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Calculus of Variations

Hessian estimate for semiconvex solutions to the sigma-2 equation

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Abstract

We derive a priori interior Hessian estimates for semiconvex solutions to the sigma-2 equation. An elusive Jacobi inequality, a transformation rule under the Legendre–Lewy transform, and a mean value inequality for the still nonuniformly elliptic equation without area structure are the key to our arguments. Previously, this result was known for almost convex solutions.

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1 Introduction

In this paper, we prove a priori Hessian estimates for semiconvex solutions to the quadratic Hessian equation

$$F\left(D^2u\right)=\sigma_2\left(\lambda\right)=\sum_{1\leq i<j\leq n}\lambda_i\lambda_j=\frac{1}{2}\left[\left(\triangle u\right)^2-\left|D^2u\right|^2\right]=1. \tag{1.1}$$

Here $\lambda_i s$ are the eigenvalues of the Hessian D^2u.

Theorem 1.1 *Let u be a smooth semiconvex solution to $\sigma_2\left(D^2u\right)=1$ on $B_R\left(0\right)\subset\mathbb{R}^n$ with $D^2u\geq -K\ I$ for any fixed $K>0$. Then*

$$\left|D^2u\left(0\right)\right|\leq C\left(n,K\right)\exp\left[C\left(n,K\right)\|Du\|^2_{L^\infty\left(B_R(0)\right)}/R^2\right].$$

Given the gradient bound in terms of K-convex function $u\left(x\right)$ (note that Trudinger's gradient estimates for σ_k equations need no semiconvexity of the solutions [12]), we can

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control D^2u in terms of the solution u in $B_{2R}(0)$ as

$$\left|D^2u(0)\right| \leq C(n,K)\exp\left[C(n,K)\,\|u\|^2_{L^\infty(B_{2R}(0))}/R^4\right].$$

One quick application of the above estimates is a rigidity result for entire semiconvex solutions with quadratic growth to (1.1): every such solution must be quadratic.

Recall any solution to the Laplace equation $\sigma_1\left(D^2u\right) = \triangle u = 1$ enjoys a priori Hessian estimates; yet there are singular solutions to the three dimensional Monge–Ampère equation $\sigma_3\left(D^2u\right) = \det D^2u = 1$ by Pogorelov [10], which automatically generalize to singular solutions to $\sigma_k\left(D^2u\right) = 1$ with $k \geq 3$ in higher dimensions $n \geq 4$.

Sixty years ago, Heinz [7] achieved a Hessian bound for solutions to equation $\sigma_2\left(D^2u\right) = 1$ in dimension two by two dimensional techniques. More than ten years ago, a Hessian bound for $\sigma_2\left(D^2u\right) = 1$ in dimension three was obtained via the minimal surface feature of the "gradient" graph $(x, Du(x))$ in the joint work with Warren [15]. Along this "integral" way, Qiu [11] has proved Hessian estimates for solutions to the three dimensional quadratic Hessian equation with $C^{1,1}$ variable right hand side. Hessian estimates for convex solutions to general quadratic Hessian equations have also been obtained via a new pointwise approach by Guan and Qiu [6]. Hessian estimates for almost convex solutions to (1.1) have been derived by a compactness argument in [9]. Hessian estimates for solutions to Monge–Ampère equation $\sigma_n\left(D^2u\right) = \det D^2u = 1$ and Hessian equations $\sigma_k\left(D^2u\right) = 1$ $(k \geq 2)$ in terms of the reciprocal of the difference between solutions and their boundary values, were derived by Pogorelov [10] and Chou–Wang [4], respectively, using Pogorelov's pointwise technique. Lastly, we also mention Hessian estimates for solutions to σ_k as well as σ_k/σ_n equations in terms of certain integrals of the Hessian by Urbas [13,14], Bao et al. [1].

Note that the almost convexity condition for solutions in Guan and Qiu ([6] (15)) and McGonagle et al. ([9], Theorem 1.1) is essential in both the respective arguments toward Hessian estimates for quadratic Hessian equations. The mean value inequality in [15] used the area structure of the equation. For semiconvex solutions, an elusive Jacobi inequality, a transformation rule under Legendre–Lewy transform, and a mean value inequality corresponding to the still nonuniformly elliptic linearized operator without area structure are essential in our proof of Theorem 1.1.

The bulk of Sect. 2 is devoted to establishing the Jacobi inequality, Proposition 2.1, $\sum F_{ij}b_{ij} \geq \sum F_{ij}b_ib_j$ with F_{ij} the linearized operator, $b = \frac{1}{4}\ln\lambda_{\max}\left(D^2u\right)$, and $u(x)$ the semiconvex solution. The difficult nature of the fully nonlinear equation (1.1) is that its linearized operator matrix $\left(F_{ij}\right)$ is not uniformly elliptic; see (2.3) and (2.4). What saves us is that the PDE for the Legendre–Lewy transform of $u(x)$ is uniformly elliptic, found in the joint work with Chang [3]. By the transformation rule Proposition 2.3, the subharmonic b in original variables corresponds to a subharmonic b in new variables for the linearized operator of the new, uniformly elliptic equation. In new variables, the local maximum principle implies a mean value inequality for the subharmonic b, which upon pulling back to original variables yields the mean value inequality in Proposition 2.4. The Hessian estimate becomes possible in Sect. 3. The Jacobi inequality combined with the divergence structure of F_{ij} allows us to bound the integral in terms of $\|Du\|_{L^\infty}$.

The Hessian estimates for general solutions ($K = \infty$) to quadratic Hessian equation $\sigma_2\left(D^2u\right) = 1$ in higher dimension $n \geq 4$ still remain an issue to us.

2 Preliminaries

Taking the gradient of both sides of the quadratic Hessian equation (1.1), we have

$$\triangle_F\ Du = 0, \tag{2.1}$$

where the linearized operator is given by

$$\triangle_F = \sum_{i,j=1}^{n} F_{ij}\partial_{ij} = \sum_{i,j=1}^{n} \partial_i \left(F_{ij}\partial_j\right), \tag{2.2}$$

with

$$\left(F_{ij}\right) = \Delta u\ I - D^2u = \sqrt{2 + \left|D^2u\right|^2}\ I - D^2u > 0. \tag{2.3}$$

Here without loss of generality, we assume $\Delta u > 0$. Otherwise the smooth Hessian D^2u would be in the $\Delta u < 0$ branch of the Eq. (1.1). Given the semiconvexity condition, the conclusion in Theorem 1.1 would be trivially true.

In passing, we add a quick proof of the quantitative ellipticity for Eq. (1.1) (again on the positive branch):

$$\frac{2}{(n+1)\lambda_1} \le F_{\lambda_1} \le (n-1)\lambda_1, \tag{2.4}$$
$$\left(\sqrt{2}-1\right)\lambda_1 \le F_{\lambda_i} \le (n-1)\lambda_1 \ \text{ for } i \ge 2,$$

which was first proved by Lin–Trudinger [8], under the convention $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. The upper bound is straightforward. For the lower bound,

$$D_1\sigma_2 = \sqrt{|\lambda|^2+2} - \lambda_1 = \frac{\left|\lambda'\right|^2+2}{\sqrt{|\lambda|^2+2}+\lambda_1} \ge \frac{2}{\sigma_1+\lambda_1} \ge \frac{2}{(n+1)\lambda_1};$$

and when $i \ge 2$,

$$D_i\sigma_2 = \sqrt{|\lambda|^2+2} - \lambda_i > \sqrt{\lambda_1^2+\lambda_i^2} - \lambda_i \ge \left(\sqrt{2}-1\right)\lambda_1,$$

since function $\sqrt{\lambda_1^2+\lambda_i^2} - \lambda_i$ is decreasing in terms of λ_i.

The gradient square $|\nabla_F v|^2$ for any smooth function v with respect to the inverse "metric" $\left(F_{ij}\right)$ is defined as

$$|\nabla_F v|^2 = \sum_{i,j=1}^{n} F_{ij}\partial_i v \partial_j v.$$

2.1 Jacobi inequality

Our objective in this subsection is to get a *quantitative* subsolution inequality for the maximum eigenvalue.

Proposition 2.1 *Let u be a smooth solution to (1.1) $\sigma_2(\lambda) = 1$. Suppose that $\lambda_1 > \lambda_2 \ge \cdots \ge \lambda_n \ge -K$ and $\lambda_1 \ge \Lambda(n,K)$ for some sufficiently large $\Lambda(n,K)$ at $x = p$. Set $b = \ln\lambda_1$. Then we have at p*

$$\triangle_F\ b \ge \varepsilon\,|\nabla_F b|^2 \tag{2.5}$$

for $\varepsilon = 1/4$, say.

Proof *Step 1. Differentiation of maximum eigenvalue.*

We derive the following formulas for smooth function $b = \ln \lambda_1$

$$|\nabla_F b|^2 = \left(b'\right)^2 \sum_{k=1}^{n} f_k u_{11k}^2 \tag{2.6}$$

and

$$\triangle_F\, b = \begin{cases} b'\left[2\sum_{i>j} -u_{ii1}u_{jj1} + \sum_{k>1} \frac{2f_k}{\lambda_1-\lambda_k} u_{kk1}^2\right] + b'' f_1 u_{111}^2 & \text{(I)} \\ +\sum_{i>1}\left[2b' + \frac{2b'}{\lambda_1-\lambda_i} f_1 + b'' f_i\right] u_{11i}^2 & \text{(II)} \\ +\sum_{i>j>1} 2b'\left(1 + \frac{f_i}{\lambda_1-\lambda_j} + \frac{f_j}{\lambda_1-\lambda_i}\right) u_{ij1}^2 & \text{(III)} \end{cases} \tag{2.7}$$

at p, where D^2u is assumed to be diagonalized and $f(\lambda) = \sigma_2(\lambda)$.

To this end, we start with the partial derivatives of the distinct eigenvalue λ_1 with respect to arbitrary unit vector $e \in \mathbb{R}^n$ at p

$$\begin{aligned} \partial_e\lambda_1 &= \partial_e u_{11}, \\ \partial_{ee}\lambda_1 &= \partial_{ee}u_{11} + \sum_{k>1} 2\frac{(\partial_e u_{1k})^2}{\lambda_1 - \lambda_k}, \end{aligned}$$

which can be reached for example by implicitly differentiating the characteristic equation

$$\det(D^2u - \lambda_1 I) = 0$$

near any point where λ_1 is distinct from the other eigenvalues.

Thus we get (2.6) at p

$$|\nabla_F b|^2 = \sum_{k=1}^{n} F_{kk}\left(b'\right)^2 u_{11k}^2 = \left(b'\right)^2 \sum_{k=1}^{n} f_k u_{11k}^2.$$

From

$$\partial_{ee}b = b'\partial_{ee}\lambda_1 + b''\left(\partial_e\lambda_1\right)^2,$$

we conclude that at p

$$\partial_{ee}b = b'\left[\partial_{ee}u_{11} + \sum_{k>1} 2\frac{(\partial_e u_{1k})^2}{\lambda_1 - \lambda_k}\right] + b''\left(\partial_e u_{11}\right)^2,$$

and

$$\begin{aligned} \triangle_F b &= \sum_{\gamma=1}^{n} F_{\gamma\gamma}\partial_{\gamma\gamma}b \\ &= \sum_{\gamma=1}^{n} F_{\gamma\gamma}b'\left(\partial_{\gamma\gamma}u_{11} + \sum_{k>1} 2\frac{\left(u_{1k\gamma}\right)^2}{\lambda_1 - \lambda_k}\right) + \sum_{\gamma=1}^{n} F_{\gamma\gamma}b'' u_{11\gamma}^2. \end{aligned} \tag{2.8}$$

Next we substitute the fourth order derivative terms $\partial_{\gamma\gamma}u_{11}$ in the above by lower order derivative terms. Differentiating equation (2.1) $\sum_{\alpha,\beta=1}^{n} F_{\alpha\beta}u_{j\alpha\beta} = 0$ and using (2.3), we obtain

$$\triangle_F u_{ij} = \sum_{\alpha,\beta=1}^{n} F_{\alpha\beta} u_{ji\alpha\beta} = \sum_{\alpha,\beta=1}^{n} -\partial_i F_{\alpha\beta} u_{j\alpha\beta} = \sum_{\alpha,\beta=1}^{n} -\left(\triangle u_i\ \delta_{\alpha\beta} - u_{i\alpha\beta}\right) u_{j\alpha\beta}$$
$$= \sum_{\alpha=1}^{n} -\left(\triangle u_i - u_{i\alpha\alpha}\right) u_{j\alpha\alpha} + \sum_{\alpha\neq\beta} u_{i\alpha\beta} u_{j\alpha\beta} = \sum_{\alpha\neq\beta} \left(u_{i\alpha\beta} u_{j\alpha\beta} - u_{i\beta\beta} u_{j\alpha\alpha}\right).$$

Plugging the above identity with $i = j = 1$ in (2.8), we have at p

$$\triangle_F b = b' \left[\sum_{i\neq j} \left(u_{ij1}^2 - u_{ii1} u_{jj1}\right) + \sum_{\gamma=1}^{n} \sum_{k>1} 2F_{\gamma\gamma} \frac{u_{1k\gamma}^2}{\lambda_1 - \lambda_k}\right] + \sum_{\gamma=1}^{n} b'' F_{\gamma\gamma} u_{11\gamma}^2.$$

Regrouping those terms $u_{\heartsuit\heartsuit 1}$ (with u_{111}), $u_{11\heartsuit}$, and $u_{\heartsuit\clubsuit 1}$ in the last expression, noting $F_{\gamma\gamma} = f_\gamma$ at p, we obtain (2.7).

Step 2. Convexity of the level set of the equation $\{M |\ F(M) = 0\}$.

We rewrite the cross terms $2\sum_{i>j} -u_{ii1}u_{jj1} - 2D^2F\left(D^2u_1, D^2u_1\right) - 2\sum_{i>j} u_{ij1}^2$ inside (I) of (2.7) in a "positive" way

$$2\sum_{i>j} -u_{ii1}u_{jj1} = \sum_{i\neq j} -t_i t_j = \frac{\left(|\lambda|^2 + 2\right)|t|^2 - \langle\lambda, t\rangle^2}{\sigma_1^2}, \tag{2.9}$$

where we denoted $t_i = u_{ii1}$. In fact, squaring the Eq. (2.1) $\sum_{i=1}^n f_i t_i = 0$ at p, or equivalently

$$\sigma_1\ \left(t_1 + \cdots + t_n\right) = \lambda_1 t_1 + \cdots + \lambda_n t_n,$$

we have

$$\sigma_1^2 \left(|t|^2 + \sum_{i\neq j} t_i t_j\right) = \langle\lambda, t\rangle^2.$$

Hence, the above "positive" way follows from Eq. (1.1) $\sigma_1^2 = |\lambda|^2 + 2$.

Step 3. Consequence of semiconvexity $\lambda_i \geq -K$

We are ready to prove the Jacobi inequality (2.5). Note that all the "off-diagonal" terms in (III) of (2.7) are nonnegative; it follows that

$$\triangle_F b - \varepsilon\,|\nabla_F b|^2 \geq (I) - \varepsilon\left(b'\right)^2 f_1 t_1^2 + (II) - \varepsilon\left(b'\right)^2 \sum_{i>1} f_i u_{11i}^2.$$

Now

$$(II) - \varepsilon\left(b'\right)^2 \sum_{i>1} f_i u_{11i}^2 = \left(b'\right)^2 \sum_{i>1} \left[2\lambda_1 + \frac{2\lambda_1}{\lambda_1 - \lambda_i} f_1 - (1+\varepsilon)\, f_i\right] u_{11i}^2$$
$$\overset{(2.3)}{\geq} \left(b'\right)^2 \sum_{i>1} \lambda_1 \left[2 - (1+\varepsilon)\,\frac{\sqrt{|\lambda|^2 + 2} - \lambda_i}{\lambda_1}\right] u_{11i}^2 \geq 0$$

for

$$\lambda_1 \geq C\,(n, K, \varepsilon) \quad \text{with } \varepsilon < 1,$$

where we used (2.11) for the last inequality.

Plugging (2.9) in (I) of (2.7), we have

$$(I) - \varepsilon b'^2 f_1 t_1^2 = \frac{1}{\lambda_1 \sigma_1^2} \left\{ \begin{array}{c} \left(|\lambda|^2 + 2\right)|t|^2 - \langle\lambda, t\rangle^2 + \sigma_1^2 \sum_{k>1} \frac{2f_k}{\lambda_1 - \lambda_k} \mathbf{t}_k^2 \\ -(1+\varepsilon)\,\frac{\sigma_1^2}{\lambda_1^2} \underbrace{\lambda_1 f_1}_{<1+0.5\lambda'^2}\ t_1^2 \end{array} \right\}.$$

Observe that

$$f_1 = \sigma_1 - \lambda_1 = \sqrt{|\lambda|^2 + 2} - \lambda_1 = \frac{|\lambda'|^2 + 2}{\sqrt{|\lambda|^2 + 2} + \lambda_1} < \frac{0.5\,|\lambda'|^2 + 1}{\lambda_1}, \tag{2.10}$$

where $\lambda' = (\lambda_2, \ldots, \lambda_n)$. We see that $\lambda_1 f_1 < 1 + 0.5\,|\lambda'|^2$ and

$$|\lambda_k| \le C\,(n, K) \quad \text{for } k \ge 2 \tag{2.11}$$

from (2.10) (cf. [3], p. 663). Indeed by the assumption $\lambda_n \ge -K$ and $|\lambda'| \le n\lambda_1$, we have

$$\begin{aligned} -\,(n-2)\,K + |\lambda'_+| &\le \lambda_n + \cdots + \lambda_2 = \frac{|\lambda'|^2 + 2}{\sqrt{|\lambda|^2 + 2} + \lambda_1} \\ &< 2 + \frac{|\lambda'|^2}{(1 + 1/n)\,|\lambda'|} \le 2 + |\lambda'_-| + \frac{|\lambda'_+|}{(1 + 1/n)}, \end{aligned}$$

where $\lambda_+ = (\lambda_2, \ldots, \lambda_m)$ and $\lambda_- = (\lambda_{m+1}, \ldots, \lambda_n)$ for $\lambda_2 \ge \cdots \ge \lambda_m \ge 0 \ge \lambda_{m+1} \ge \cdots \ge \lambda_n$. Solving the above inequality for $|\lambda_+|$, we get (2.11)

$$|\lambda_+| < (n+1)\,[2 + 2\,(n-2)\,K] = C\,(n, K)\,.$$

Consequently, $\lambda_1\,(x)$ is a distinct eigenvalue, thus smooth near $x = p$ if

$$\lambda_1\,(p) > C_{\text{smooth}}\,(n, K)\,; \tag{2.12}$$

$$c\,(n) \le \lambda_1 f_1 \le C\,(n, K)\,; \tag{2.13}$$

and also

$$\frac{\sigma_1^2}{\lambda_1^2} = 1 + o\,(1) \text{ and } \sigma_1^2 \frac{2 f_k}{\lambda_1 - \lambda_k} = [2 + o\,(1)]\,\lambda_1^2$$

for large enough λ_1 and $k \ge 2$, after recalling (2.4). Denoting $t' = (t_2, \ldots, t_n)$. It follows that

$$\begin{aligned} &\lambda_1 \sigma_1^2 \left[(I) - \varepsilon b'^2 f_1 t_1^2\right] \ge \\ &\left\{\underline{\underline{\lambda_1^2}} + [1 - \varepsilon - o\,(1)]\left(1 + 0.5\,|\lambda'|^2\right)\right\} t_1^2 + \left\{[\mathbf{3} + o\,(1)]\,\lambda_1^2 + \underline{|\lambda'|^2} + 2\right\} |t'|^2 \\ &\quad -\underline{\underline{\lambda_1^2 t_1^2}} - \underline{|\lambda'|^2\,|t'|^2} - \underbrace{2 t_1\,|\lambda'|\,\lambda_1\,|t'|}_{\text{redistribute}} \\ &\ge \begin{array}{l} \left\{[1 - \varepsilon - o\,(1)]\left(1 + 0.5\,|\lambda'|^2\right)\right\} t_1^2 + \left\{[\mathbf{3} + o\,(1)]\,\lambda_1^2 + 2\right\} |t'|^2 \\ -\left[(1 - \varepsilon - o\,(1))\left(1 + 0.5\,|\lambda'|^2\right)\right] t_1^2 - \frac{|\lambda'|^2}{(1 - \varepsilon - o(1))\left(1 + 0.5|\lambda'|^2\right)} \lambda_1^2\,|t'|^2 \end{array} \\ &\ge \left\{[\mathbf{3} + o\,(1)] - \frac{2}{1 - \varepsilon - o\,(1)}\right\} \lambda_1^2\,|t'|^2 \ge 0 \end{aligned}$$

for $\varepsilon < 1/3$, say $\varepsilon = 1/4$ for simple notation and large enough smooth

$$\lambda_1 \ge \Lambda\,(n, K) > C_{\text{smooth}}\,(n, K) \tag{2.14}$$

with $C_{\text{smooth}}\,(n, K)$ from (2.12).

We have proved the pointwise Jacobi inequality (2.5) in Proposition 2.1. □

2.2 Integral Jacobi inequality

Eventually in the proof of our Theorem 1.1, we use the following integral form of (2.5).

From now on, repeated indices represent summation, unless otherwise indicated.

Proposition 2.2 *Let u be a smooth K-convex (namely, $D^2u \geq -KI$) solution to $F\left(D^2u\right) = \sigma_2 = 1$ on B_3, and define the Lipschitz quantity*

$$b = \varepsilon \ln \max(\Lambda, \lambda_{max}) = \frac{1}{4} \ln \max(\Lambda, \lambda_{max}),$$

where the sufficiently large $\Lambda = \Lambda(n, K)$ is from (2.14). Then for all nonnegative $\varphi \in C_c^\infty(B_3)$, there holds the inequality

$$0 \geq \int_{B_3} F_{ij}\varphi_i b_j + \varphi F_{ij} b_i b_j \, dx. \tag{2.15}$$

Proof It is easy to see that the Lipschitz function $b(x)$ is smooth away from the level set $\{x | \lambda_{\max}(x) = \Lambda\}$. By Sard's theorem, we perturb Λ a tiny bit, still denoted by Λ, so that the Lipschitz $b(x)$ is smooth away from a zero measure set. Integrating by parts the pointwise Jacobi inequality (2.5) multiplied by φ, over a family of approximated domains of B_3 from the complement of the above zero measure set, we reach the integral Jacobi inequality (2.15). □

2.3 Legendre–Lewy transform

In the integral approach of [15] toward Hessian estimates for (1.1) with $n = 3$, a mean value inequality, pertaining to the area structure on the Lagrangian minimal surface $(x, Du(x)) \in \mathbb{R}^{2n}$, is used to bound $b(x)$ at $x = 0$ by its integral. However, for $n > 3$, an area-like structure is unclear to us.

To construct a mean value inequality for subsolution b, in principle, we would apply the local maximum principle, but the ellipticity constants for the linearized operator of $\sigma_2 = 1$ are not uniform. To circumvent this, we show that b is a subsolution of a new uniformly elliptic operator after a change of variables, which we describe below.

The K-convexity of u ensures that the smallest canonical angle of the "Lewy-sheared" "gradient" graph $(x, Du(x) + Kx)$ is uniformly positive, i.e. $\theta_{min} := \arctan(\lambda_{min} + K) > 0$. This means we can make a well defined Legendre reflection about the origin,

$$(x, Du(x) + Kx) = (Dw(y), y), \tag{2.16}$$

where $w(y)$ is the Legendre transform of $u + \frac{K}{2}|x|^2$. Note that $y(x) = Du(x) + Kx$ is a diffeomorphism.

We show here that this transformation preserves the linearized operator of any fully nonlinear PDE, *not just* $F\left(D^2u\right) = \sigma_2$. Geometrically, this is clear for $K = 0$ and the special Lagrangian equation $\sum_{i=1}^n \arctan(\lambda_i) = \Theta$, since at the level of "gradient" graphs, the transformation is just a reflection, or a $\pi/2$-$U(n)$ rotation followed by a conjugation, so it only changes the constant phase Θ.

Proposition 2.3 [Transformation rule] *Let u solve $F(D^2u(x)) = 0$, and its Legendre–Lewy transform $w(y)$ solve $G(D^2w(y) = -F(-KI + D^2w(y)^{-1}) = 0$. Then $L_F \approx L_G$, in the sense that for all smooth functions φ, we have*

$$\frac{\partial F}{\partial M_{ij}}(D^2u(x))\frac{\partial^2\varphi}{\partial x^i \partial x^j}(x) = \frac{\partial G}{\partial N_{ij}}(D^2w(y))\frac{\partial^2\varphi^*}{\partial y^i \partial y^j}(y), \tag{2.17}$$

where $\varphi^*(y) = \varphi(x(y))$.

Equivalently, the right hand side will not have first order terms $\partial\varphi(x(y))/\partial x^i$.

Proof We will transform the left hand side of (2.17) into its right. First,

$$\frac{\partial\varphi}{\partial x^j} = \frac{\partial y^k}{\partial x^j}\frac{\partial\varphi^*}{\partial y^k} = (K\delta_{jk} + u_{jk})\varphi_k^*.$$

Consequently,

$$F_{ij}\partial_i(\partial_j\varphi) = F_{ij}u_{ijk}\varphi_k^* + F_{ij}(K\delta_{jk} + u_{jk})\varphi_{k\ell}^*(K\delta_{i\ell} + u_{i\ell}).$$

The first term on the right hand side vanishes via the equation:

$$F_{ij}u_{ijk}\varphi_k^* = \varphi_k^*\frac{\partial}{\partial x^k}F(D^2u(x)) = 0.$$

So it remains to verify that

$$(K\delta_{i\ell} + u_{i\ell})\frac{\partial F}{\partial M_{ij}}(D^2u(x))(K\delta_{jk} + u_{jk}) = \frac{\partial G}{\partial N_{ij}}(D^2w(y)), \tag{2.18}$$

which is a little clearer in the eigenvalue dependent case

$$F(M) = f(\lambda(M)),\, G(N) = g(\mu(N)) = -f(-K + 1/\mu(N)),$$

since if the Hessian $D^2u(p)$ is diagonal at $x = p$, then $K\delta_{i\ell} + u_{i\ell} = (K + \lambda_i)\delta_{i\ell}$, so that at p, the putative equality is

$$(K + \lambda_i)^2 f_i = g_i. \tag{2.19}$$

Since $(\lambda_i + K)^2 f_i = (1/\mu_i)^2\partial f/\partial\lambda_i = \partial g/\partial\mu_i$, the result follows in this case.

Let us now return to the general situation. Using the chain rule for $F(M - KI) = -G(M^{-1})$, we get

$$\begin{aligned} F_{ij}(M - KI) &= \frac{\partial}{\partial M_{ij}}(-G(M^{-1})) \\ &= -\left.\frac{\partial G}{\partial N_{k\ell}}\right|_{M^{-1}}\frac{\partial(M^{-1})_{k\ell}}{\partial M_{ij}} \\ &= \left.\frac{\partial G}{\partial N_{k\ell}}\right|_{M^{-1}}(M^{-1})_{ki}(M^{-1})_{\ell j}, \end{aligned}$$

so upon multiplying by $K\delta_{i\ell} + u_{i\ell} = M_{i\ell}$, we obtain (2.18); in turn, the equivariance (2.17). □

Remark 2.1 The disappearance of gradient terms $D\varphi^*(y)$ in the right hand side of (2.17) depends on u solving $F(D^2u) = 0$. For comparison, without any equation for function $u(x)$, the Laplace–Beltrami operator

$$\Delta_{g(x)}\varphi(x) := \frac{1}{\sqrt{\det g(x)}}\frac{\partial}{\partial x^i}\left(\sqrt{\det g(x)}\, g^{ij}(x)\frac{\partial}{\partial x^j}\varphi(x)\right)$$

corresponding to the induced metric

$$g(x) = dx^2 + dy^2\big|_L = \left(I + (D^2u(x))^T D^2u(x)\right)dx^2$$

on the "gradient" graph $L = (x, Du(x)) \in \mathbb{R}^n \times \mathbb{R}^n$ is invariant under any rotation in $\mathbb{R}^{2n}$, in particular the Legendre transform,

$$\Delta_{g(x)}\varphi(x) = \Delta_{g(y)}\varphi^*(y).$$

This is because, by design, the invariant Laplace-Beltrami operator

$$\Delta_g = g^{ij}\partial_{ij} + g^{ij}\Gamma_{ij}^k\partial_k,$$

carries over those first order derivative terms.

We now prove the mean value inequality using a transformation argument. We suppose $Du(0) = 0$ and K is $K + 1$ in transform (2.16) for simplicity.

Proposition 2.4 [Mean value inequality] *Let u be a smooth K-convex solution to* (1.1) *on $B_3(0)$. If $b \in C(B_3)$ is a viscosity subsolution of the linearized operator* (2.2)*, then the following inequality holds:*

$$b(0) \leq C(n, K)\int_{B_1} b(x)\Delta u(x)\, dx. \tag{2.20}$$

Proof Let us first verify that transformed viscosity subsolution $b^*(y) := b(x(y))$ is a viscosity subsolution of the transformed linearized operator (2.17). We denote $y(B_3) := (K(\cdot) + Du)(B_3)$ the dual domain under the coordinate inversion. Suppose that $\psi(y) \in C^2(y(B_3))$ touches $b^*(y)$ from above near $y_0 \in y(B_3)$. Then $\psi_*(x) := \psi(y(x))$ touches $b(x)$ from above near $x_0 = x(y_0)$, so

$$F_{ij}\frac{\partial^2\psi_*}{\partial x^i\partial x^j}(x_0) \geq 0.$$

We recall $x \mapsto Kx + Du(x)$ is a diffeomorphism: letting $\varphi(x) = \psi(y(x)) \in C^2$, it follows from transformation rule (2.17) that $\varphi^*(y) = \psi(y)$ satisfies the desired inequality at y_0.

It was first shown in [3] that the equation solved by the vertical coordinate Lagrangian potential $w(y)$,

$$G(D^2w) = -F(D^2u) = -\sigma_2(-KI + (D^2w)^{-1}) = -1,$$

is conformally, uniformly elliptic for K-convex solutions u, in the sense that for $H_{ij} := \sigma_n(\lambda(D^2w))G_{ij}$, the operator $H_{ij}\partial_{ij}$ is uniformly elliptic:

$$c(n, K)I \leq (H_{ij}) = \sigma_n(1/\lambda)(G_{ij}) \leq C(n, K)I.$$

This can also be seen from (2.13) and (2.4) using the change of variables (2.19).

Since $u \in C^\infty(B_3)$ is K-convex, the gradient map $y(x) = Du(x) + Kx$ is uniformly monotone, and we have a lower bound $|y(x_I) - y(x_{II})| \geq |x_I - x_{II}|$ for each $x_I, x_{II} \in B_3$, if, abusing notation for simplicity, K is $K + 1$ in our Legendre–Lewy transform (2.16). It follows that the dual domain contains the unit ball:

$$y(B_1) = (K(\cdot) + Du)(B_1(0)) \supset B_1(0).$$

Since $b^*(y)$ is a subsolution of a uniformly elliptic operator, it follows from the local maximum principle [2, p. 36] that b^* satisfies a mean value inequality:

$$b^*(0) \leq C(n, K)\int_{B_1^y} b^*(y)dy.$$

Returning to x variables and using $x(B_1^y) \subset B_1$, we obtain

$$b(0) \leq C(n, K) \int_{B_1} b(x) \det(D^2u(x) + KI)dx.$$

Using $\lambda_i \leq C(n, K)$ with $i \geq 2$ (for the small eigenvalues) from (2.11), as well as $c(n) \leq \lambda_{max} = \lambda_1 < \Delta u$, we get

$$b(0) \leq C(n, K) \int_{B_1} b(x)\lambda_{max}(x)\, dx \leq C(n, K) \int_{B_1} b(x)\Delta u(x)\, dx,$$

as required. □

Remark 2.2 Without going through the Legendre–Lewy transform, we do not see a direct proof for Proposition 2.4. In the original x-coordinates, in general, without the K-convexity assumption on the solution $u(x)$, we have a weaker-quadratic-weight mean value inequality than the one with the linear weight Δu in Proposition 2.4. In fact, given any *smooth* positive subsolution $a(x)$, such as Δu, of linearized operator (2.2), an easy modification of the local maximum principle [5, Theorem 9.20] yields the weighted mean value inequality

$$a(0) \leq C(n) \int_{B_1} \left(\frac{\|DF\|^n}{\det DF}\right) a(x)dx,$$

where $\|DF\|$ is the maximum eigenvalue of (F_{ij}). By the eigenvalue bounds (2.4) of (F_{ij}), we have

$$\frac{\|DF\|^n}{\det DF} \leq C(n)\lambda_1^2 < C(n)(\Delta u)^2,$$

leading to the (ineffective!) mean value inequality

$$a(0) \leq C(n) \int_{B_1} a(x)(\Delta u)^2 dx.$$

Still, there follows an L^∞ Hessian bound for the solutions $u(x)$ to (1.1) in terms of the L^3 norm of the Hessian D^2u, improving a result in [14].

3 Proof of Theorem 1.1

By scaling $v(x) = u(Rx)/R^2$, we assume $R = 3$, and we assume $Du(0) = 0$ and K is $K + 1$ in (2.16) for simplicity. By Proposition 2.1, $b(x)$ is a smooth subsolution of linearized operator (2.2) when $\lambda_{max}(x) \geq \Lambda(n, K)$ is sufficiently large. Redefining it as

$$b(x) = \max\left\{\frac{1}{4}\ln \lambda_{\max}(x), \frac{1}{4}\ln \Lambda(n, K)\right\},$$

we see that $b(x)$, as the maximum of two smooth subsolutions, is a viscosity subsolution of linearized operator (2.2). By Proposition 2.4, we conclude it satisfies the mean value inequality

$$b(0) \leq C(n, K) \int_{B_1} b(x)\Delta u(x)dx.$$

Our next step is to apply the integral Jacobi inequality. Introducing a cutoff function $\varphi = 1$ on B_1 and $\varphi = 0$ outside B_2, we integrate by parts:

$$\int_{B_1} b(x)\Delta u(x)\, dx \leq \int_{B_2} \varphi^2 b(x)\Delta u(x)\, dx$$

$$\leq -\int_{B_2} \varphi^2 Db \cdot Du\,dx - 2\int_{B_2} (\varphi b) D\varphi \cdot Du\,dx.$$

The second term is easy to control if we invoke $b \leq C(n, K)\ln\lambda_{max} \leq C(n, K)\lambda_{max} \leq C(n, K)\Delta u$:

$$-2\int_{B_2} (\varphi b) D\varphi \cdot Du\,dx \leq C(n, K)\|Du\|_{L^\infty(B_2)} \int_{B_2} \Delta u\,dx \leq C(n, K)\|Du\|^2_{L^\infty(B_2)}.$$

For the first term, we start with

$$-\int_{B_2} \varphi^2 Db \cdot Du\,dx \leq C(n)\|Du\|_{L^\infty(B_2)} \int_{B_2} |Db|\,dx.$$

Next, the idea is to bound $|Db|$ by $F_{ij}b_ib_j$. Assume that $D^2u(x)$ is diagonal at $x = p$, with $u_{ii} = \lambda_i$ and $\lambda_1 \geq \cdots \geq \lambda_n$. Write $|Db(p)| \leq \sum_{i=1}^n |b_i(p)|$. For $i = 1$:

$$|b_1| \leq f_1 b_1^2 + 1/f_1 \leq f_1 b_1^2 + C(n)\lambda_{max},$$

since $f_1 \geq c(n)/\lambda_1$ from (2.4). For *each fixed* $i \geq 2$:

$$|b_i| \leq f_i b_i^2 + 1/f_i \leq f_i b_i^2 + C(n),$$

since $f_i \geq \left(\sqrt{2}-1\right)\lambda_1 \geq \left(\sqrt{2}-1\right)\sqrt{2}/n$ from (2.4) and (1.1). We conclude that in B_2,

$$|Db| \leq F_{ij}b_ib_j + C(n)\Delta u,$$

where we used $\Delta u \geq \sqrt{2n/(n-1)}$. Therefore, we see there is one term left to estimate:

$$\int_{B_2} |Db|\,dx \leq \int_{B_2} F_{ij}b_ib_j\,dx + C(n)\|Du\|_{L^\infty(B_2)}.$$

Let Φ be another cutoff, defined by $\Phi(x) = 1$ on B_2, and $\Phi = 0$ outside B_3. Applying the integral Jacobi inequality (2.15) with $\varphi = \Phi^2$, we can write

$$\int_{B_3} \Phi^2 F_{ij}b_ib_j\,dx \leq -2\int_{B_3} F_{ij}\Phi_i(\Phi b_j)\,dx \leq \frac{1}{2}\int_{B_3} \Phi^2 F_{ij}b_ib_j\,dx + 2\int_{B_3} F_{ij}\Phi_i\Phi_j\,dx$$

or

$$\int_{B_3} \Phi^2 F_{ij}b_ib_j\,dx \leq 4\int_{B_3} F_{ij}\Phi_i\Phi_j\,dx.$$

Thus, it remains to estimate this final integral. Assume again that $D^2u(x)$ is diagonal at $x = p$. Then at p, it is easy to estimate the integrand:

$$F_{ij}\Phi_i\Phi_j = f_i\Phi_i^2 \leq C\sum_{i=1}^n f_i = C\cdot(n-1)\ \Delta u.$$

We conclude the final integral has the desired bound:

$$\int_{B_3} F_{ij}\Phi_i\Phi_j\,dx \leq C(n)\|Du\|_{L^\infty(B_3)}.$$

Putting all the above pieces together, we conclude

$$b(0) \leq C(n, K)\|Du\|^2_{L^\infty(B_3)}.$$

which completes the proof of Theorem 1.1.

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