Hessian estimates for the sigma-2 equation in dimension four

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Abstract

We derive a priori interior Hessian estimates and interior regularity for the σ_2 equation in dimension four. Our method provides respectively a new proof for the corresponding three dimensional results and a Hessian estimate for smooth solutions satisfying a dynamic semi-convexity condition in higher $n \geq 5$ dimensions.

1 Introduction

In this article, we resolve the question of the interior a priori Hessian estimate and regularity for the σ_2 equation

$$\sigma_2\left(D^2 u\right) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j = 1 \tag{1.1}$$

in dimension $n = 4$, where $\lambda_i' s$ are the eigenvalues of the Hessian $D^2 u$.

Theorem 1.1. *Let u be a smooth solution to (1.1) in the positive branch $\triangle u > 0$ on $B_1(0) \subset \mathbb{R}^4$. Then u has an implicit Hessian estimate*

$$|D^2 u(0)| \leq C(\|u\|_{C^1(B_1(0))})$$

with $\|u\|_{C^1(B_1(0))} = \|u\|_{L^\infty(B_1(0))} + \|Du\|_{L^\infty(B_1(0))}$.

From the gradient estimate for σ_k-equations by Trudinger [T2] and also Chou-Wang [CW] in the mid 1990s, we can bound $D^2 u$ in terms of the solution u in $B_2(0)$ as

$$|D^2 u(0)| \leq C(\|u\|_{L^\infty(B_2(0))}).$$

In higher $n \geq 5$ dimensions, our method provides a Hessian estimate for smooth solutions satisfying a semi-convexity type condition with movable lower bound (1.2), which is unconditionally valid in four dimensions by (2.3).

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Theorem 1.2. *Let u be a smooth solution to (1.1) in the positive branch $\triangle u > 0$ on $B_1(0) \subset \mathbb{R}^n$ with $n \geq 5$, satisfying a dynamic semi-convex condition*

$$\lambda_{\min}\left(D^2 u\right) \geq -c\,(n) \triangle u \quad \textit{with} \quad c\,(n) = \frac{\sqrt{3n^2+1}-n+1}{2n}. \tag{1.2}$$

Then u has an implicit Hessian estimate

$$|D^2 u(0)| \leq C(n, \|u\|_{L^\infty(B_1(0))}).$$

One application of the above estimates is the interior regularity (analyticity) of C^0 viscosity solutions to (1.1) in four dimensions, when the estimates are combined with the solvability of the Dirichlet problem with C^4 boundary data by Caffarelli-Nirenberg-Spruck [CNS] and also Trudinger [T1]. In particular, the solutions of the Dirichlet problem with C^0 boundary data to four dimensional (1.1) of both positive branch $\triangle u > 0$ and negative branch $\triangle u < 0$ respectively, enjoy interior regularity.

Another consequence is a rigidity result for entire solutions to (1.1) of both branches with quadratic growth, namely all such solutions must be quadratic, provided the smooth solutions in dimension $n \geq 5$ also satisfying the dynamic semi-convex assumption (1.2), or the symmetric one $\lambda_{\max}\left(D^2 u\right) \leq -c\,(n) \triangle u$ in the symmetric negative branch case. Warren's rare saddle entire solution to (1.1) shows certain convexity condition is necessary [W]. Other earlier related results can be found [BCGJ] [Y1] [CY] [CX] [SY3].

In two dimensions, an interior Hessian bound for (1.1), the Monge-Ampère equation $\sigma_2 = \det D^2 u = 1$ was found via isothermal coordinates, which are readily available under Legendre-Lewy transform, by Heinz [H] in the 1950s. The dimension three case was done via the minimal surface structure of equation (1.1) and a full strength Jacobi inequality by Warren-Yuan in the late 2000s [WY]. In higher dimensions $n \geq 4$ any effective geometric structure of (1.1) appears hidden, although the level set of non-uniformly elliptic equation (1.1) is convex.

In recent years, Hessian estimates for convex smooth solutions of (1.1) have been obtained via a pointwise approach by Guan and Qiu [GQ]. Hessian estimates for almost convex smooth solutions of (1.1) have been derived by a compactness argument in [MSY], and for semi-convex smooth solutions in [SY1] by an integral method. However, we cannot extend these a priori estimates, including Theorem 1.2, to interior regularity statements for viscosity solutions of (1.1), because the smooth approximations may not preserve the convexity or semi-convexity constraints. Taking advantage of an improved regularity property for the equation satisfied by the Legendre-Lewy transform of almost convex viscosity solutions, interior regularity was reached in [SY2].

For higher order $\sigma_k\left(D^2 u\right) = 1$ with $k \geq 3$ equations, which is the Monge-Ampère equation in k dimensions, there are the famous singular solutions constructed by Pogorelov [P] in the 1970s, and later generalized in [U1]. Worse singular solutions have been produced in recent years. Hessian estimates for solutions with certain strict k-convexity constraints to Monge-Ampère equations and σ_k equation ($k \geq 2$) were derived by Pogorelov [P] and Chou-Wang [CW] respectively using the Pogorelov's pointwise technique. Urbas [U2] [U3] obtained (pointwise) Hessian estimates in term of certain integrals of the Hessian for σ_k equations. Recently, Mooney [M] derived the strict 2-convexity of convex viscosity solutions to (1.1), consequently, relying on the solvability [CNS] and a priori estimates [CW], gave a different proof of the interior regularity of those convex viscosity solutions.

Our proof of Theorem 1.1 synthesizes the ideas of Qiu [Q1] with Chaudhuri-Trudinger [CT] and Savin [S]. Qiu showed that in dimension three, where a Jacobi inequality is valid (see Section 2 for definitions of the operators)

$$F_{ij}\partial_{ij}\ln\Delta u \geq \varepsilon F_{ij}(\ln\Delta u)_i(\ln\Delta u)_j,$$

a maximum principle argument leads to a doubling, or "three-sphere" inequality:

$$\sup_{B_1(0)}\Delta u \leq C(n,\|u\|_{C^1(B_2(0))})\sup_{B_{1/2}(0)}\Delta u.$$

(A lower bound condition on $\sigma_3(D^2u)$, satisfied by convex solutions of (1.1) in general dimensions permitted Guan-Qiu to exclude the inner "sphere" term $B_{1/2}(0)$ in the above inequality for their eventual Hessian estimates earlier in [GQ].) Iterating this "three-sphere" inequality shows that the Hessian is controlled by its maximum on any arbitrarily small ball. To put it another way, any blowup point propagates to a dense subset of $B_1(0)$. To rule out Weierstrass nowhere twice differentiable counterexamples, it suffices to find a single smooth point; Savin's small perturbation theorem [S] guarantees a smooth ball if there is a smooth point. It more than suffices to establish partial regularity, such as Alexandrov's theorem. Chaudhuri and Trudinger [CT] showed that k-convex functions have an Alexandrov theorem if $k > n/2$. This gives a new compactness proof of Hessian estimate and regularity for (1.1) in dimension three without minimal surface arguments, and also Hessian estimate for (1.1) in general dimensions with semi-convexity assumption in [SY1], where a Jacobi inequality and Alexandrov twice differentiability are available.

In higher dimensions $n \geq 4$, there are three new difficulties. Although the Hölder estimate for k-convex *functions* may not be valid for $k \leq n/2$, we can replace it with the interior gradient estimate for 2-convex *solutions* in [T2] [CW]; this gives an Alexandrov theorem. The main hurdle is the Jacobi inequality, which fails for four and higher dimensions without a priori control on the minimum eigenvalue λ_{min} of D^2u; the Jacobi inequality was discovered in [SY1, SY3] for semi-convex solutions. Instead, we can only establish an "almost-Jacobi inequality", where $\varepsilon \sim 1+2\lambda_{min}/\Delta u$ in four dimensions. This choice of ε degenerates to zero for the extreme configurations $(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = (a,a,a,-a+O(1/a))$. At first glance, $\varepsilon \to 0$ means Qiu's maximum principle argument fails; the positive term $\varepsilon|\nabla_F b|^2$ can no longer absorb bad terms. On the other hand, for the extreme configurations, the equation becomes conformally uniformly elliptic. The, usually defective, lower order term $\Delta_F|Du|^2 \gtrsim \sigma_1\lambda_{min}^2$, is large enough to take control of the bad terms. The dynamic semi-convexity assumption (1.2) allows the outlined four dimensional arguments to continue working in higher $n \geq 5$ dimensions.

Using similar methods, a new proof of regularity for strictly convex solutions to the Monge-Ampère equation is found in [SY4]. Extrinsic curvature estimates for the scalar curvature equation in dimension four are found in [Sh], extending the dimension three result of Qiu [Q2]. In forthcoming work, investigation will be done on conformal geometry's σ_2 Schouten tensor equation with negative scalar curvature and the improvement of the $W^{2,6+\delta}$ to $C^{1,1}$ estimate in [D] to a $W^{2,6}$ to $C^{1,1}$ estimate.

In still higher dimensions $n \geq 5$, we are not even able to establish that $\ln \Delta u$ is subharmonic, $\varepsilon \geq 0$, without a priori conditions on the Hessian. There is still the problem of regularity for such solutions. Combining the Alexandrov theorem with small perturbation [S, Theorem 1.3] only shows that the singular set is closed with Lebesgue measure zero.

2 Almost Jacobi inequality

In [SY3], we established a Jacobi inequality for $b = \ln(\Delta u + C(n, K))$ under the semi-convexity assumption $\lambda_{min}(D^2u) \geq -K$, namely the quantitative subsolution inequality

$$\Delta_F b := F_{ij}\partial_{ij} b \geq \varepsilon\, F_{ij} b_i b_j =: \varepsilon |\nabla_F b|^2,$$

where $\varepsilon = 1/3$, and for the sigma-2 equation $F(D^2u) = \sigma_2(\lambda) = 1$, we denote the linearized operator by the positive definite matrix

$$(F_{ij}) = \Delta u\, I - D^2u = \sqrt{2\sigma_2 + |D^2u|}\, I - D^2u > 0. \tag{2.1}$$

In dimension three, the above Jacobi inequality holds for $C(3, K) = 0$ unconditionally; see [SY3, p. 3207] and Remark 2.2. In dimension four, we can establish an inequality for $b = \ln \Delta u$ without any Hessian conditions. The cost is that ε depends on the Hessian, and $\varepsilon(D^2u) \to 0$ is allowed. We obtain an "almost" Jacobi inequality.

Proposition 2.1. *Let u be a smooth solution to $\sigma_2(\lambda) = 1$, and $b = \ln \Delta u$. In dimension $n = 4$, we have*

$$\Delta_F b \geq \varepsilon\, |\nabla_F b|^2, \tag{2.2}$$

where

$$\varepsilon = \frac{2}{9}\left(\frac{1}{2} + \frac{\lambda_{min}}{\Delta u}\right) > 0.$$

In higher dimension $n \geq 5$, (2.2) holds for

$$\varepsilon = \frac{\sqrt{3n^2+1} - (n+1)}{3(n-1)}\left(\frac{\sqrt{3n^2+1} - (n-1)}{2n} + \frac{\lambda_{min}}{\Delta u}\right)$$

under the condition

$$\frac{\lambda_{min}}{\Delta u} \geq -\frac{\sqrt{3n^2+1} - (n-1)}{2n}$$

Here, λ_{min} is the minimum eigenvalue of D^2u.

An important ingredient for Proposition 2.1 is the following sharp control on the minimum eigenvalue.

Lemma 2.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ solve $\sigma_2(\lambda) = 1$ with $\lambda_1 + \cdots + \lambda_n > 0$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the following bound holds for $n > 2$ and is sharp:*

$$\sigma_1(\lambda) > \frac{n}{n-2}|\lambda_n|. \tag{2.3}$$

Proof. The sharpness follows from the configurations

$$\lambda = \left(a, a, \dots, a, -\frac{(n-2)}{2}a + \frac{1}{(n-1)a}\right). \tag{2.4}$$

Next, if $\lambda_n \geq 0$, we have

$$\sigma_1 = \lambda_1 + \cdots + \lambda_n \geq n\lambda_n.$$

For $\lambda_n < 0$, we write $\lambda' = (\lambda_1, \dots, \lambda_{n-1})$ and observe that $\lambda_n = (1 - \sigma_2(\lambda'))/\sigma_1(\lambda')$. We must have $\sigma_2(\lambda') > 1$, as $\sigma_1(\lambda') > 0$ from (2.1), so we write

$$\frac{\sigma_1(\lambda)}{-\lambda_n} = -1 + \frac{\sigma_1(\lambda')^2}{\sigma_2(\lambda') - 1} > -1 + \frac{\sigma_1(\lambda')^2}{\sigma_2(\lambda')}.$$

We write $\sigma_1(\lambda')^2$ in terms of the traceless part $\lambda'^{\perp}$ of λ' and $\sigma_2(\lambda')$:

$$\sigma_1(\lambda')^2 = \frac{n-1}{n-2}(2\sigma_2(\lambda') + |\lambda'^{\perp}|^2).$$

It then follows

$$\frac{\sigma_1(\lambda)}{-\lambda_n} > -1 + \frac{2(n-1)}{n-2} = \frac{n}{n-2}.$$

□

As a consequence, we obtain the following quantitative ellipticity for equation (1.1).

Corollary 2.1. *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ solve $f(\lambda) = \sigma_2(\lambda) = 1$, with $\lambda_1 + \cdots + \lambda_n > 0$. For $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $f_i = \partial f/\partial\lambda_i$, we have*

$$\begin{aligned} \frac{1}{\sigma_1} \leq f_1 \leq \left(\frac{n-1}{n}\right)\sigma_1, \\ \left(1 - \frac{1}{\sqrt{2}}\right)\sigma_1 \leq f_i \leq 2\left(\frac{n-1}{n}\right)\sigma_1, \qquad i \geq 2. \end{aligned} \tag{2.5}$$

Proof. The upper bound for $f_1 = \sigma_1 - \lambda_1$ comes from the easy bound $n\lambda_1 \geq \sigma_1$. The sharp upper bound for f_n follows from (2.3):

$$f_i \leq f_n = \sigma_1 - \lambda_n < \left(1 + \frac{n-2}{n}\right)\sigma_1.$$

The $i = 1$ lower bound goes as follows:

$$f_1 = \sigma_1 - \lambda_1 = \frac{2 + |(0, \lambda_2, \dots, \lambda_n)|^2}{\sigma_1 + \lambda_1} > \frac{2}{\sigma_1 + \lambda_1} > \frac{1}{\sigma_1}.$$

The $i \geq 2$ lower bounds for $f_i = \sigma_1 - \lambda_i$ are true if $\lambda_i \leq 0$. For $\lambda_i > 0$,

$$f_i = \sigma_1 - \lambda_i > \sigma_1 - \sqrt{\frac{\lambda_1^2 + \cdots + \lambda_i^2}{i}} > \left(1 - i^{-1/2}\right)\sigma_1,$$

where we used

$$\sigma_1 = \sqrt{2 + |\lambda|^2} > \sqrt{\lambda_1^2 + \cdots + \lambda_i^2},$$

in the last inequality. □

Remark 2.1. A sharp form of (2.5) for the $i \geq 2$ lower bounds and rougher upper bounds was first shown in [LT, (16)]. A rougher form of the lower bounds in (2.5), enough for our proof of doubling Proposition 3.1, also follows from [CW, Lemma 3.1], [CY, Lemma 2.1], and [SY1, (2.4)].

Proof of Proposition 2.1. Step 1. Expression of the Jacobi inequality. After a rotation at $x = p$, we assume that $D^2u(p)$ is diagonal. Then $(F_{ij}) = \mathrm{diag}(f_i)$, where $f(\lambda) = \sigma_2(\lambda)$. The following calculation was performed in [SY3, p. 4] for $b = \ln(\Delta u + J)$ for some constant J. We repeat it below with $J = 0$, for completeness. We start with the following formulas at $x = p$:

$$|\nabla_F b|^2 = \sum_{i=1}^n f_i \frac{(\Delta u_i)^2}{(\Delta u)^2}, \tag{2.6}$$

$$\Delta_F b = \sum_{i=1}^n f_i \left[\frac{\partial_{ii}\Delta u}{\Delta u} - \frac{(\partial_i \Delta u)^2}{(\Delta u)^2} \right] \tag{2.7}$$

Next, we replace the fourth order derivatives $\partial_{ii}\Delta u = \sum_{k=1}^n \partial_{ii} u_{kk}$ in (2.7) by third derivatives. By differentiating (1.1), we have

$$\Delta_F Du = (F_{ij} u_{ijk})_{k=1}^n = 0. \tag{2.8}$$

Differentiating (2.8) and using (2.1), we obtain at $x = p$,

$$\begin{aligned} \sum_{i=1}^n f_i \partial_{ii}\Delta u = \sum_{k=1}^n \Delta_F u_{kk} = \sum_{i,j,k=1}^n F_{ij}\partial_{ij} u_{kk} = -\sum_{i,j,k=1}^n \partial_k F_{ij} \partial_{ij} u_k \\ = \sum_{i,j,k=1}^n -(\Delta u_k \delta_{ij} - u_{kij}) u_{kij} = \sum_{i,j,k=1}^n u_{ijk}^2 - \sum_{k=1}^n (\Delta u_k)^2. \end{aligned}$$

Substituting this identity into (2.7) and regrouping terms of the forms $u^2_{\clubsuit\heartsuit\spadesuit}$, $u^2_{\clubsuit\clubsuit\heartsuit}$, $u^2_{\heartsuit\heartsuit\heartsuit}$, and $(\Delta u_\clubsuit)^2$, we obtain

$$\Delta_F b = \frac{1}{\sigma_1} \left\{ 6 \sum_{i<j<k} u_{ijk}^2 + \left[3 \sum_{i \neq j} u_{jji}^2 + \sum_i u_{iii}^2 - \sum_i \left(\left(1 + \frac{f_i}{\sigma_1}\right) (\Delta u_i)^2 \right] \right\}$$

Accounting for (2.6), we obtain the following quadratic:

$$(\Delta_F b - \varepsilon |\nabla_F b|^2)\sigma_1 \geq 3 \sum_{i \neq j} u_{jji}^2 + \sum_i u_{iii}^2 - \sum_i (1 + \delta f_i/\sigma_1)(\Delta u_i)^2,$$

where $\delta := 1 + \varepsilon$ here. As in [SY3], we fix i and denote $t_i = (u_{11i}, \ldots, u_{nni})$ and e_i the i-th basis vector of $\mathbb{R}^n$. Then we recall equation (2.9) from [SY3] for the i-th term above:

$$Q := 3|t| - 2\langle e_i, t\rangle^2 - (1 + \delta f_i/\sigma_1)\langle (1, \ldots, 1), t\rangle^2.$$

The objective is to show that $Q \geq 0$. The idea in [SY3] was to reduce the quadratic form to a two dimensional subspace. In that paper, $Q \geq 0$ was shown under a semi-convexity assumption of the Hessian. Here, we show how to remove this assumption in dimension four. For completeness, we repeat that reduction below.

Step 2. Anisotropic projection. Equation (2.8) at $x = p$ shows that $\langle Df, t_i\rangle = 0$, so Q is zero along a subspace. We can thus replace the vectors e_i and $(1, \dots, 1)$ in Q with their projections:

$$Q = 3|t|^2 - 2\langle E, t\rangle^2 - (1 + \delta f_i/\sigma_1)\langle L, t\rangle^2,$$

where

$$E = e_i - \frac{\langle e_i, Df\rangle}{|Df|^2} Df, \qquad L = (1, \dots, 1) - \frac{\langle (1, \dots, 1), Df\rangle}{|Df|^2} Df.$$

Their rotational invariants can be calculated as in [SY3, equation (2.10)]:

$$|E|^2 = 1 - \frac{f_i^2}{|Df|^2}, \qquad |L|^2 = 1 - \frac{2(n-1)}{|Df|^2}, \qquad E \cdot L = 1 - \frac{(n-1)\sigma_1 f_i}{|Df|^2}. \tag{2.9}$$

The quadratic is mostly isotropic: if t is orthogonal to both E and L, then $Q = 3|t|^2 \geq 0$, so it suffices to assume that t lies in the $\{E, L\}$ subspace. The matrix associated to the quadratic form is

$$Q = 3I - 2E \otimes E - \eta L \otimes L,$$

where $\eta = 1 + \delta f_i/\sigma_1 = 1 + (1+\varepsilon) f_i/\sigma_1$. Since Q is a quadratic form, its matrix is symmetric and has real eigenvalues. In the non-orthogonal basis $\{E, L\}$, the eigenvector equation is

$$\begin{pmatrix} 3 - 2|E|^2 & -2E \cdot L \\ -\eta L \cdot E & 3 - \eta |L|^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \xi \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

The real eigenvalues of this matrix have the explicit form

$$\xi = \frac{1}{2}\left(tr \pm \sqrt{tr^2 - 4det} \right),$$

where the trace and determinant are given by

$$tr = 6 - 2|E|^2 - \eta |L|^2, \qquad det = 9 - 6|E|^2 - 3\eta |L|^2 + 2\eta \left[|E|^2 |L|^2 - (E \cdot L)^2 \right].$$

It thus suffices to show that $tr \geq 0$ and $\det \geq 0$.

Step 3. Non-negativity of the trace of the quadratic form. In [SY3], the trace was shown positive; indeed, by (2.9),

$$\begin{aligned} tr &= 6 - 2\left(1 - \frac{f_i^2}{|Df|^2}\right) - \left(1 + \delta \frac{f_i}{\sigma_1}\right)\left(1 - \frac{2(n-1)}{|Df|^2}\right) \\ &> 3 - \delta \frac{f_i}{\sigma_1} = \frac{(3-\delta)\sigma_1 + \delta \lambda_i}{\sigma_1} \\ &\geq 3 - \delta\left(1 + \frac{n-2}{n}\right) \geq 0, \end{aligned}$$

for any

$$\delta \leq \frac{3n}{2(n-1)}, \tag{2.10}$$

using the bound (2.3) in the case that $\lambda_i < 0$.

Step 4. Non-negativity of the determinant of the quadratic form. Our new contribution here is to analyze the determinant in general. Again by (2.9), the determinant is

$$\begin{aligned} det &= \frac{6f_i^2}{|Df|^2} - \frac{3\delta f_i}{\sigma_1} + 3\left(1+\frac{\delta f_i}{\sigma_1}\right)\boxed{\frac{2(n-1)}{|Df|^2}} \\ &+ 2\left(1+\frac{\delta f_i}{\sigma_1}\right)\left[\frac{2(n-1)\sigma_1 f_i}{|Df|^2} - \frac{nf_i^2}{|Df|^2} - \boxed{\frac{2(n-1)}{|Df|^2}}\right] \\ &> -\frac{3\delta f_i}{\sigma_1} + 4\left(1+\frac{\delta f_i}{\sigma_1}\right)\frac{(n-1)\sigma_1 f_i}{|Df|^2} + \left[6-2n\left(1+\frac{\delta f_i}{\sigma_1}\right)\right]\frac{f_i^2}{|Df|^2}. \end{aligned}$$

Since $f_i = \sigma_1 - \lambda_i$ and $\sigma_1^2 = 2+|\lambda|^2$, we get $|Df|^2 = (n-1)\sigma_1^2 - 2$, so we obtain an inequality in terms of $y := f_i/\sigma_1$:

$$\begin{aligned} det\cdot\frac{|Df|^2}{\sigma_1 f_i} &> \boxed{\frac{6\delta}{\sigma_1^2}} - 3(n-1)\delta + 4(n-1)\left(1+\delta\frac{f_i}{\sigma_1}\right) + \left[6-2n\left(1+\delta\frac{f_i}{\sigma_1}\right)\right]\frac{f_i}{\sigma_1} \\ &> (n-1)\left(4-3\delta\right) + \Big[6-2n+4\left(n-1\right)\delta\Big]y - 2n\delta y^2 \\ &=: q_\delta(y). \end{aligned} \tag{2.11}$$

Remark 2.2. In three dimensions, the almost Jacobi inequality (2.2) becomes a full strength one $\triangle_F b \geq \frac{1}{3}\left|\nabla_F b\right|^2$, because in (2.11), $q_{4/3}\left(y\right) = \frac{8}{3}\frac{f_i}{\sigma_1}\left(1+3\lambda_i/\sigma_1\right) > 0$ by (2.3). This was observed in [SY3, p. 3207].

We write $q_\delta(y) = q_1(y) + \varepsilon\, r(y)$. The remainder:

$$r(y) = -3(n-1) + 4(n-1)y - 2ny^2 = -3(n-1) + 2ny\left(\frac{2(n-1)}{n} - y\right) > -3(n-1),$$

where we used $0 < y = f_i/\sigma_1 \leq f_n/\sigma_1 < 2(n-1)/n$; see (2.5) in Corollary 2.1. To estimate $q_1(y)$, let us solve $0 = q_1(y) = n-1+2(n+1)y-2ny^2$:

$$y_n^\pm := \frac{n+1\pm\sqrt{1+3n^2}}{2n}, \qquad y_n^+ \overset{n=4}{=} \frac{3}{2}. \tag{2.12}$$

Then $q_1(y)/(y_n^+ - y) = 2n(y - y_n^-)$. This linear function is minimized at the endpoint $y=0$, so if $y_n^+ - y \geq 0$, we conclude

$$q_\delta(y) \geq -2ny_n^-(y_n^+ - y) - 3(n-1)\varepsilon \geq -2ny_n^-\left(y_n^+ - \frac{f_n}{\sigma_1}\right) - 3(n-1)\varepsilon = 0,$$

provided

$$\begin{aligned} \varepsilon &:= -\frac{2ny_n^-}{3(n-1)}\left(y_n^+ - \frac{f_n}{\sigma_1}\right) \\ &= \frac{\sqrt{3n^2+1}-(n+1)}{3(n-1)}\left(\frac{\sqrt{3n^2+1}-(n-1)}{2n} + \frac{\lambda_n}{\sigma_1}\right) \\ &\overset{n=4}{=} \frac{2}{9}\left(\frac{1}{2} + \frac{\lambda_n}{\sigma_1}\right). \end{aligned} \tag{2.13}$$

The condition $y_n^+ - y = y_n^+ - \frac{f_i}{\sigma_1} \geq 0$ for all i is equivalent to dynamic semi-convexity,

$$\frac{\lambda_n}{\sigma_1} \geq -\frac{\sqrt{3n^2+1}-(n-1)}{2n}.$$

If $n = 4$, all solutions satisfy this unconditionally, using (2.3).

Let us now check that the trace condition (2.10) is also satisfied. It suffices to have $\varepsilon < 1/2$. Writing $\varepsilon = c(n)(c_n + \lambda_n/\sigma_1)$, it can be shown that $c(n)$ is an increasing function of n bounded by $(\sqrt{3}-1)/3 < 1/4$, and c_n is a decreasing function bounded by $(\sqrt{13}-1)/4 < 2/3$. Combined with $\lambda_n/\sigma_1 \leq 1/n \leq 1/2$ (see Lemma 2.1), we find that $\varepsilon < 7/24$ for $n \geq 2$.

This completes the proof of Proposition 2.1 in dimension $n = 4$ and higher dimension $n \geq 5$. $\square$

3 The doubling inequality

We now use the almost-Jacobi inequality in Proposition 2.1 to show an a priori doubling inequality for the Hessian.

Proposition 3.1. *Let u be a smooth solution of sigma-2 equation* (1.1) *on $B_4(0) \subset \mathbb{R}^n$. If $n = 4$, then the following inequality is valid:*

$$\sup_{B_2(0)} \Delta u \leq C(n) \exp\left(C(n)\|u\|^2_{C^1(B_3(0))}\right) \sup_{B_1(0)} \Delta u.$$

If $n \geq 5$, the inequality is true, if we suppose also that on $B_3(0)$, there is a semi-convexity type condition

$$\frac{\lambda_{min}(D^2u)}{\Delta u} \geq -c_n, \qquad c_n := \frac{\sqrt{3n^2+1}-n+1}{2n}. \tag{3.1}$$

Proof. The following test function on $B_3(0)$ is taken from [GQ, Theorem 4] and [Q1, Lemma 4]:

$$P_{\alpha\beta\gamma} := 2\ln\rho(x) + \alpha(x \cdot Du - u) + \beta|Du|^2/2 + \ln\max(\bar{b}, \gamma^{-1}). \tag{3.2}$$

Here, $\rho(x) = 3^2 - |x|^2$, and $\bar{b} = b - \max_{B_1(0)} b$ for $b = \ln \Delta u$. We also define $\Gamma := 4 + \|u\|_{L^\infty(B_3(0))} + \|Du\|_{L^\infty(B_3(0))}$ to gauge the lower order terms, and denote by $C = C(n)$ a dimensional constant which changes line by line and will be fixed in the end. Small dimensional positive γ, and smaller positive constants α, β depending on γ and Γ, will be chosen later. We also assume summation over repeated indices for simplicity of notation, where it is impossible in Section 2.

Suppose the maximum of $P_{\alpha\beta\gamma}$ occurs at $x^* \in B_3(0)$. If $\bar{b}(x^*) \leq \gamma^{-1}$, then we conclude that for C large enough,

$$\max_{B_2(0)} P_{\alpha\beta\gamma} \leq C + 3\alpha\Gamma + \frac{1}{2}\beta\Gamma^2 + \ln\gamma^{-1}. \tag{3.3}$$

So we suppose that $\bar{b}(x^*) > \gamma^{-1}$. If $|x^*| \leq 1$, then again we obtain (3.3), so we also assume that $1 < |x^*| < 3$.

After a rotation about $x = 0$, we assume that $D^2u(x^*)$ is diagonal, $u_{ii} = \lambda_i$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. At the maximum point x^*, we have $DP_{\alpha\beta\gamma} = 0$,

$$
\begin{aligned}
-\frac{\bar{b}_i}{\bar{b}} &= 2\frac{\rho_i}{\rho} + \alpha x_k u_{ik} + \beta u_k u_{ik} \\
&= 2\frac{\rho_i}{\rho} + \alpha x_i \lambda_i + \beta u_i \lambda_i,
\end{aligned}
\tag{3.4}
$$

and for $0 \geq D^2 P_{\alpha\beta\gamma} = (\partial_{ij} P_{\alpha\beta\gamma})$, we get

$$
0 \geq \Big(-\frac{4\delta_{ij}}{\rho} - 2\frac{\rho_i\rho_j}{\rho^2} + \alpha(x_k u_{ijk} + u_{ij}) + \beta(u_k u_{ijk} + u_{ik}u_{jk}) + \frac{\bar{b}_{ij}}{\bar{b}} - \frac{\bar{b}_i\bar{b}_j}{\bar{b}^2} \Big) \tag{3.5}
$$

Contracting with $F_{ij} = \partial\sigma_2/\partial u_{ij}$ and using

$$
F_{ij}u_{ijk} = 0, \qquad F_{ij}u_{ij} = 2\sigma_2 = 2, \qquad F_{ij}\delta_{ij} = (n-1)\sigma_1,
$$

as well as diagonality at x^*, $(F_{ij}) = (f_i\delta_{ij})$ for $f(\lambda) = \sigma_2(\lambda)$, we obtain at maximum point x^*,

$$
0 \geq F_{ij}\partial_{ij}P_{\alpha\beta\gamma} > -4(n-1)\frac{\sigma_1}{\rho} - 2\frac{f_i\rho_i^2}{\rho^2} + \beta f_i\lambda_i^2 + \frac{f_i\bar{b}_{ii}}{\bar{b}} - \frac{f_i\bar{b}_i^2}{\bar{b}^2}.
$$

Under the assumption that $n = 3, 4$, or instead that $n \geq 5$ with Hessian constraint (3.1), almost-Jacobi inequality Proposition 2.1 is valid, and we get for larger C,

$$
0 \geq -C\frac{\sigma_1}{\rho} - 2\frac{f_i\rho_i^2}{\rho^2} + \beta f_i\lambda_i^2 + \left(c_n + \frac{\lambda_n}{\Delta u}\right)\frac{f_i\bar{b}_i^2}{\bar{b}} - \frac{f_i\bar{b}_i^2}{\bar{b}^2}. \tag{3.6}
$$

If the nonnegative coefficient of $f_i\bar{b}_i^2/\bar{b}$ is positive, we can proceed as in Qiu's proof. In the alternative case, we must use the β term. We start with the latter case. Note that from (2.3) in Lemma 1, condition (3.1) $\lambda_n/\Delta u > -1/2 = -c_n$ is automatically satisfied for $n = 4$, and $\lambda_n/\Delta u > -1/3 > -c_n/2$ for $n = 3$.

CASE $-c_n \leq \lambda_n/\Delta u \leq -c_n/2$: It follows from (2.5) that $f_n\lambda_n^2 \geq c(n)\sigma_1^3$. For larger C,

$$
0 \geq -C\frac{\sigma_1}{\rho^2} + \beta\sigma_1^3 - Cf_i\frac{\bar{b}_i^2}{\bar{b}^2}.
$$

Using (3.4) and ellipticity (2.5), we obtain

$$
\beta\sigma_1^3 \leq C\frac{\sigma_1}{\rho^2} + C(\alpha^2 + \beta^2\Gamma^2)\sigma_1^3.
$$

If the small parameters satisfy

$$
\alpha^2 \leq \beta/(3C), \qquad \beta \leq 1/(3C\Gamma^2), \tag{3.7}
$$

we obtain $\rho^2\sigma_1^2 \leq C/\beta$. Since $\sigma_1 = \sqrt{2 + |\lambda|^2} > \sqrt{2}$, we have $\sigma_1^2 > 2\ln\sigma_1$, and we conclude from (3.2) and (3.7) that

$$
P_{\alpha\beta\gamma} \leq C + \ln\beta^{-1}. \tag{3.8}
$$

We next show that Qiu's argument goes through, in the case that "almost" Jacobi becomes a regular Jacobi.

CASE $\lambda_n/\Delta u \geq -c_n/2$**.** It follows that, after enlarging C, (3.6) can be reduced to

$$0 \geq -C\frac{\sigma_1}{\rho} - C\frac{f_i\rho_i^2}{\rho^2} + \beta f_i\lambda_i^2 + (\bar{b} - C)f_i\frac{\bar{b}_i^2}{\bar{b}^2}.$$

Using $\bar{b}(x^*) \geq \frac{1}{2}\bar{b}(x^*) + \frac{1}{2}\gamma^{-1}$, we assume that γ satisfies

$$\frac{1}{2}\gamma^{-1} \geq C, \tag{3.9}$$

so after enlarging C again, we can further reduce it to

$$0 \geq -C\frac{\sigma_1}{\rho} - C\frac{f_i\rho_i^2}{\rho^2} + \beta f_i\lambda_i^2 + \bar{b}f_i\frac{\bar{b}_i^2}{\bar{b}^2}. \tag{3.10}$$

SUBCASE $1 < |x^*| < 3$ and $x_1^2 > 1/n$: If the small parameters satisfy the condition

$$\beta \leq \alpha/(2n\Gamma), \tag{3.11}$$

we then obtain from (3.4),

$$\frac{\bar{b}_1^2}{\bar{b}^2} \geq \frac{1}{2}(\alpha/n - \beta\Gamma)^2\lambda_1^2 - \frac{C}{\rho^2} \geq \frac{1}{8n^2}\alpha^2\lambda_1^2 - \frac{C}{\rho^2}.$$

We assume that this gives a lower bound, or that $C/\rho^2 \leq \alpha^2\lambda_1^2/(16n^2)$:

$$\frac{\bar{b}_1^2}{\bar{b}^2} \geq \frac{1}{16n^2}\alpha^2\lambda_1^2. \tag{3.12}$$

For if not, we get $\rho^2\lambda_1^2 \leq C/\alpha^2$. Since $\lambda_1 \geq \sigma_1/n$, we can get $\rho^2 \ln\sigma_1 \leq C/\alpha^2$. Using (3.2) and (3.7), we would obtain

$$P_{\alpha\beta\gamma} \leq C + 2\ln\alpha^{-1}. \tag{3.13}$$

It follows then, from (3.12) and (2.5), that (3.10) can be simplified to

$$0 \geq -C\frac{\sigma_1}{\rho^2} + \bar{b}\,f_1(\alpha^2\lambda_1^2).$$

From (2.5), there holds $f_1\lambda_1^2 \geq \sigma_1/n^2$, so we conclude $\rho^2\bar{b} \leq C/\alpha^2$. By (3.2) and (3.7), we conclude a similar bound (3.13):

$$P_{\alpha\beta\gamma} \leq C + 2\ln\alpha^{-1}.$$

SUBCASE $1 < |x^*| < 3$ and $x_k^2 > 1/n$ for some $k \geq 2$: Let us first note that $\sigma_1/\rho \leq Cf_k\rho_k^2/\rho^2$, by (2.5). We apply $\bar{b} > \gamma^{-1}$ to (3.10):

$$0 \geq -C\frac{f_i\rho_i^2}{\rho^2} + \beta f_i\lambda_i^2 + \gamma^{-1}f_i\frac{\bar{b}_i^2}{\bar{b}^2}.$$

Using the $DP = 0$ equation (3.4) and enlarging C, we obtain

$$\begin{aligned} 0 &\geq -C\frac{f_i\rho_i^2}{\rho^2} + \beta f_i\lambda_i^2 + \gamma^{-1} f_i \frac{\rho_i^2}{\rho^2} - C\gamma^{-1}\alpha^2 f_i x_i^2 \lambda_i^2 - C\gamma^{-1}\Gamma^2\beta^2 f_i\lambda_i^2 \\ &\geq \frac{f_i\rho_i^2}{\rho^2}(\gamma^{-1} - C) + \Gamma^{-2} f_i\lambda_i^2\Big((\Gamma^2\beta) - C\gamma^{-1}(\Gamma\alpha)^2 - C\gamma^{-1}(\Gamma^2\beta)^2\Big). \end{aligned} \tag{3.14}$$

The first term is handled if γ^{-1} is large enough:

$$\gamma^{-1} \geq 2C.$$

We choose α, β as follows:

$$\alpha = \gamma^4/\Gamma, \qquad \beta = \gamma^6/\Gamma^2. \tag{3.15}$$

Let us check that the previous α, β conditions (3.7) and (3.11) are satisfied for any $\gamma^{-1} \geq 2C$, if C is large enough:

$$\frac{\alpha^2}{\beta} = \gamma^2 \leq \frac{1}{4C^2} < \frac{1}{3C}, \qquad \frac{\Gamma\beta}{\alpha} = \gamma^2 \leq \frac{1}{4C^2} < \frac{1}{2n}.$$

Finally, the coefficient of $\Gamma^{-2} f_i\lambda_i^2$ in (3.14) is

$$\gamma^6 - C\gamma^7 - C\gamma^{11} = \gamma^6(1 - C\gamma - C\gamma^5) \geq \gamma^6\left(1 - \frac{1}{2} - \frac{\gamma^4}{2}\right) > 0.$$

Overall, we obtain a contradiction to (3.14).

We conclude that for all large $\gamma^{-1} \geq 2C$ and α, β satisfying (3.15), the maximum of $P_{\alpha\beta\gamma}$ obeys the largest of the P bounds (3.3), (3.8), and (3.13):

$$\max_{B_2(0)} P_{\alpha\beta\gamma} \leq C + \ln\max(\gamma^{-1}, \beta^{-1}, \alpha^{-2}) = C + \ln(\Gamma^2\gamma^{-8}).$$

We now choose large $\gamma^{-1} = 2C = C(n)$. By (3.2), we obtain the doubling estimate

$$\frac{\max_{B_2(0)}\sigma_1}{\max_{B_1(0)}\sigma_1} \leq \exp\exp\left(C + \ln\Gamma^2\right) = \exp(C\Gamma^2).$$

□

We now modify the doubling inequality to account for "moving centers". We may control the global maximum by the maximum on any small ball.

Corollary 3.1. *Let u be a smooth solution of sigma-2 equation* (1.1) *on $B_4(0) \subset \mathbb{R}^n$. If $n = 4$, or if lower bound* (3.1) *holds for $n \geq 5$, then the following inequality is true for any $y \in B_{1/3}(0)$ and $0 < r < 4/3$:*

$$\sup_{B_2(0)} \Delta u \leq C(n, r, \|u\|_{C^1(B_3(0))}) \sup_{B_r(y)} \Delta u. \tag{3.16}$$

Proof. We first note that

$$B_1(0) \subset B_{4/3}(y) \subset B_{5/3}(y) \subset B_2(0),$$

for any $|y| < 1/3$. By Proposition 3.1, we find an inequality independent of the center:

$$\sup_{B_{5/3}(y)} \Delta u \leq C(n) \exp\left(C(n)\|u\|^2_{C^1(B_3(0))}\right) \sup_{B_{4/3}(y)} \Delta u. \tag{3.17}$$

We iterate this inequality about y using the rescalings

$$u_{k+1}(\bar{x}) = \left(\frac{5}{4}\right)^2 u_k\left(\frac{4}{5}(\bar{x} - y) + y\right), \qquad u_0 = u, \qquad k = 0, 1, 2, \ldots$$

It follows that each u_k satisfies (3.17). Denoting

$$C_k = C(n)\exp\left[C(n)\|u_k\|^2_{C^1(B_3(0))}\right] \leq C(n)\exp\left[\left(\frac{5}{4}\right)^{2k} C(n)\|u\|_{C^1(B_3(0))}\right],$$

we obtain for $k = 1, 2, \ldots,$

$$\sup_{B_{5/3}(y)} \Delta u \leq C_0 C_1 \cdots C_k \sup_{B_{r_{k+1}}(y)} \Delta u \leq C(k, n, \|u\|_{C^1(B_3(0))}) \sup_{B_{r_{k+1}}(y)} \Delta u, \qquad r_k = \frac{5}{3}\left(\frac{4}{5}\right)^k$$

Letting $r_{k+1} \leq r < r_k$ for some k, we combine this inequality with Proposition 3.1 again, to arrive at (3.16). □

Remark 3.1. In the uniformly elliptic case, or $a^{ij}b_{ij} \geq a^{ij}b_ib_j$ for $\lambda I \leq (a^{ij}) \leq \Lambda I$, it follows from Trudinger [T3, p. 70] that a local Alexandrov maximum principle argument gives an integral doubling inequality:

$$\sup_{B_1(0)} b \leq C\left(n, r, \frac{\Lambda}{\lambda}\right)\left(1 + \|b\|_{L^n(B_r(0))}\right).$$

In the σ_2 case, we can find an integral doubling inequality by modifying Qiu's argument, but the non-uniform ellipticity adds a nonlinear weight to the integral:

$$\sup_{B_1(0)} \ln \Delta u \leq C(n, r)\Gamma^2\left(1 + \|(\Delta u)^{2/n} \ln \Delta u\|_{L^n(B_r(0))}\right).$$

This *nonlinear* doubling inequality can be employed to reach Theorems 1.1 and 1.2, as in Section 5, Step 3.

4 Alexandrov regularity for viscosity solutions

We modify the approach of Evans-Gariepy [EG] and Chaudhuri-Trudinger [CT] to show the following Alexandrov regularity. In [EG, Theorem 1, section 6.4], the Alexandrov theorem is seen to arise from combining a gradient estimate with a “$W^{2,1}$ estimate” for convex functions. The latter can be heuristically understood from the a priori divergence structure calculation

$$\int_{B_1(0)} |D^2u|\, dx \le \int_{B_1(0)} \Delta u\, dx \le C(n)\|u\|_{L^\infty(B_2(0))}.$$

However, for k-convex functions, there is no gradient estimate, in general, and only Hölder and $W^{1,n+}$ estimates for $k > n/2$. We are not able to use Chaudhuri and Trudinger’s result in dimension $n = 4$. Yet, 2-convex solutions of $\sigma_2 = 1$ have an even stronger interior Lipschitz estimate, by Trudinger [T2], and also Chou-Wang [CW], with a similar “$W^{2,1}$ estimate” from $\Delta u = \sqrt{2 + |D^2u|^2}$, so the method of [EG] and [CT] can be applied verbatim. We record the modifications below, for completeness.

Proposition 4.1. *Let u be a viscosity solution of sigma-2 equation* (1.1) *on $B_4(0)$ with $\Delta u > 0$. Then u is twice differentiable almost everywhere in $B_4(0)$, or for almost every $x \in B_4(0)$, there is a quadratic polynomial Q such that*

$$\sup_{y\in B_r(x)} |u(y) - Q(y)| = o(r^2).$$

We begin the proof of this proposition by first recalling the weighted norm Lipschitz estimate [TW, Corollary 3.4, p. 587] for smooth solutions of $\sigma_2 = 1, \Delta u > 0$ on a smooth, strongly convex domain $\Omega \subset \mathbb{R}^n$:

$$\sup_{x,y\in\Omega: x\neq y} d_{x,y}^{n+1} \frac{|u(x) - u(y)|}{|x - y|} \le C(n) \int_\Omega |u| dx, \tag{4.1}$$

where $d_{x,y} = \min(d_x, d_y)$, and $d_x = \mathrm{dist}(x, \partial\Omega)$. By solving the Dirichlet problem [CNS] with smooth approximating boundary data, this pointwise estimate holds for viscosity solution u, if $\Omega \subset\subset B_4(0)$, i.e. u is locally Lipschitz. By Rademacher’s theorem, u is differentiable almost everywhere, with $Du \in L^\infty_{loc}$ equal the weak (distribution) gradient. By Lebesgue differentiation, for almost every $x \in B_4(0)$,

$$\lim_{r\to 0} \fint_{B_r(x)} |Du(y) - Du(x)| dy = 0. \tag{4.2}$$

For second order derivatives, we next recall the definition [CT, p. 306] that a continuous 2-convex function satisfies both $\Delta u > 0$ and $\sigma_2 > 0$ in the viscosity sense. Since viscosity solution u to $\sigma_2 = 1$ and $\Delta u > 0$ is 2-convex, we deduce from [CT, Theorem 2.4] that the weak Hessian $\partial^2 u$, interpreted as a vector-valued distribution, gives a vector-valued Radon measure $[D^2u] = [\mu^{ij}]$:

$$\int u\, \varphi_{ij}\, dx = \int \varphi\, d\mu^{ij}, \qquad \varphi \in C_0^\infty(B_4(0)).$$

Let us outline another proof. Noting that $\sum_{j\neq i} D_{jj}u \ge \Delta u - \lambda_{\max} \ge 0$ for 2-convex smooth function u, where the last inequality follows from (2.1) with $2\sigma_2 \ge 0$, via smooth approximation in C^0/L^∞ norm, we see that μ^i and also μ^e for any unit vector on $\mathbb{R}^n$ in [CT, (2.7)], are non-negative Borel measures, in turn, bounded on compact sets, that is, Radon measures for 2-convex continuous u. Readily μ^{I_n} for 2-convex continuous u in [CT, (2.6)] is also a Radon measure. Consequently, for 2-convex continuous u, $D_{ii}u = \mu^{I_n} - \mu^i$ and also $D_{ee}u = \mu^{I_n} - \mu^e$ in [CT, (2.8)] are Radon measures. This leads to another way in showing that the Hessian measures $D_{ij}u = \mu^{ij} = (\mu^{e_+e_+} - \mu^{e_-e_-})/2$ with $e_+ = (\partial_i + \partial_j)/\sqrt{2}$ and $e_- = (\partial_i - \partial_j)/\sqrt{2}$ in [CT, (2.9)], are Radon measures for all $1 \le i, j \le n$ and 2-convex continuous u.

By Lebesgue decomposition, we write $[D^2u] = D^2u\, dx + [D^2u]_s$, where $D^2u \in L^1_{loc}$ denotes the absolutely continuous part with respect to dx, and $[D^2u]_s$ is the singular part. In particular, for dx-almost every x in $B_4(0)$,

$$\lim_{r\to 0} ⨍_{B_r(x)} |D^2u(y) - D^2u(x)|dy = 0, \tag{4.3}$$

$$\lim_{r\to 0} \frac{1}{r^n} \|[D^2u]_s\|(B_r(x)) = 0. \tag{4.4}$$

Here, we denote by $\|[D^2u]_s\|$ the total variation measure of $[D^2u]_s$. In fact, these conditions plus (4.2) are precisely conditions (a)-(c) in [EG, Theorem 1, section 6.4]. We state their conclusion as a lemma, and include their proof of this fact in the Appendix.

Lemma 4.1. *Let $u \in C(B_4(0))$ have a weak gradient $Du \in L^1_{loc}$ which satisfies* (4.2) *for a.e. x, and a weak Hessian $\partial^2 u$ which induces a Radon measure $[D^2u] = D^2u\, dx + [D^2u]_s$ obeying conditions* (4.3) *and* (4.4) *for a.e. x. Then for a.e. $x \in B_4(0)$, it follows that*

$$⨍_{B_r(x)} \left| u(y) - u(x) - (y-x)\cdot Du(x) - \frac{1}{2}(y-x)D^2u(x)(y-x) \right| dy = o(r^2). \tag{4.5}$$

Choose x for which conditions (4.2), (4.3), and (4.4) are valid. Let $h(y) = u(y) - u(x) - (y-x)\cdot Du(0) - (y-x)\cdot D^2u(0)\cdot(y-x)/2$. Using

$$⨍_{B_r(x)} |h(y)|dy = o(r^2), \tag{4.6}$$

we will upgrade this to the desired $\|h\|_{L^\infty(B_{r/2}(x))} = o(r^2)$. The crucial ingredient is a pointwise estimate: for $0 < 2r < 4 - |x|$,

$$\sup_{y,z\in B_r(x), y\ne z} \frac{|h(y)-h(z)|}{|y-z|} \le \frac{C(n)}{r} ⨍_{B_{2r}(x)} |h(y)|dy + Cr, \tag{4.7}$$

where $C = C(n)|D^2u(x)|$. This was shown as [CT, Lemma 3.1] for k-convex functions with $k > n/2$ using the Hölder estimate [TW, Theorem 2.7], and [EG, Claim #1, p. 244] for convex functions using a gradient estimate, respectively.

Proof of (4.7). To establish (4.7), we first let $g(y) = u(y) - u(x) - (y-x)\cdot Du(x)$; then

$\sigma_2(D^2 g(y)) = 1$ with $\Delta g(y) > 0$, so gradient estimate (4.1) yields

$$\begin{aligned} r^{n+1} & \sup_{y,z\in B_r(x), y\neq z} \frac{|g(y)-g(z)|}{|y-z|} \\ & = \operatorname{dist}(\boxed{\partial B_r(x)}, \partial B_{2r}(x))^{n+1} \sup_{y,z\in \boxed{B_r(x)}, y\neq z} \frac{|g(y)-g(z)|}{|y-z|} \\ & \leq \sup_{y,z\in B_{2r}(x), y\neq z} d_{y,z}^{n+1} \frac{|g(y)-g(z)|}{|y-z|} \\ & \overset{(4.1)}{\leq} C(n) \int_{B_{2r}(x)} |g(y)| dy \\ & \leq C(n) \int_{B_{2r}(x)} |h(y)| dy + C(n) |D^2 u(x)|\, r^{n+2}, \end{aligned} \tag{4.8}$$

where $d_{y,z} := \min(2r - |y-x|, 2r - |z-x|)$. Next, we polarize

$$(y-x)\cdot D^2u(x)\cdot(y-x) - (z-x)\cdot D^2u(x)\cdot(z-x) = (y-x+z-x)\cdot D^2u(x)\cdot(y-z),$$

which gives

$$r^{n+1} \sup_{y,z\in B_r(x), y\neq z} \frac{|h(y)-h(z)|}{|y-z|} \leq r^{n+1} \sup_{y,z\in B_r(x), y\neq z} \frac{|g(y)-g(z)|}{|y-z|} + C(n) r^{n+2} |D^2u(x)|.$$

This inequality and (4.8) lead to (4.7). □

The rest of the proof follows [EG, Claim #2, p. 244] or [CT, Proof of Theorem 1.1, p. 311] verbatim. We summarize the conclusion as a lemma and include its proof in the appendix.

Lemma 4.2. *Let $h(y) \in C(B_4(0))$ and $x \in B_4(0)$ satisfy integral* (4.6) *and pointwise* (4.7) *bounds for $0 < 2r < 4 - |x|$. Then* $\sup_{B_{r/2}(x)} |h(y)| = o(r^2)$.

This completes the proof of Proposition 4.1.

Remark 4.1. In fact, Proposition 4.1 holds true for (continuous) viscosity solutions to $\sigma_k(D^2u) = 1$ for $2 \leq k \leq n/2$ in n dimensions, because the needed conditions (4.1)-(4.4) in the proof are all available. The twice differentiability a.e. for all k-convex functions and $k > n/2$, without satisfying any equation in n dimensions, is the content of the theorems by Alexandrov [EG, p. 242] and Chaudhuri-Trudinger [CT].

5 Proof of Theorems 1.1 and 1.2

Step 1. After scaling $4^2 u(x/4)$, we claim that the Hessian $D^2u(0)$ is controlled by $\|u\|_{C^1(B_4(0))}$. Otherwise, there exists a sequence of smooth solutions u_k of (1.1) on $B_4(0)$ with bound $\|u_k\|_{C^1(B_3(0))} \leq A$, but $|D^2 u_k(0)| \to \infty$, in either dimension $n = 4$, or in higher dimension $n \geq 5$ with dynamic semi-convexity (3.1). By Arzela-Ascoli, a subsequence, still denoted by u_k, uniformly converges on $B_3(0)$. By the closedness of viscosity solutions (cf.[CC]), the subsequence u_k converges uniformly to a continuous viscosity solution, abusing notation, still denoted by u, of (1.1) on $B_3(0)$; we included the non-uniformly elliptic convergence proof in the appendix, Lemma 6.1. By Alexandrov Proposition 4.1, we deduce that u is second order differentiable almost everywhere on $B_3(0)$. We fix such a point $x = y$ inside $B_{1/3}(0)$, and let $Q(x)$ be such that $u - Q = o(|x-y|^2)$.

Step 2. We apply Savin's small perturbation theorem [S] to $v_k = u_k - Q$. Given small $0 < r < 4/3$, we rescale near y:

$$\bar{v}_k(\bar{x}) = \frac{1}{r^2} v_k(r\bar{x} + y).$$

Then

$$\begin{aligned}\|\bar{v}_k\|_{L^\infty(B_1(0))} &\leq \frac{\|u_k(r\bar{x}+y) - u(r\bar{x}+y)\|_{L^\infty(B_1(0))}}{r^2} + \frac{\|u(r\bar{x}+y) - Q(r\bar{x}+y)\|_{L^\infty(B_1(0))}}{r^2} \\ &\leq \frac{\|u_k(r\bar{x}+y) - u(r\bar{x}+y)\|_{L^\infty(B_1(0))}}{r^2} + \sigma(r)\end{aligned}$$

for some modulus $\sigma(r) = o(r^2)/r^2$. And also $\bar{v}_k$ solves the elliptic PDE in $B_1(0)$

$$G(D^2\bar{w}) = \Delta\bar{w} + \Delta Q - \sqrt{2 + |D^2\bar{w} + D^2Q|^2} = 0.$$

Note that $\sigma_2(D^2Q) = 1$ with $\Delta Q > 0$, so $G(0) = 0$ with $G(M)$ smooth. Moreover, $|D^2G| \leq C(n)$, and $G(M)$ is uniformly elliptic for $|M| \leq 1$, with elliptic constants depending on n, Q.

Now we fix $r = r(n, Q, \sigma) =: \rho$ small enough such that $\sigma(\rho) < c_1/2$, where c_1 is the small constant in [S, Theorem 1.3]. As u_k uniformly converges to v, we have $\|\bar{v}_k\|_{L^\infty(B_1(0))} \leq c_1$ for all large enough k. It follows from [S, Theorem 1.3] that

$$\|u_k - Q\|_{C^{2,\alpha}(B_{\rho/2}(y))} \leq C(n, Q, \sigma),$$

with $\alpha = \alpha(n, Q, \sigma) \in (0, 1)$. This implies $\Delta u_k \leq C(n, Q, \sigma)$ on $B_{\rho/2}(y)$, uniform in k.

Step 3. Finally we apply doubling inequality (3.16) in Corollary 3.1 to u_k with $r = \rho/2$:

$$\sup_{B_2(0)} \Delta u_k \leq C(n, \rho/2, \|u_k\|_{C^1(B_3(0))}) C(n, Q, \sigma) \leq C(n, Q, \sigma, A).$$

We deduce a contradiction to the "otherwise blowup assumption" at $x = 0$.

Remark 5.1. In fact, a similar proof directly establishes interior regularity for viscosity solution u of (1.1) in four dimensions, and then the Hessian estimate, instead of first obtaining the Hessian estimate, then the interior regularity as indicated in the introduction. By rescaling $\bar{u}(\bar{x}) = u(r\bar{x} + x_0)/r^2$ at various centers, it suffices to show smoothness in $B_1(0)$, if $u \in C(B_5(0))$. By Alexandrov Proposition 4.1, we let $x = y$ be a second order differentiable point of u in $B_{1/3}(0)$, with quadratic approximation $Q(x)$ and error σ at y. By Savin's small perturbation theorem [S, Theorem 1.3], we find a ball $B_\rho(y)$ with $\rho = \rho(n, Q, \sigma)$ on which u is smooth, with estimates depending on n, Q, σ. Using [CNS], we find smooth approximations $u_k \to u$ uniformly on $B_4(0)$, with $|Du_k(x)| \leq C(\|u\|_{L^\infty(B_4(0))})$ in $B_3(0)$ by the gradient estimate in [T2] and also [CW]. By the small perturbation theorem [S, Theorem 1.3], it follows that $u_k \to u$ in $C^{2,\alpha}$ on $B_{\rho/2}(y)$. Applying doubling (3.16) to u_k with $r = \rho/2$, we find that $\Delta u_k \le C(n, Q, \sigma, \|u\|_{L^\infty(B_4(0))})$ on $B_2(0)$. By Evans-Krylov, $u_k \to u$ in $C^{2,\alpha}(B_1(0))$. It follows that u is smooth on $B_1(0)$.

From interior regularity, a compactness proof for a Hessian estimate would then follow by an application of the small perturbation theorem. Suppose $u_k \to u$ uniformly but $|D^2 u_k(0)| \to \infty$. We observe that the limit u is interior smooth. Applying Savin's small perturbation theorem to $u_k - u$, which solves a fully nonlinear elliptic PDE with smooth coefficients, implies a uniform bound on $D^2 u_k(0)$ for large k, a contradiction.

Remark 5.2. By combining Alexandrov Proposition 4.1 with [S, Theorem 1.3] as above, we find that general viscosity solutions of $\sigma_2 = 1$ on $B_1(0) \subset \mathbb{R}^n$ with $\Delta u > 0$ have partial regularity: the singular set is closed with Lebesgue measure zero. The same partial regularity also holds for (k-convex) viscosity solutions of equation $\sigma_k = 1$, because Alexandrov Proposition 4.1 is valid for such solutions as noted in Remark 4.1.

6 Appendix

Proof of Lemma 4.1. Choose $x \in B_4(0)$ for which conditions (4.2), (4.3), and (4.4) are valid. Given $r > 0$ small enough for $B_{2r}(x) \subset B_4(0)$, we just assume $x = 0$. Letting $\eta_\varepsilon(y) = \varepsilon^{-n}\eta(y/\varepsilon)$ be the standard mollifier, we set $u^\varepsilon(y) = \eta_\varepsilon * u(y)$ for $|y| < r$. Letting $Q^\varepsilon(y) = u^\varepsilon(0) + y \cdot Du^\varepsilon(0) + y \cdot D^2u(0) \cdot y/2$, we use Taylor's theorem for the linear part:

$$u^\varepsilon(y) - Q^\varepsilon(y) = \int_0^1 (1-t) y \cdot [D^2 u^\varepsilon(ty) - D^2 u(0)] \cdot y \, dt.$$

Letting $\varphi \in C_c^2(B_r(0))$ with $|\varphi(y)| \le 1$, we average over $B_r = B_r(0)$:

$$\begin{aligned} \fint_{B_r} \varphi(y)(u^\varepsilon(y) - Q^\varepsilon(y)) dy &= \int_0^1 (1-t) \left(\fint_{B_r} \varphi(y) y \cdot [D^2 u^\varepsilon(ty) - D^2 u(0)] \cdot y \, dy \right) dt \\ &= \int_0^1 \frac{1-t}{t^2} \left(\fint_{B_{rt}} \varphi(t^{-1}z) z \cdot [D^2 u^\varepsilon(z) - D^2 u(0)] \cdot z \, dz \right) dt. \end{aligned} \tag{6.1}$$

The first term converges to the Radon measure representation of the Hessian:

$$\begin{aligned} g^\varepsilon(t) &:= \int_{B_{rt}} \varphi(t^{-1}z) z \cdot D^2 u^\varepsilon(z) \cdot z \, dz \\ &\to \int_{B_{rt}} u(z) \partial_{ij}(z^i z^j \varphi(t^{-1}z) dz \qquad \text{as } \varepsilon \to 0 \\ &= \int_{B_{rt}} \varphi(t^{-1}z) z^i z^j d\mu^{ij} \\ &= \int_{B_{rt}} \varphi(t^{-1}z) z \cdot D^2 u(z) \cdot z \, dz + \int_{B_{rt}} \varphi(t^{-1}z) z^i z^j d\mu_s^{ij}. \end{aligned}$$

It also has a bound which is uniform in ε:

$$\begin{aligned}
\frac{g^\varepsilon(t)}{r^n t^{n+2}} &\le \frac{r^2}{(rt)^n}\int_{B_{rt}} |D^2u^\varepsilon(z)|dz \\
&= \frac{r^2}{(rt)^n}\int_{B_{rt}} \left|\int_{\mathbb{R}^n} D^2\eta_\varepsilon(z-\zeta)u(\zeta)\right| dz \\
&= \frac{r^2}{(rt)^n}\int_{B_{rt}} \left|\int_{\mathbb{R}^n} \eta_\varepsilon(z-\zeta)d[D^2u](\zeta)\right| dz \\
&\le \frac{Cr^2}{\varepsilon^n(rt)^n}\int_{B_{rt+\varepsilon}} |B_{rt}(0)\cap B_\varepsilon(\zeta)|\, d\|D^2u\|(\zeta) \\
&\le \frac{Cr^2}{\varepsilon^n(rt)^n}\min(rt,\varepsilon)^n\|D^2u\|(B_{rt+\varepsilon}) \\
&\le Cr^2\frac{\|D^2u\|(B_{rt+\varepsilon})}{(rt+\varepsilon)^n} \\
&\le Cr^2.
\end{aligned}$$

In the last inequality, we used (4.3) and (4.4), and denoted by $\|D^2u\|$ the total variation measure of $[D^2u]$. Note also, by (4.2),

$$\begin{aligned}
|Du^\varepsilon(0) - Du(0)| &\le \int_{B_\varepsilon} \eta_\varepsilon(z)|Du(z)-Du(0)|dz \\
&\le C⨍_{B_\varepsilon} |Du(z)-Du(0)|dz \\
&= o(1)_\varepsilon.
\end{aligned}$$

By the dominated convergence theorem, we send $\varepsilon \to 0$ in (6.1):

$$\begin{aligned}
⨍_{B_r}\varphi(y)(u(y)-Q(y))dy &\le Cr^2\int_0^1 ⨍_{B_{rt}} |D^2u(z)-D^2u(0)|dzdt + Cr^2\int_0^1 \frac{\|[D^2u]_s\|(B_{rt})}{(rt)^n}dt \\
&= o(r^2),
\end{aligned}$$

using (4.3) and (4.4). Taking the supremum over all such $|\varphi(y)| \le 1$, we conclude $⨍_{B_r}|h(y)|dy = o(r^2)$. This completes the proof. □

Proof of Lemma 4.2. Given $x \in B_4(0)$ such that (4.6) and (4.7) are true, we let $0 < 2r < 4-|x|$ and $0<\varepsilon<1/2$. Then by (4.6),

$$\begin{aligned}
\left|\{z\in B_r(x) : |h(z)| \ge \varepsilon r^2\}\right| &\le \frac{1}{\varepsilon r^2}\int_{B_r(x)} |h(z)|dz \\
&= \varepsilon^{-1}o(r^n) \\
&< \varepsilon|B_r(x)|,
\end{aligned}$$

provided $r < r_0(\varepsilon, n, h)$. Then for each $y \in B_{r/2}(x)$, there exists $z \in B_r(x)$ such that

$$|h(z)| \le \varepsilon r^2 \qquad \text{and} \qquad |y-z| \le \varepsilon r.$$

By (4.7) and (4.6), we obtain for such y,

$$\begin{aligned}|h(y)| &\le |h(z)| + \frac{|h(y)-h(z)|}{|y-z|}\varepsilon r \\ &\le \varepsilon r^2 + C(n)\varepsilon \fint_{B_{2r}(x)} |h(\zeta)| d\zeta + C(n,h)\varepsilon r^2 \\ &\le C(n,h)\varepsilon r^2.\end{aligned}$$

We conclude $\sup_{B_{r/2}(x)} |h(y)| = o(r^2)$. □

The following is standard, but for lack of reference, we include a proof.

Lemma 6.1. *If $u_k \to u$ is a uniformly convergent sequence of viscosity solutions on $B_1(0)$ of a fully nonlinear elliptic equation $F(D^2u, Du, u, x) = 0$ continuous in all variables, then u is a viscosity solution of F on $B_1(0)$.*

Proof. We show it is a subsolution. Suppose for some $x_0 \in B_1(0)$, $0 < r < \operatorname{dist}(x_0, \partial B_1(0))$, and smooth Q that $Q \ge u$ on $B_r(x_0)$ with equality at x_0. Set

$$Q_\varepsilon = Q + \varepsilon|x - x_0|^2 - \varepsilon^4.$$

We observe that

$$u_k(x_0) - Q_\varepsilon(x_0) \ge u(x_0) - Q(x_0) + \varepsilon^4 - o(1)_k > 0$$

for $k = k(\varepsilon)$ large enough. In the ring $B_\varepsilon(x_0) \setminus B_{\varepsilon/2}(x_0)$, we have

$$u_k(x) - Q_\varepsilon(x) < u(x) - Q(x) - \varepsilon^3/4 + \varepsilon^4 + o(1)_k < 0$$

for $\varepsilon = \varepsilon(r)$ small enough, and $k = k(\varepsilon)$ large enough. This means the maximum of $u_k - Q_\varepsilon$ occurs at some in $x_\varepsilon \in B_{\varepsilon/2}(x_0)$. Since u_k is a subsolution, we get

$$0 \le F(D^2Q_\varepsilon(x_\varepsilon), DQ_\varepsilon(x_\varepsilon), Q_\varepsilon(x_\varepsilon), x_\varepsilon) \to F(D^2Q(x_0), DQ(x_0), Q(x_0), x_0),$$

as $\varepsilon \to 0$. This completes the proof. □

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