \leq n} \lambda_i \lambda_j = 1 \text{ with } \lambda_1 + \cdots + \lambda_n > 0, \tag{1.2}$$

where $\lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of the Hessian $D^2 u$ and Θ is constant.

A constant rank theorem asserts that the minimum or maximum eigenvalue of the Hessian of a solution to a certain elliptic equation obeys the strong minimum or maximum principle. In other words, the Hessian shifted by the corresponding scalar matrix, $D^2 u - \lambda_{\min} I$ or $\lambda_{\max} I - D^2 u$, has constant rank. That certain condition for fully nonlinear elliptic equations is the inverse-convexity of Alvarez-Lasry-Lions in [1], employed to obtain the constant rank result for those various equations by Caffarelli-Guan-Ma in [5].

By connecting the convexity of the level sets of equation (1.1) at critical or supercritical phase $|\Theta| \geq (n-2)\pi/2$ [22], to the inverse-convexity of (1.1) at subcritical phase

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$|\Theta| < (n-2)\pi/2$, we obtain the following constant rank theorem for the special Lagrangian equation (1.1), which is new for saddle solutions.

Theorem 1.1 *If u is a smooth solution of (1.1) in a (connected) domain with* $\arctan\lambda_{\min} \geq (\Theta-\pi)/n$ *and $\lambda_{\min}$ attains a local minimum at an interior point, then $\lambda_{\min}$ is constant. Furthermore, if in addition the local minimum* $\arctan\lambda_{\min} > (\Theta-\pi)/n$, *then there is a fixed direction e such that the Hessian D^2u splits as $\lambda_{\min} \equiv u_{ee} \equiv const$. If $\lambda_{\max}$ attains a local interior maximum, the corresponding conclusions for $\lambda_{\max}$ also hold, provided* $\arctan\lambda_{\max} \leq (\Theta+\pi)/n$ *and* $\arctan\lambda_{\max} < (\Theta+\pi)/n$ *respectively.*

One application of Theorem 1.1 is the following rigidity result for homogeneous degree 2 solutions.

Corollary 1.1 *If u is a smooth degree 2 homogeneous solution of (1.1) on $\mathbb{R}^n \setminus \{0\}$, and* $\arctan\lambda_{\min} > (\Theta-\pi)/n$ *or* $\arctan\lambda_{\max} < (\Theta+\pi)/n$, *then u is a quadratic polynomial.*

As a more general by-product of the proof of Theorem 1.1, we find a Liouville type result for entire solutions of (1.1), which is new at subcritical and critical phases $|\Theta| \leq (n-2)\pi/2$ in dimension $n \geq 5$.

Theorem 1.2 *If u is an entire smooth solution of (1.1) on $\mathbb{R}^n$ with* $\arctan\lambda_{\min} \geq \gamma > (\Theta-\pi)/n$ *or* $\arctan\lambda_{\max} \leq \gamma < (\Theta+\pi)/n$, *then u is a quadratic polynomial.*

Relying on the same inverse-convexity used to derive the Liouville type result for almost convex entire solutions to (1.2) in [6], we have the following constant rank result for the quadratic Hessian equation (1.2).

Theorem 1.3 *If u is a smooth solution of (1.2) in a (connected) domain with $\lambda_{\min} \geq -\sqrt{2/[n(n-1)]}$ and $\lambda_{\min}$ attains a local minimum at interior point, then $\lambda_{\min}$ is constant. Furthermore, if in addition $\lambda_{\min} > -\sqrt{2/[n(n-1)]}$, then there is a fixed direction e such that the Hessian D^2u splits as $\lambda_{\min} \equiv u_{ee} \equiv const$.*

A sufficient condition for fully nonlinear elliptic equations to have a lower constant rank theorem is convexity. But (1.1) with convex solutions and the equivalent equation of (1.2), $\Lambda\left(D^2u\right) = \sqrt{\sigma_2\left(D^2u\right)} = 1$, are concave, not convex. A relaxed convexity, as alluded to earlier, the inverse-convexity of a fully nonlinear elliptic equation $G\left(D^2u\right) = c$, that is, convexity of $G\left(A^{-1}\right)$ in terms of A, permits a strong minimum principle for the minimum eigenvalue of the Hessian of solutions. In terms of the Legendre transform $u^* = w(y)$ of strongly convex solution $u(x)$ to $G\left(D^2u\right) = c$, as

$$(x, Du(x)) = (Dw(y), y) \text{ and then } D^2u = \left(D^2w\right)^{-1},$$

the equation $G\left(D^2u\right) = c$ becomes a concave fully nonlinear elliptic equation

$$\Lambda\left(D^2w\right) = -G\left(\left(D^2w\right)^{-1}\right) = -c.$$

Now given $\lambda_{\min}\left(D^2u\right)$ attains its local minimum, we have $\lambda_{\max}\left(D^2w\right) = \lambda_{\min}^{-1}\left(D^2u\right)$ reaching its local maximum, and say, $\lambda_{\max}\left(D^2w\right) = w_{11}$ at the local maximum point. By the strong maximum principle, subsolution $w_{11} \equiv l \equiv \lambda_{\max}\left(D^2w\right)$. The gradient graphs split off a line

$$\left(x_1, x', u_1(x), D'u(x)\right) = \left(ly_1, D'w(y'), y_1, y'\right).$$

Accordingly, $u_1(x) = l^{-1}x_1$, hence $u_{11} \equiv \lambda_{\min}\left(D^2u\right)$ is constant.

The subtlety lies in the case $\lambda_{\min}\left(D^2u\right) = 0$, where the Legendre transformation is not possible. The strong minimum principle for $\lambda_{\min}$ was achieved in [5] by realizing its superharmonicity in a hidden smooth form of $\triangle_G\lambda_{\min} \le B(x) \cdot D\lambda_{\min}$. Moreover, the Hessian D^2u may no longer split, that is, the the minimum-eigendirection may not be fixed, as counterexamples by Korevaar-Lewis in [7, p.31] and also by Székelyhidi-Weinkove in [15, p.6530] indicate. A quantitative version of the constant rank theorem was discovered in [16].

The lower constant rank results for convex solutions in Theorem 1.1 and Theorem 1.3 are contained in [5, p.1772], as the two corresponding equations are inverse-convex then. The inverse convexity was also employed in deriving the upper, or equivalently lower, constant rank result for solutions of (1.1) with the eigenvalues of the Hessian between -1 and 1 by Bhattacharya-Shankar in [4, p.18]. The connection to the inverse-convexity of (1.1) with saddle solutions in Theorem 1.1 and [4] is via the rotation developed in [21], once recognized as the Legendre transformation up to translation and scaling.

Note that the lower bound in Theorem 1.1 with $\Theta = \pi/2$ and $n = 3$ is exactly the one $\lambda_{\min} > \tan(-\pi/6) = -1/\sqrt{3} = -m$ in Theorem 1.3 with $n = 3$. This is because here the algebraic form of (1.1) is (1.2). In this instance, the new rotated equation $\sum_{i=1}^n \arctan\bar{\lambda}_i = -\pi/2$ is equivalent to $\sigma_2\left(D^2\bar{u}\right) = 1$ on the negative branch $\bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_3 < 0$. In terms of $\mu_i = (\lambda_i + m)^{-1}$, equation $\sigma_2\left(\bar{\lambda}\right) = 1$ becomes $\Lambda(\mu) = \sigma_2(\mu)/\sigma_1(\mu) = 1/(2m)$, which is concave. In other words, $D^2u + mI$ satisfies an inverse-convex equation, with abused notation, $-\Lambda\left(\left(D^2u + mI\right)^{-1}\right) = -1/(2m)$. This particular three dimensional case inspired the Legendre-Lewy transformation, $(u+m|x|^2/2)^*$ with $m = \sqrt{2/[n(n-1)]}$, used to derive the Liouville type result in [6], and the constant rank result Theorem 1.3 for almost convex solutions of (1.2) with $D^2u > -mI$ in general dimensions. The rigidity result for general semiconvex entire solutions to (1.2) needed more effort in [17].

The above rotation used in proving Theorem 1.1 turns an entire solution to (1.1) with the specific lower or upper Hessian bound into a new entire one with bounded Hessian on a concave/convex level set, Evans-Krylov's theorem then leads to the rigidity result Theorem 1.2. Coupled with the quadratic rigidity of 2-homogeneous analytic solutions to a uniformly elliptic Hessian equation in $\mathbb{R}^4 \setminus \{0\}$ in [14], the argument in [21, p.124–125] also shows the Liouville type result for entire solutions to (1.1) with arbitrary lower or upper Hessian bound now in four dimensions. A Bernstein type result, namely a quadratic rigidity for entire solutions to (1.1) with supercritical phase $|\Theta| > (n-2)\pi/2$ was derived in [22]. Precious entire solutions $W(x) = \left(x_1^2 + x_2^2 - 1\right)e^{x_3} + e^{-x_3}/4$ and $L(x) = x_1^2x_2 - \frac{2}{3}x_2^3 - x_2x_3$ to (1.1) with critical $\Theta = \pi/2$ or $\sigma_2\left(D^2W\right) = 1$ and (1.1) with subcritical $\Theta = 0$ or $\triangle L = \det D^2L$ were constructed by Warren in [18] and by C.-Y. Li in [9], respectively.

Special Lagrangian equation (1.1) is the potential equation for minimal gradient graph or special Lagrangian graph $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$. While there are singular $C^{1,\alpha}$ or Lipschitz viscosity solutions to (1.1) with subcritical phase [11, 13, 19], regularity for $C^{1,1}$ solutions in five dimensions and higher is an open question, which is equivalent to the quadratic rigidity of 2-homogeneous solutions to (1.1) with subcritical phase. Nontrivial 2-homogeneous solutions were found to a saddle uniformly elliptic Hessian equation in five dimensions [12]. Should a nontrivial 2-homogeneous solution exist to (1.1) with subcritical phase, say for example $\Theta = 0$, the lower bound for the Hessian must be less than or equal to $-\tan(\pi/5)$. Otherwise, a splitting of the Hessian by Corollary 1.1 implies the triviality of the supposed example. It would be marvelous that the lower bound of the Hessian is exactly the $-\tan(\pi/5)$ in Theorem 1.1.

In the appendix of this paper, we present another proof of a general constant rank theorem for inverse-convex elliptic Hessian equations, in viscosity way, besides the smooth [2, 5] and approximation [15, 16] ones. The argument is to demonstrate the next smallest nonidentically vanishing eigenvalue is a supersolution in viscosity sense. In particular, the demonstration of the constancy of the smallest eigenvalue is simple. After our paper was finished, we found a similar viscosity approach for the constant rank theorem had appeared in the work of Bryan-Ivaki-Scheuer [3].

2 Proof of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1 We show the assertions for $\lambda_{\min}$. These assertions applied to the solution $v=-u$ of the equation $\sum_{i=1}^n \arctan\lambda_i\left(D^2v\right)=-\Theta$ yield the corresponding conclusions for $\lambda_{\max}\left(D^2u\right)$. We consider three cases: $\Theta\leq(2-n)\pi/2$, $\Theta\in((2-n)\pi/2,\pi]$, and $\Theta\in(\pi,n\pi/2)$.

Case $\Theta\leq(2-n)\pi/2$: By [22, Lemma 2.1], the level set of equation $F\left(D^2u\right)=\sum_{i=1}^n\arctan\lambda_i=\Theta$ is concave as a graph along the gradient $\nabla_\lambda\sum_{i=1}^n\arctan\lambda_i$, then $\triangle_F u_{ee}=F_{ij}\partial_{ij}u_{ee}\leq 0$. Therefore, a strong minimum principle holds for u_{ee}. If λ_1 reaches its local minimum ($>-\infty$), then after change of x-coordinates if necessary, u_{11} reaches a local minimum of the same value of λ_1, at the same point, so $u_{11}\equiv\lambda_1$ is constant.

Case $\Theta\in(\pi,n\pi/2)$: The hypothesis $\arctan\lambda_1\geq\frac{\Theta-\pi}{n}$ implies $\lambda_n\geq\cdots\geq\lambda_1>0$, so u is strongly convex. Let u^* be the Legendre transform of u. The eigenvalues μ_i of $D^2u^*=\left(D^2u\right)^{-1}$ satisfy $\mu_i=\lambda_i^{-1}>0$ and (1.1) becomes

$$\sum_{i=1}^n\arctan\mu_i=n\frac{\pi}{2}-\Theta.$$

Since $\mu_i>0$, this elliptic equation for u^* is concave, so we have a strong maximum principle for pure second derivatives of u^*. In turn, this also yields a strong minimum principle for pure second derivatives of u. The conclusion of Theorem 1.1 follows.

Case $\Theta\in((2-n)\pi/2,\pi]$: Let $\alpha=(\Theta-\pi)/n\in(-\pi/2,0]$ and $\beta=\frac{\pi}{2}+\alpha\in(0,\pi/2]$. First we assume that the attained local minimum is strictly larger than the assumed lower bound, that is

$$\frac{\pi}{2}>\theta_n\geq\cdots\geq\theta_1>\alpha.$$

where $\theta_i=\arctan\lambda_i$. As in [21, p.124–125], we are able to represent the Lagrangian graph $(x,y)=(x,Du(x))\subset\mathbb{R}^n\times\mathbb{R}^n$ with new anti-clockwise β-rotated coordinate axes $\partial_{\bar{x}}+\sqrt{-1}\partial_{\bar{y}}=\left(c+\sqrt{-1}s\right)\left(\partial_x+i\partial_y\right)$, where $c=\cos\beta$ and $s=\sin\beta$,

$$(cx+sDu(x),-sx+cDu(x))=(\bar{x},D\bar{u}(\bar{x}))\tag{2.1}$$

such that

$$\bar{\theta}_i=\theta_i-\beta=\theta_i-\frac{\pi}{2}-\alpha\in(-\pi/2,\pi/2)$$

and

$$\bar{F}\left(D^2\bar{u}\right)=\sum_{i=1}^n\arctan\bar{\lambda}_i=(2-n)\frac{\pi}{2},\tag{2.2}$$

where $\bar{\theta}_i=\arctan\bar{\lambda}_i$ and $\bar{\lambda}_i's$ are the eigenvalues of the Hessian $D^2\bar{u}$ of the new potential $\bar{u}$.

Again by [22, Lemma 2.1], the level set of the equation (2.2) is concave as a graph along the gradient $\nabla_{\bar\lambda}\sum_{i=1}^{n}\arctan\bar\lambda_i$, then $\triangle_{\bar F}\bar u_{\bar e\bar e}=\bar F_{ij}\partial_{ij}\bar u_{\bar e\bar e}\le 0$. Thus a strong minimum principle holds for $\bar u_{\bar e\bar e}$.

Now $\lambda_{\min}=\lambda_1$ reaches its local minimum, which is strictly larger than $\tan(\alpha-\beta)=-\infty$. Correspondingly, $\bar\theta_{\min}=\bar\theta_1=\theta_1-\beta$ reaches its local minimum. By the strong minimum principle, $\bar u_{\bar 1\bar 1}\equiv\tan\bar\delta\equiv\bar\lambda_1$, under change of $\bar x$-coordinates if necessary. Then the new Lagrangian graph (2.1) becomes

$$\begin{aligned}&\left(cx_1+su_1(x),cx'+sD'u(x),-sx_1+cu_1(x),-sx'+cD'u(x)\right)\\&=\left(\bar x_1,\bar x',\tan\bar\delta\,\bar x_1,D'\bar u\left(\bar x'\right)\right)\end{aligned}$$

Accordingly, $u_1(x)=\tan\left(\bar\delta+\beta\right)x_1$, hence $u_{11}\equiv\lambda_1$ is constant.

Remark. The Legendre transform u^* of strongly convex u in case $\Theta\in(\pi,n\pi/2)$ is obtained by performing the rotation (2.1) with $\beta=\pi/2$ followed by a reflection. In fact, $\bar u=-u^*$ satisfies $\sum_{i=1}^{n}\arctan\bar\lambda_i\left(D^2\bar u\right)=\Theta-n\pi/2$ with $\bar\lambda_i\left(D^2\bar u\right)<0$. This equation's level set is concave on the part with all $\bar\lambda_i(D^2\bar u)<0$ as a graph along the gradient $\nabla_{\bar\lambda}\sum_{i=1}^{n}\arctan\bar\lambda_i$. The above argument for case $\Theta\in((2-n)\pi/2,\pi]$ also leads to the conclusions in the case $\Theta\in(\pi,n\pi/2)$.

Next we consider the borderline situation, the attained local minimum $\arctan\lambda_1=\alpha=(\Theta-\pi)/n$. The previous new representation by rotation is no longer valid, because $\bar\theta_1=-\pi/2$, or equivalently $\bar\lambda_1=-\infty$. Whenever this rotation is possible, that is, when $\arctan\lambda_1>\alpha$, we have

$$\begin{aligned}&\bar\lambda_i=\tan(\theta_i-\beta)=-a-\frac{1+a^2}{\lambda_i-a}\quad\text{for }\tan\beta=-\tan^{-1}\alpha=-a^{-1}\\&\text{or}\quad D^2\bar u=-aI-\left(1+a^2\right)\left(D^2u-aI\right)^{-1}.\end{aligned}$$

By the above mentioned concavity of the level set of $\bar F\left(D^2\bar u\right)=(2-n)\pi/2$, which is preserved under translation and scaling, the level set of

$$\tilde F(A)\overset{\text{def}}{=}\bar F\left(-aI-\left(1+a^2\right)A\right)=(2-n)\frac{\pi}{2}$$

is also concave as a graph along $\nabla_A\tilde F$. Hence the condition for constant rank in [CGM, (2.2)] with $A=D^2u-aI$ and $X=D^2u_e$ holds. Effectively, the condition (4.2) for constant rank with $F(A)$ and $u(x)$ replaced by $\tilde F\left(A^{-1}\right)$ and $u(x)-a|x|^2/2$ respectively holds. Therefore, by [5, Theorem 1.1] or Theorem 4.1, D^2u-aI has constant rank, and in turn, $\lambda_{\min}=\lambda_1\equiv\tan\alpha=a$. □

The Hessian splitting conclusion of Theorem 1.1 is the main tool used to prove Corollary 1.1. The idea is to split off a quadratic function in one variable from u and then observe that the remaining part of the solution also satisfies the hypotheses of Theorem 1.1.

Proof of Corollary 1.1 We prove the quadratic rigidity under the assumption $\arctan\lambda_{\min}>(\Theta-\pi)/n$. This result applied to the solution $v=-u$ of the equation $\sum_{i=1}^{n}\arctan\lambda_i(D^2v)=-\Theta$ yields the same rigidity when $\arctan\lambda_{\max}(D^2u)<(\Theta+\pi)/n$.

Since u is homogeneous of degree 2, D^2u is homogeneous of degree 0, so $\arctan\lambda_{\min}$ achieves its minimum value, $\arctan\lambda_{\min}\ge\arctan m_1>(\Theta-\pi)/n$. By Theorem 1.1, $\lambda_{\min}$ is constant and the Hessian D^2u splits such that, after a change of coordinates if necessary, $u_{11}\equiv m_1$. Then,

$$u=\frac{1}{2}m_1x_1^2+\tilde u(x_2,\dots,x_n).$$

The function $\tilde{u}$ satisfies the equation

$$\sum_{i=1}^{n-1} \arctan \lambda_i (D^2\tilde{u}) = \Theta - \arctan m_1, \tag{2.3}$$

and $\arctan \lambda_i (D^2\tilde{u}) \geq \arctan m_1 > (\Theta - \pi)/n$ for all $1 \leq i \leq n-1$.

By Theorem 1.1 in dimension $n-1$, if $\arctan \lambda_{\min}(D^2\tilde{u}) > (\tilde{\Theta} - \pi)/(n-1)$, with $\tilde{\Theta} = \Theta - \arctan m_1$, then $\lambda_{\min}(D^2\tilde{u})$ is constant and the Hessian $D^2\tilde{u}$ splits. Indeed, by the assumption $\arctan m_1 > (\Theta - \pi)/n$, or equivalently

$$\frac{\Theta - \pi}{n} > \frac{\Theta - \arctan m_1 - \pi}{n-1},$$

we have

$$\lambda_{\min}(D^2\tilde{u}) \geq \arctan m_1 > \frac{\Theta - \pi}{n} > \frac{\tilde{\Theta} - \pi}{n-1}.$$

Inductively, we conclude that u is a quadratic polynomial. □

Remark. The assumption that u is homogeneous degree 2 is used to ensure that every λ_i attains an interior local minimum. In fact, if u is a smooth solution of (1.1) with $\arctan \lambda_1 \geq (\Theta - \pi)/n$, and every λ_i for $1 \leq i \leq k$, attains an interior local minimum, then λ_i is constant for $1 \leq i \leq k$. If all the local minima $\arctan \lambda_i > (\Theta - \pi)/n$, then the Hessian splits as $\lambda_i \equiv u_{ii} \equiv const.$ for $1 \leq i \leq k$.

We now proceed with the proof of Theorem 1.2, which is an application of the Evans-Krylov theorem to an equation obtained by a rotation of the form (2.1).

Proof of Theorem 1.2 We show the quadratic rigidity under the condition $\arctan \lambda_{\min} \geq \gamma > (\Theta - \pi)/n$. The same rigidity for the solution $v = -u$ to the equation $\sum \arctan \lambda_i \left(D^2 v\right) = -\Theta$ under the condition $\arctan \lambda_{\min}\left(D^2 v\right) \geq -\gamma > (-\Theta - \pi)/n$ or equivalently $\arctan \lambda_{\max}\left(D^2 u\right) \leq \gamma < (\Theta + \pi)/n$ yields the corresponding conclusion under the $\lambda_{\max}$ condition.

The result is known if $\Theta \geq \pi$ as then $\lambda_{\min} \geq 0$ [21, Theorem 1.1], or if $\Theta \geq \pi - n\pi/6 - \varepsilon'(n)$ as then $\lambda_{\min} \geq -1/\sqrt{3} - \varepsilon(n)$ [20, p.924] [10, p.22], or if $\Theta < (2-n)\pi/2$ [22, Theorem 1.1]. We consider more than the remaining range, $\Theta \in [(2-n)\pi/2, \pi)$, where our unified argument is in the spirit of [22].

First, each Lagrangian angle $\theta_i = \arctan \lambda_i$ satisfies

$$\frac{\pi}{2} > \theta_i \geq \gamma > \frac{\Theta - \pi}{n} = \alpha \in \left[-\frac{\pi}{2}, 0\right),$$

where we may and do assume $\gamma \leq 0$. In order to include the negative critical phase, set $\delta = (\gamma - \alpha)/2 > 0$. Then the new representation (2.1) with β replaced by $\tilde{\beta} = \pi/2 + \alpha + \delta \in (\delta, \pi/2)$ is valid, at least locally, such that

$$\bar{\theta}_i = \theta_i - \frac{\pi}{2} - \alpha - \delta \in \left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right)$$

and

$$\bar{F}\left(D^2\bar{u}\right) = \sum_{i=1}^{n} \arctan \bar{\lambda}_i = (2-n)\frac{\pi}{2} - n\delta. \tag{2.4}$$

The new representation (2.1) of the original entire special Lagrangian graph $(x, Du(x))$ is in fact valid for all $\bar{x} \in \mathbb{R}^n$. This is because of the following distance expansion property

$$\begin{aligned}
\left|\bar{x}^2 - \bar{x}^1\right| &= \tilde{c}^2 \left|\left(x^2 - x^1\right) + \tilde{t}\left[Du\left(x^2\right) - Du\left(x^1\right)\right]\right|^2 \\
&= \tilde{c}^2 \left|\left(1 - \tilde{t}m\right)\left(x^2 - x^1\right) + \tilde{t}\left[Du\left(x^2\right) + mx^2 - Du\left(x^1\right) - mx^1\right]\right|^2 \\
&\geq \tilde{c}^2 \left|\left(1 - \tilde{t}m\right)\left(x^2 - x^1\right)\right|^2 + \tilde{t}^2 \left|Du\left(x^2\right) + mx^2 - Du\left(x^1\right) - mx^1\right|^2 \\
&\geq \tilde{c}^2 \left|\left(1 - \frac{\tan|\gamma|}{\tan|\gamma+\delta|}\right)\left(x^2 - x^1\right)\right|^2,
\end{aligned}$$

where $(\tilde{c}, \tilde{s}) = \left(\cos\left(\frac{\pi}{2} + \alpha + \delta\right), \sin\left(\frac{\pi}{2} + \alpha + \delta\right)\right)$, $\tilde{t} = \tilde{s}/\tilde{c}$, $m = \tan|\gamma|$, and convexity of $u(x) + m|x|^2/2$ is used in the first inequality.

Finally, we have an entire solution $\bar{u}$ with bounded Hessian on the concave level set [22, Lemma 2.1] of the now uniformly elliptic equation (2.4). Scaling Evans-Krylov's Hölder seminorm estimate for the Hessian $D^2\bar{u}$ and then taking limit yields the constancy of $D^2\bar{u}$. Thus $(\bar{x}, D\bar{u}(\bar{x}))$, or equivalently $(x, Du(x))$, is a plane, and in turn, the original potential u is a quadratic polynomial. $\square$

3 Proof of Theorem 1.3

Proof of Theorem 1.3 First we assume the attained local minimum is strictly larger than the assumed lower bound, $\lambda_{\min} > -\sqrt{2/[n(n-1)]} \overset{\text{def}}{=} m$. Let $w(y)$ be the Legendre-Lewy transform, $(u(x) + m|x|^2/2)^*$. In terms of gradient graphs in $\mathbb{R}^n \times \mathbb{R}^n$,

$$(x, Du(x) + mx) = (Dw(y), y). \tag{3.1}$$

A tangent plane of $(x, Du(x) + mx)$ is formed by tangent vectors $\left(e, \left(D^2u(x) + mI\right)e\right)$ with $e \in \mathbb{R}^n$; correspondingly, tangent vectors $\left(\left(D^2u(x) + mI\right)^{-1} e, e\right)$ form the tangent plane of $(Dw(y), y)$ at $y = Du(x) + mx$. Thus

$$D^2w(y) = \left(D^2u(x) + mI\right)^{-1},$$

and in terms of eigenvalues,

$$\mu_i = (\lambda_i + m)^{-1} > 0.$$

It follows that μ satisfies

$$1 = \sum_{1 \leq i < j \leq n} (\mu_i^{-1} - m)(\mu_j^{-1} - m) = \frac{\sigma_{n-2}(\mu)}{\sigma_n(\mu)} - (n-1)m\frac{\sigma_{n-1}(\mu)}{\sigma_n(\mu)} + 1$$

or

$$\Lambda\left(D^2w\right) = \Lambda(\mu) \overset{\text{def}}{=} \frac{\sigma_{n-1}(\mu)}{\sigma_{n-2}(\mu)} = \frac{1}{(n-1)m}, \tag{3.2}$$

where $m = \sqrt{2/[n(n-1)]}$ is used. The function $\Lambda(\mu)$ is concave and $\nabla_\mu \Lambda(\mu) > 0$ componentwise [L, Theorem 15.18], therefore the level set $\Lambda(\mu) = 1/[(n-1)m]$ is convex as a graph along $\nabla_\mu \Lambda$. Hence

$$\triangle_\Lambda w_{ee} = \Lambda_{ij}\partial_{ij} w_{ee} \geq 0.$$

Consequently, a strong maximum principle holds for w_{ee}.

Now $\mu_{\max} = (\lambda_{\min} + m)^{-1}$ reaches its local maximum. By the strong maximum principle, $w_{11} \equiv l \equiv \mu_{\max}$, under change of y-coordinates if necessary. Then the gradient graph (3.1) reads

$$\left(x_1, x', u_1(x) + mx_1, D'u(x) + mx'\right) = \left(ly_1, D'w(y'), y_1, y'\right).$$

Accordingly, $u_1(x) = \left(l^{-1} - m\right) x_1$, hence $u_{11} \equiv \lambda_{\min}$ is constant.

Next we consider the borderline situation, the attained local minimum of $\lambda_{\min} = m$. The previous Legendre-Lewy transformation is no longer valid. Whenever the transformation is possible, that is, when $\lambda_{\min}(D^2u) > m$, we have (3.2). By the above proved convexity of the level set of $\Lambda(D^2w) = 1/\left[(n-1)\, m\right]$ as a graph along $\nabla_{D^2w}\Lambda$, the level set of

$$\tilde{F}(D^2u) = -\Lambda\left(\left(D^2u + mI\right)^{-1}\right) = -1/\left[(n-1)\, m\right]$$

is concave as a graph along $\nabla_{D^2u}\tilde{F}\left(D^2u\right)$. As in the proof of Theorem 1.1, the condition for constant rank in [5, (2.2)] with $A = D^2u + mI$ and $X = D^2u_e$ holds. Effectively, the condition (4.2) for constant rank with $F(A)$ and $u(x)$ replaced by $\tilde{F}\left(A^{-1}\right)$ and $u(x) + m\,|x|^2/2$ respectively holds. Therefore, by [5, Theorem 1.1] or Theorem 4.1, $D^2u + mI$ has constant rank, and in turn, $\lambda_{\min} = \lambda_1$ is constant. $\square$

4 Appendix

In this section we present a viscosity approach to prove the following constant rank theorem for nonlinear elliptic Hessian equations. As mentioned in the introduction, a similar viscosity proof appeared in [3]. For completeness, we include our proof here. For convenience, we state and prove the theorem under the assumption that $F(D^2u) = f(\lambda_1, \ldots, \lambda_n)$ where f is a symmetric function of its arguments, but the proof can be adapted to accommodate a general Hessian equation.

Theorem 4.1 *Assume $u \in C^3$ is a convex solution of the elliptic equation $F(D^2u) = 0$ where $F(M^{-1})$ is a convex function on $\mathrm{Sym}^+(n)$. Then D^2u has constant rank.*

One of the obstacles in proving a constant rank theorem is that the eigenvalues of the Hessian matrix need not be differentiable. Previous works deal with this issue by applying strong minimum principle arguments to smooth quantities [2, 5], or by approximations coupled with weak Harnack inequalities [15, 16]. The current approach is to compute directly with the eigenvalues of the Hessian in order to apply the strong minimum principle in the viscosity sense. In particular, this approach gives a simple argument for establishing the constancy of the smallest eigenvalue.

We begin with a lemma establishing a formula for 1-sided derivatives of eigenvalues at matrices in $\mathrm{Sym}(n)$ with repeated eigenvalues.

Lemma 4.1 *Assume $M = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \leq \cdots \leq \lambda_n$. Furthermore, assume $\lambda_{j-1} < \lambda_j = \cdots = \lambda_k < \lambda_{k+1}$. Then, if $j \leq i \leq k$ and $A \in \mathrm{Sym}(n)$, then*

$$\left.\frac{d^+}{dt}\right|_{t=0} \lambda_i(M + tA) = \lim_{t \to 0^+} \frac{\lambda_i(M + tA) - \lambda_i}{t}$$

exists and

$$\left.\frac{d^+}{dt}\right|_{t=0} \lambda_i(M + tA) = \lambda_{i-j+1}(A|_E),$$

where $E = \text{span}(e_j, \ldots, e_k)$ *is the* λ_i*-eigenspace of* M *and* $A|_E$ *is the* $(k-j+1)\times(k-j+1)$ *matrix with entries* $(A_{rs})_{r,s=j}^k$.

Proof We may choose an orthogonal basis for E to diagonalize $A|_E$. Once $A|_E$ is diagonal, it is clear that for small nonzero t, the eigenvalues of the perturbed matrix $(M+tA)|_E$ satisfy

$$\left.\frac{d^+}{dt}\right|_{t=0} \lambda_{i-j+1}((M+tA)|_E) = \lambda_{i-j+1}(A|_E).$$

The entries of tA outside of the block $tA|_E$ make contributions on the order of t^2 to $\lambda_i(M + tA)$, so this proves the lemma. □

A consequence of Lemma 4.1 is that if M has repeated eigenvalues $\lambda_j = \cdots = \lambda_k$, then λ_i $(j \le i \le k)$ is differentiable in the direction A at M if and only if $\lambda_{i-j+1}(A|_E) = \lambda_{k-i+1}(A|_E)$. To see this, note that if $\frac{d}{dt}|_{t=0}\lambda_i(M+tA)$ exists, then

$$\begin{aligned}\left.\frac{d^+}{dt}\right|_{t=0} \lambda_i(M+tA) &= \left.\frac{d^-}{dt}\right|_{t=0} \lambda_i(M+tA)\\ &= \lim_{t\to 0^-} \frac{\lambda_i(M+tA)-\lambda_i}{t}\\ &= -\lim_{t\to 0^+} \frac{\lambda_i(M-tA)-\lambda_i}{t}\\ &= -\left.\frac{d^+}{dt}\right|_{t=0} \lambda_i(M-tA)\\ &= -\lambda_{i-j+1}(-A|_E).\end{aligned}$$

Note that $-\lambda_{i-j+1}(-A|_E) = \lambda_{k-i+1}(A|_E)$.

In particular, λ_j is differentiable in the direction A if and only if $A|_E$ is a scalar matrix.

Proof of Theorem 4.1 Recall that λ_j is a Lipschitz function, C^1 on the set $\{p : \lambda_{j-1} < \lambda_j < \lambda_{j+1}\}$. We assume that $0 = \lambda_1(0) = \cdots = \lambda_{n-r}(0)$, so that the rank of D^2u reaches its minimum at 0. All computations will be performed at a fixed point p, assuming that $D^2u(p)$ is diagonal with $\lambda_j(p) = u_{jj}(p)$. Note that the assumption that F is a symmetric function of the eigenvalues implies that $F_{ij}(D^2u(p)) = f_{\lambda_i}\delta_{ij}$. For a positive integer $t \le n$, let

$$I_t = \{(i, j, k, l) \,:\, 1 \le i, j, k, l \le n, \text{ at least one of } i, j, k, l \le t\}.$$

Step 1: Superharmonicity of λ_1.

Assume first that $\lambda_1(p) < \lambda_2(p)$. At p, we have

$$\begin{aligned}\partial_i\lambda_1 &= \partial_i u_{11},\\ \partial_{ii}\lambda_1 &= \partial_{ii}u_{11} + \sum_{j>1}\frac{2}{\lambda_1-\lambda_j}(\partial_i u_{1j})^2.\end{aligned} \tag{4.1}$$

Differentiating the equation $F(D^2u) = 0$, we obtain

$$0 = \partial_{11}F(D^2u) = \sum_{i,j,k,l} F_{ij,kl}u_{ij1}u_{kl1} + \sum_{i,j} F_{ij}u_{ij11},$$

so

$$\begin{aligned}\triangle_F u_{11} &= -\sum_{i,j,k,l} F_{ij,kl} u_{ij1} u_{kl1} \\ &= -\sum_{i,j,k,l>1} F_{ij,kl} u_{ij1} u_{kl1} - \sum_{(i,j,k,l)\in I_1} F_{ij,kl} u_{ij1} u_{kl1}.\end{aligned}$$

where $\triangle_F = \sum_{i,j} F_{ij}\partial_{ij}$.

By the inverse-convexity we have

$$-\sum_{i,j,k,l>1} F_{ij,kl} u_{ij1} u_{kl1} \leq 2\sum_{i,j>1} F_{ii}\lambda_j^{-1} u_{ij1}^2. \tag{4.2}$$

The sum of terms with at least one index equal to 1 is of the form $B \cdot Du_{11}$, and at p, $Du_{11} = D\lambda_1$ by (4.1). Therefore, at p

$$\triangle_F u_{11} \leq 2\sum_{i,j>1} F_{ii}\lambda_j^{-1} u_{ij1}^2 + B \cdot D\lambda_1.$$

Then,

$$\begin{aligned}\triangle_F \lambda_1 &= \triangle_F u_{11} + 2\sum_i \sum_{j>1} F_{ii} \frac{1}{\lambda_1 - \lambda_j} u_{ij1}^2 \\ &\leq B \cdot D\lambda_1 + 2\sum_{i,j>1} F_{ii}\left(\frac{1}{\lambda_j} + \frac{1}{\lambda_1 - \lambda_j}\right) u_{ij1}^2 + 2\sum_{j>1} F_{11}\frac{1}{\lambda_1 - \lambda_j} u_{1j1}^2 \\ &\leq B \cdot D\lambda_1\end{aligned} \tag{4.3}$$

because $\frac{1}{\lambda_j} - \frac{1}{\lambda_j - \lambda_1} \leq 0$.

Now assume that $\lambda_1(p) = \cdots = \lambda_s(p) < \lambda_{s+1}(p)$. Suppose ϕ is a smooth function supporting λ_1 from below at p. Then

$$\phi \leq \lambda_1 \leq \frac{1}{s}\sum_{m\leq s} \lambda_m$$

with equality at p. Observe that $\sum_{m\leq s} \lambda_m$ is differentiable in a neighborhood of p. It follows from the assumption that ϕ supports λ_1 from below that λ_1 is also differentiable at p. Therefore Lemma 4.1 implies that $u_{ije}(p) = 0$ if $i \neq j$ for $i, j \leq s$.

We now compute $\triangle_F \sum_{m\le s} \lambda_m$. We have

$$
\begin{aligned}
\triangle_F \sum_{m\le s} \lambda_m &= \triangle_F \sum_{m\le s} u_{mm} + 2\sum_i \sum_{j>s} \sum_{m\le s} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} \\
&= -\sum_{i,j,k,l} \sum_{m\le s} F_{ij,kl} u_{ijm} u_{klm} + 2\sum_i \sum_{j>s} \sum_{m\le s} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} \\
&= -\sum_{i,j,k,l>s} \sum_{m\le s} F_{ij,kl} u_{ijm} u_{klm} + 2\sum_i \sum_{j>s} \sum_{m\le s} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} \\
&\quad - \sum_{(i,j,k,l)\in I_s} \sum_{m\le s} F_{ij,kl} u_{ijm} u_{klm} \\
&\le 2\sum_{i,j>s} \sum_{m\le s} F_{ii} \frac{u_{ijm}^2}{\lambda_j} + 2\sum_i \sum_{j>s} \sum_{m\le s} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} \\
&\quad + \sum_{m\le s} B_m \cdot D\lambda_m .
\end{aligned}
$$

Lemma 4.1 also implies that $D\lambda_i = D\lambda_j$ for $i, j \le s$, so

$$
\triangle_F \sum_{m\le s} \lambda_m \le B \cdot D\lambda_1 . \tag{4.4}
$$

Combining (4.3) with (4.4) shows that λ_1 is a viscosity supersolution of the equation $\triangle_F w - B \cdot Dw = 0$, so by the strong minimum principle and the assumption that $\lambda_1(0) = 0$ it follows that $\lambda_1 \equiv 0$.

Step 2: Superharmonicity of λ_{a+1} given $\lambda_1 = \cdots = \lambda_a = 0$.

Assume that for some $a < r$, $\lambda_a \equiv 0$. Fix $p \in \Omega$ and assume that $\lambda_{a+1}(p) = \cdots = \lambda_{a+s}(p) < \lambda_{a+s+1}(p)$. Let $\mu = \sum_{a<m\le a+s} \lambda_m$.

If $\lambda_{a+1}(p) > 0$, then we compute

$$
\begin{aligned}
\triangle_F \mu &= \triangle_F \mu + 2\triangle_F \sum_{m\le a} \lambda_m \\
&= \triangle_F \sum_{a<m\le a+s} u_{mm} + 2\sum_i \sum_{j>a+s} \sum_{a<m\le a+s} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} \\
&\quad + 2\sum_i \sum_{j\le a} \sum_{a<m\le a+s} F_{ii} \frac{u_{ijm}^2}{\lambda_m} + 2\triangle_F \sum_{m\le a} u_{mm} \\
&\quad + 4\sum_i \sum_{j>a} \sum_{m\le a} F_{ii} \frac{u_{ijm}^2}{-\lambda_j}
\end{aligned}
$$

$$= -\sum_{i,j,k,l}\sum_{a<m\le a+s} F_{ij,kl}u_{ijm}u_{klm} - 2\sum_{i,j,k,l}\sum_{m\le a} F_{ij,kl}u_{ijm}u_{klm}$$
$$+ 2\sum_i\sum_{j>a+s}\sum_{a<m\le a+s} F_{ii}\frac{u_{ijm}^2}{\lambda_m-\lambda_j} + 2\sum_i\sum_{j\le a}\sum_{a<m\le a+s} F_{ii}\frac{u_{ijm}^2}{\lambda_m}$$
$$+ 4\sum_i\sum_{j>a}\sum_{m\le a} F_{ii}\frac{u_{ijm}^2}{-\lambda_j}$$

By the inverse-convexity assumption,

$$-\sum_{i,j,k,l}\sum_{a<m\le a+s} F_{ij,kl}u_{ijm}u_{klm} \le 2\sum_{i,j>a+s}\sum_{a<m\le a+s} F_{ii}\frac{1}{\lambda_j}u_{ijm}^2 - \sum_{(i,j,k,l)\in I_{a+s}}\sum_{a<m\le a+s} F_{ij,kl}u_{ijm}u_{klm}$$

and

$$-2\sum_{i,j,k,l}\sum_{m\le a} F_{ij,kl}u_{ijm}u_{klm} \le 4\sum_{i,j>a+s}\sum_{m\le a} F_{ii}\frac{1}{\lambda_j}u_{ijm}^2 - 2\sum_{(i,j,k,l)\in I_{a+s}}\sum_{m\le a} F_{ij,kl}u_{ijm}u_{klm}.$$

Therefore

$$\begin{aligned}\triangle_F\mu \le\; & 2\sum_{i,j>a+s}\sum_{a<m\le a+s} F_{ii}\frac{1}{\lambda_j}u_{ijm}^2 - \sum_{(i,j,k,l)\in I_{a+s}}\sum_{a<m\le a+s} F_{ij,kl}u_{ijm}u_{klm}\\ & + 4\sum_{i,j>a+s}\sum_{m\le a} F_{ii}\frac{1}{\lambda_j}u_{ijm}^2 - 2\sum_{(i,j,k,l)\in I_{a+s}}\sum_{m\le a} F_{ij,kl}u_{ijm}u_{klm}\\ & + 2\sum_i\sum_{j>a+s}\sum_{a<m\le a+s} F_{ii}\frac{u_{ijm}^2}{\lambda_m-\lambda_j} + 2\sum_i\sum_{j\le a}\sum_{a<m\le a+s} F_{ii}\frac{u_{ijm}^2}{\lambda_m}\\ & + 4\sum_i\sum_{j>a}\sum_{m\le a} F_{ii}\frac{u_{ijm}^2}{-\lambda_j}\\ =\; & 2\sum_{i,j>a+s}\sum_{a<m\le a+s} F_{ii}\frac{1}{\lambda_j}u_{ijm}^2 - \sum_{(i,j,k,l)\in I_{a+s}}\sum_{a<m\le a+s} F_{ij,kl}u_{ijm}u_{klm}\\ & + 4\sum_{i,j>a+s}\sum_{m\le a} F_{ii}\frac{1}{\lambda_j}u_{ijm}^2 - 2\sum_{(i,j,k,l)\in I_{a+s}}\sum_{m\le a} F_{ij,kl}u_{ijm}u_{klm}\end{aligned} \tag{4.5}$$

$$
\begin{aligned}
&+2\sum_i \sum_{j>a+s} \sum_{a<m\le a+s} F_{ii}\frac{u_{ijm}^2}{\lambda_m-\lambda_j} + 2\sum_i \sum_{j\le a} \sum_{a<m\le a+s} F_{ii}\frac{u_{ijm}^2}{\lambda_m} \\
&+4\sum_i \sum_{a<j\le a+s} \sum_{m\le a} F_{ii}\frac{u_{ijm}^2}{-\lambda_j} + 4\sum_i \sum_{j>a+s} \sum_{m\le a} F_{ii}\frac{u_{ijm}^2}{-\lambda_j} \\
\le& -\sum_{(i,j,k,l)\in I_{a+s}} \sum_{a<m\le a+s} F_{ij,kl}u_{ijm}u_{klm} - 2\sum_{(i,j,k,l)\in I_{a+s}} \sum_{m\le a} F_{ij,kl}u_{ijm}u_{klm} \\
&+2\sum_i \sum_{j>a} \sum_{m\le a} F_{ii}\frac{u_{ijm}^2}{-\lambda_j},
\end{aligned}
$$

because

$$\sum_{i,j>a+s} \sum_{a<m\le a+s} F_{ii}\frac{1}{\lambda_j}u_{ijm}^2 + \sum_i \sum_{j>a+s} \sum_{a<m\le a+s} F_{ii}\frac{u_{ijm}^2}{\lambda_m-\lambda_j} \le 0$$

and

$$\sum_{i,j>a+s} \sum_{m\le a} F_{ii}\frac{1}{\lambda_j}u_{ijm}^2 + \sum_i \sum_{j>a+s} \sum_{m\le a} F_{ii}\frac{u_{ijm}^2}{-\lambda_j} \le 0.$$

We now estimate the terms $F_{ij,kl}u_{ijm}u_{klm}$. We use the same technique as [15, p.6527]. First,

$$
\begin{aligned}
&-\sum_{(i,j,k,l)\in I_{a+s}} \sum_{a<m\le a+s} F_{ij,kl}u_{ijm}u_{klm} - 2\sum_{(i,j,k,l)\in I_{a+s}} \sum_{m\le a} F_{ij,kl}u_{ijm}u_{klm} \\
&\qquad \le C(\|F\|_{C^2}, \|u\|_{C^3}, n, a, s)\sum_i \sum_{j,k\le a+s} |u_{ijk}| \\
&\qquad \le C\left(\sum_i \sum_{j<k\le a+s} |u_{ijk}| + \sum_i \sum_{j\le a+s} |u_{ijj}|\right).
\end{aligned}
$$

For $j\le a$, $\partial_i\lambda_j=0$. If ϕ is a smooth function supporting λ_{a+1} from below at p, then λ_{a+1} is differentiable at p because μ/s supports λ_{a+1} from above. Therefore Lemma 4.1 implies that $D\lambda_{a+1}(p)=\cdots=D\lambda_{a+s}(p)$. We then have $u_{ijj}(p)=\partial_i\lambda_{a+1}(p)$ if $a<j\le a+s$.

For the remaining terms, we use the Cauchy-Schwarz inequality. It follows from Lemma 4.1 that the matrix $(\partial_i u_{jk})_{j,k}$ is 0 in the upper left $a\times a$ block and the $s\times s$ block $(\partial_i u_{jk})_{j,k=a+1}^{a+s}$ is diagonal. Therefore the only nonzero terms of the sum $\sum_i\sum_{j<k\le a+s}|u_{ijk}|$ are those with $j\le a<k$. By assumption, $\mu>0$ at p, so

$$|u_{ijk}| \le \frac{u_{ijk}^2}{M\mu} + M\mu$$

where M is a large constant to be chosen. By ellipticity,

$$\frac{u_{ijk}^2}{M\mu} \le C\sum_i F_{ii}\frac{u_{ijk}^2}{M\mu}\frac{\mu}{\lambda_k-\lambda_j}.$$

It follows that

$$\sum_i \sum_{j\le a} \sum_{a<k\le a+s} |u_{ijk}| \le \frac{C}{M} \sum_i \sum_{j\le a} \sum_{a<k\le a+s} F_{ii} \frac{u_{ijk}^2}{\lambda_k - \lambda_j}. \tag{4.6}$$

Combining (4.6) with the result of (4), we have

$$\begin{aligned} \triangle_F \mu \le\ & 2\sum_i \sum_{j>a} \sum_{m\le a} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} + \frac{C}{M} \sum_i \sum_{j\le a} \sum_{a<k\le a+s} F_{ii} \frac{u_{ijk}^2}{\lambda_k - \lambda_j} \\ & + CM\mu + \sum_i \sum_{j\le a+s} |u_{ijj}| \\ =\ & 2\sum_i \sum_{j>a} \sum_{m\le a} F_{ii} \frac{u_{ijm}^2}{\lambda_m - \lambda_j} + \frac{C}{M} \sum_i \sum_{j\le a} \sum_{a<k\le a+s} F_{ii} \frac{u_{ijk}^2}{\lambda_k - \lambda_j} \\ & + CM\mu + B|D\mu|. \end{aligned} \tag{4.7}$$

Provided M is large enough, the sum of the first two terms in (4.7) is nonpositive, so

$$\triangle_F \mu \le CM\mu + B|D\mu|.$$

It then follows that

$$\triangle_F \lambda_{a+1} \le C\lambda_{a+1} + B|D\lambda_{a+1}|$$

in the viscosity sense on the set $\{\lambda_{a+1} > 0\}$. That the differential inequality holds also on the set $\{\lambda_{a+1} = 0\}$ follows from (4.4).

Now the theorem follows by the strong minimum principle and induction on a. □

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Declarations

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