Special Lagrangian Equations

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Dedicated to Professor Gang Tian on the occasion of his 60th birthday

Abstract. We survey special Lagrangian equation and its related fully nonlinear elliptic and parabolic equations: definition, geometric background, basic properties, and progress. These include the rigidity of entire solutions, a priori Hessian estimates, construction of singular solutions, existence, the counterparts in the parabolic-curvature flow-settings, and open problems.

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1. Introduction

1.1. Definition of the equation

We start with a scalar function u with its gradient Du and Hessian D^2u. The real symmetric matrix D^2u has n many real eigenvalues $\lambda_1, \ldots, \lambda_n$. Adding them together, we have the Laplace equation

$$\Delta u = \lambda_1 + \cdots + \lambda_n = c;$$

multiplying them together, we have the Monge–Ampère equation

$$\ln \det D^2u = \ln \lambda_1 + \cdots + \ln \lambda_n = c. \tag{1.1}$$

Switching from the logarithm function to the inverse tangent function, we then have the special Lagrangian equation

$$\arctan D^2u = \arctan \lambda_1 + \cdots + \arctan \lambda_n = \Theta. \tag{1.2}$$
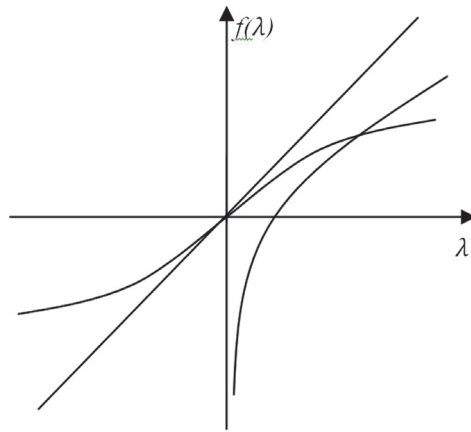

FIGURE 1. Elliptic equation corresponds to a monotonic function

The fundamental symmetric algebraic combination of those eigenvalues forms the general σ_k-equation

$$\sigma_k(\lambda) := \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k} = c.$$

General analytic combinations generate general second-order equations

$$F(D^2 u) = f(\lambda) = 0. \tag{1.3}$$

If $f(\lambda)$ is monotonic in λ_i, then the equation is elliptic (Figure 1). In principle, when the defining function f is convex (or concave), the regularity of solutions is easier to study; otherwise, it is more complicated.

1.2. Special Lagrangian submanifold background of the equation

If a half-codimensional graph $(x, F(x)) \in \mathbb{R}^n \times \mathbb{R}^n$ has a potential u such that $F(x) = Du(x)$, then it is called a Lagrangian graph. Certainly, the vector field $F(x)$ having a potential is equivalent to it being irrotational. Meanwhile, if the tangent space T of the Lagrangian submanifold is perpendicular to JT at each point, with J being the complex structure of $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{C}^n$, then $F(x)$ has a potential. Special Lagrangian submanifold means its volume is minimizing compared to all submanifolds (Lagragian or not) with the same boundary.

Harvey–Lawson[14] showed that the "gradient" graph $(x, Du(x))$ is volume minimizing if and only if u satisfies special Lagrangian equation (1.2), by applying the fundamental theorem of calculus to a calibration, namely the real closed n form $\mathrm{Re}(e^{-\sqrt{-1}\Theta} dz_1 \wedge \cdots \wedge dz_n)$. One obtains odd- as well as even-dimensional volume minimizing submanifolds from solving the special Lagrangian equation. Previously, the only known high-codimensional volume minimizing submanifolds were real even-dimensional complex submanifolds; the volume minimality was proved through applying the fundamental theorem of calculus to the real closed $2k$ form
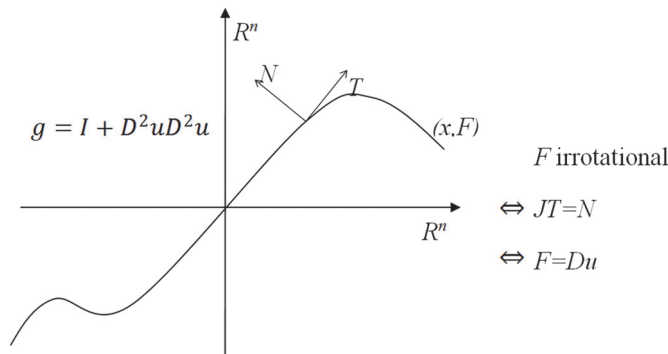
FIGURE 2. Lagrangian submanifold

$\frac{1}{k!}\omega^k$ by Wirtinger, where $\omega = \frac{1}{2\sqrt{-1}}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$. Moreover, the volume minimality of codimensional one minimal graph $(x, f(x))$ over convex domains can also be proved by applying the fundamental theorem of calculus to variable coefficient n form

$$\frac{1}{\sqrt{1+|Df|^2}}\left[dx_1 \wedge \cdots \wedge dx_n + \sum_{i=1}^{n}(-1)^{i-1}f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \wedge dx_{n+1}\right].$$

This form is closed because f satisfies the minimal surface equation

$$\mathrm{div}\left(Df/\sqrt{1+|Df|^2}\right) = 0.$$

Interestingly, there is an analogous presentation for the Monge–Ampère equation. Indeed, consider spacelike Lagrangian submanifolds in $\mathbb{R}^n \times \mathbb{R}^n$ with pseudo-Euclidean ambient metric $dx^2 - dy^2$ or $dxdy$; we can show that a spacelike "gradient" graph of u is volume maximizing if and only if u satisfies Monge–Ampère equation (1.1). In passing, let us recall the potential $|x|^{-1}$ for the three-dimensional gravitational field $-(x_1, x_2, x_3)|x|^{-3}$ satisfies the Laplace equation $\Delta |x|^{-1} = 0$.

1.3. Algebraic form of the equation

From the eigenvalues $\lambda_1, \ldots, \lambda_n$ of D^2u we define a complex number

$$z := (1 + \sqrt{-1}\lambda_1)\cdots(1 + \sqrt{-1}\lambda_n) = (1 - \sigma_2 + \cdots) + \sqrt{-1}(\sigma_1 - \sigma_3 + \cdots).$$

Denoting the phase by $\Theta = \arctan D^2u$, z can also be written as

$$z = \sqrt{(1+\lambda_1^2)\cdots(1+\lambda_n^2)}(\cos\Theta + \sqrt{-1}\sin\Theta).$$

Obviously, z is perpendicular to complex number $-\sin\Theta + \sqrt{-1}\cos\Theta$ (Figure 3), such that u satisfies

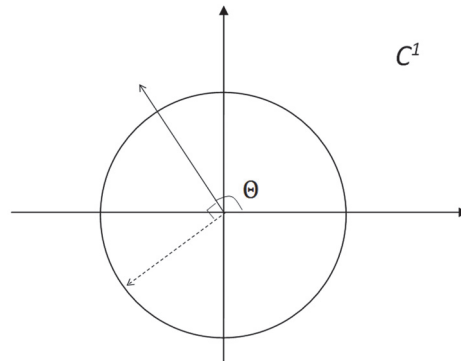$$\Sigma := \cos\Theta(\sigma_1 - \sigma_3 + \cdots) - \sin\Theta(1 - \sigma_2 + \cdots) = 0. \tag{1.4}$$

FIGURE 3. Phase $\Theta = \arctan D^2 u$

Note that σ_k has a divergence structure; thus, when u satisfies (1.2), that is, Θ is a constant, (1.4) is also an equation in divergence form. In particular, equation (1.4) has the following special forms:

- $n = 2, \Theta = 0$: $\sigma_1 = 0$;
- $n = 2$ or 3, $\Theta = \pm\frac{\pi}{2}$: $\sigma_2 = 1$;
- $n = 3, \Theta = 0$ or $\pm\pi$: $\sigma_3 = \sigma_1$, that is $\det D^2 u = \Delta u$.

It is worth noticing that the induced metric of the "gradient" graph of u is $g = I + (D^2 u) D^2 u$, such that its volume element becomes

$$\sqrt{\det g} = \sqrt{(1 + \lambda_1^2) \cdots (1 + \lambda_n^2)} = \cos \Theta (1 - \sigma_2 + \cdots) + \sin \Theta (\sigma_1 - \sigma_3 + \cdots).$$

When Θ is constant, the above volume element also has a divergence structure.

1.4. Level set of the equation

As mentioned above, the ellipticity of equation (1.3) means the defining function f is monotonic. Actually, we can also give a geometric description of the ellipticity. Consider the level set of f in λ-space; the ellipticity of the equation is equivalent to the fact that the normal of the level set $N := D_\lambda f$ falls into the positive cone Γ; namely all components of N are positive. Further, uniform ellipticity means N is uniformly inside the positive cone Γ, or all components of the unit normal $N/|N|$ have a fixed lower and upper bound. For example, Figure 4 illustrates the level sets of the three-dimensional special Lagrangian equations. In [32] we observed that the level set of the special Lagrangian equation is convex if and only if $|\Theta| \geq (n-2)\frac{\pi}{2}$. Naturally, $(n-2)\frac{\pi}{2}$ is called the critical phase. Solutions are better behaved when their equations are convex. Indeed, we have Bernstein type results for special Lagrangian equations with supercritical phase, and *a priori* estimates and regularity in the critical and supercritical cases. On the other hand, singular solutions do exist in the subcritical case.
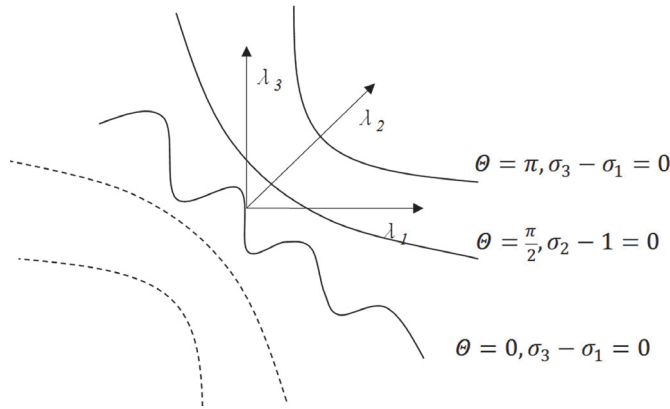
$\Theta = \pi, \sigma_3 - \sigma_1 = 0$

$\Theta = \frac{\pi}{2}, \sigma_2 - 1 = 0$

$\Theta = 0, \sigma_3 - \sigma_1 = 0$

FIGURE 4. Level sets of Θ in λ-space $(n = 3)$

2. Results

2.1. Outline

Once equations are given, the first question to answer is the existence of solutions. Smooth ones cannot be reached at once, in general; worse, they may not even exist. The usual way to compromise is to first search for weak solutions, in the integral sense if the equation has divergence structure, or in the "pointwise integration by parts sense", namely, in the viscosity sense if the equation enjoys a comparison principle. After obtaining those weak solutions, one studies the regularity and other properties of the solutions, such as Liouville or Bernstein type results for entire solutions. All these depend on *a priori* estimates of derivatives of solutions:

$$\|D^2 u\|_{L^\infty(B_1)} \le C(\|Du\|_{L^\infty(B_2)}) \le C(\|u\|_{L^\infty(B_3)}).$$

Given the L^∞ bound of the Hessian, the ellipticity of the above fully nonlinear equations becomes uniform, we can apply Evans–Krylov–Safonov theory (for the ones with convexity/concavity, possibly without divergence structure) or Evans–Krylov–De Giorgi–Nash theory (for the ones with convexity/concavity and divergence structure) to obtain $C^{2,\alpha}$ estimates of solutions. For the special Lagrangian equation, this $C^{2,\alpha}$ estimate can also be achieved via geometric measure theory; for the Monge–Ampère equation, earlier in the 1950s, Calabi reached C^3 estimates by interpreting the cubic derivatives in terms of the curvature of the corresponding Hessian metric $g = D^2 u$. In turn, iterating the classic Schauder estimates, one gains smoothness of the solutions, and even analyticity, if the smooth equations are also analytic.

2.2. Rigidity of entire solutions

The classic Liouville theorem asserts every entire harmonic function bounded from below or above is a constant. Thus every semiconvex harmonic function is a quadratic one, as its double derivatives are all harmonic with lower bounds, hence constants. Similarly, every entire (convex) solution to the Monge–Ampère equation $\det D^2 u = 1$ is quadratic. This was first proved in two-dimensional case by Jörgens, later in low dimensions by Calabi, and in all dimensions by Pogorelov. Also, Cheng–Yau had a geometric proof. For the special Lagrangian equation $\arctan D^2 u = \Theta$, Yuan [31] showed every entire convex solution is quadratic. Actually the convexity condition can be relaxed to a semiconvex one

$$D^2 u \geq -\tan \frac{\pi}{6} - \varepsilon(n),$$

where $\varepsilon(n)$ is a small-dimensional constant. On the other hand, Yuan [32] replaced the convexity condition of solutions with the phase condition on the equation

$$|\Theta| > (n-2)\frac{\pi}{2}$$

for a rigidity result. This shows the phase $(n-2)\frac{\pi}{2}$ is indeed a critical one: all entire solutions to the special Lagrangian equation with supercritical phase must be quadratic. It is a Bernstein type result. Chang–Yuan [4] proved a similar Liouville type result for the σ_2-equation: If u is an entire solution to $\sigma_2(D^2 u) = 1$ such that

$$D^2 u \geq \left(\delta - \sqrt{\frac{2}{n(n-1)}} \right) I,$$

for any small fixed $\delta > 0$, then u is quadratic. In all the above rigidity results, certain convexity of the solutions u or lower bound of the Hessian $D^2 u$ is needed. Otherwise, there are counterexamples. For example, when $n = 2$, $u = \sin x_1 e^{x_2}$ is a nontrivial solution to $\arctan D^2 u = 0$. Whereas for $n = 3$, Warren [27] found a precious explicit solution

$$u = (x_1^2 + x_2^2)e^{x_3} - e^{x_3} + \frac{1}{4}e^{-x_3}$$

to the equation $\arctan D^2 u = \frac{\pi}{2}$ or $\sigma_2(D^2 u) = 1$.

In the following, we present the idea of showing the rigidity of entire solutions to special Lagrangian equation in the two-dimensional case as an example. Given an entire solution u to $\arctan \lambda_1 + \arctan \lambda_2 = \Theta > 0$. First, notice that every dihedral angle $\arctan \lambda_1$ or $\arctan \lambda_2$ between the tangent plane of the "gradient" graph $(x, Du) \subset \mathbb{R}^2 \times \mathbb{R}^2$ and x plane has a lower bound $\Theta - \pi/2$. So after we rotate the x coordinate plane to another one $\bar{x} = x \cos \Theta/2 + y \sin \Theta/2$, the original tangent plane and the new coordinate $\bar{x}$ plane form the new dihedral angles $(\arctan \lambda_1 - \Theta/2, \arctan \lambda_2 - \Theta/2)$. Those two angles fall into the interval $(-\pi/2 + \Theta/2, \pi/2 - \Theta/2)$. This means the old "gradient" graph is still a graph in the new coordinate system $\bar{x}$ and $\bar{y} = -x \sin \Theta/2 + y \cos \Theta/2$. Further, it is another "gradient" graph $(\bar{x}, D\bar{u})$ corresponding to a new potential $\bar{u}$. It is easy to see the Hessian $D^2 \bar{u}$ of the new potential $\bar{u}$ is bounded, and moreover, its

eigenvalues satisfy equation $\arctan \bar{\lambda}_1 + \arctan \bar{\lambda}_2 = 0$. Thus, we have obtained an entire harmonic function $\bar{u}$ with bounded Hessian; in turn, $\bar{u}$ is quadratic. From this, we know the "gradient" graph is a plane. Therefore, the original entire solution u is quadratic.

For higher-dimensional special Lagrangian equation with supercritical phase, via a similar coordinate rotation, we get a new entire solution to special Lagrangian equation with critical phase. Applying Evans–Krylov's $C^{2,\alpha}$ estimates (really its scaled version in the entire space), we know the new Hessian is a constant matrix. Therefore, the original entire solution u is quadratic.

The above Liouville type result for the σ_2-equation can be proved in a similar way. As for the rigidity of entire semiconvex solutions to the special Lagrangian equation with subcritical phase, more effort is required, because the new equation loses convexity.

2.3. A priori estimates for Monge–Ampère equation

In the 1950s, Heinz [15] studied *a priori* estimates for the two-dimensional Monge–Ampère equation, a particular case is the following: If u is a solution to the equation $\det D^2 u = 1$ in the unit ball, then

$$|D^2 u(0)| \leq C(\|u\|_{L_\infty(B_1)}).$$

Later, this result was achieved in the higher-dimensional case by Pogorelov [19], but with a strict convexity restriction. Chou–Wang [10] proved similar estimates for "k-strictly" convex solutions to σ_k-equation by adapting Pogorelov's technique. Trudinger [22], Urbas [23], and Bao–Chen [1] obtained *a priori* Hessian bound in terms of the integral of the Hessian for solutions to σ_k-equation and its quotient forms. Bao–Chen–Guan–Ji [2] proved *a priori* Hessian estimates for strictly convex solutions to the quotient σ_n/σ_k type equations. If no strict convexity restriction is assumed, then Pogorelov [19] constructed his famous singular $C^{1,1-\frac{2}{n}}$ solution to the Monge–Ampère equation $\det D^2 u = 1$. Caffarelli provided merely Lipschitz solution to the Monge–Ampère equation with variable right-hand side. Furthermore, Caffarelli–Yuan obtained Lipschitz and $C^{1,\alpha}$, with α being any rational number in $(0, 1 - \frac{2}{n}]$, singular solutions to the Monge–Ampère equation $\det D^2 u = 1$.

2.4. A priori estimates for special Lagrangian equation with critical and supercritical phases

For special Lagrangian equation with critical and supercritical phases

$$\arctan D^2 u = \Theta, \quad |\Theta| \geq (n-2)\frac{\pi}{2}, \tag{2.1}$$

Wang–Yuan [24] proved the following *a priori* estimates for the Hessian (Figure 5): Suppose u is a smooth solution to special Lagrangian equation (2.1) in n-dimensional ($n \geq 3$) unit ball $B_1 \subset \mathbb{R}^n$. Then for $|\Theta| \geq (n-2)\frac{\pi}{2}$,

$$|D^2 u(0)| \leq C(n) \exp\left(C(n)\|Du\|_{L^\infty(B_1)}^{2n-2}\right); \tag{2.2}$$
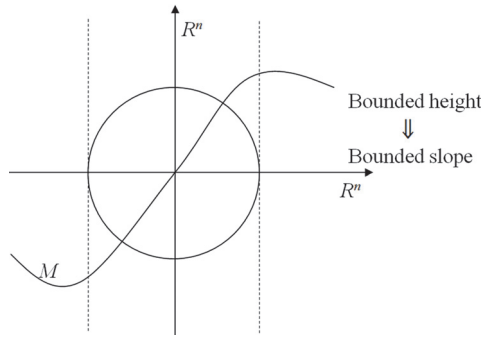
FIGURE 5. *A priori* estimate for Hessian D^2u

and for $|\Theta| = (n-2)\frac{\pi}{2}$,

$$|D^2u(0)| \leq C(n) \exp\left(C(n)\|Du\|_{L^\infty(B_1)}^{2n-4} \right). \tag{2.3}$$

Combined with the gradient estimates for equation (2.1) by Warren–Yuan [30]

$$\max_{B_R(0)} |Du| \leq C(n)\left(\mathrm{osc}_{B_{2R}(0)}\frac{u}{R} + 1 \right),$$

we immediately obtain the estimate for D^2u in terms of solution u itself. Actually the gradient estimates for equation (2.1) can be improved slightly [33]

$$\max_{B_R(0)} |Du| \leq C(n)\,\mathrm{osc}_{B_{2R}(0)}\frac{u}{R}.$$

For $n = 3$, earlier on Warren–Yuan [29, 30] proved *a priori* Hessian estimates in the critical and supercritical cases. Chen–Warren–Yuan [9] showed similar estimates for convex solutions to the special Lagrangian equation. Warren–Yuan [28] derived Hessian estimates for solutions to two-dimensional special Lagrangian equation

$$|D^2u(0)| \leq C(2) \exp\left(\frac{C(2)}{|\sin\Theta|^{\frac{3}{2}}}\|Du\|_{L^\infty(B_1)} \right).$$

From the minimal surface example by Finn [13] via Heinz transformation [16], one sees that the above Hessian bound in terms of linear exponential of gradient is sharp. For $n \geq 3$, corresponding sharp Hessian estimates are not known. As applications of the above *a priori* estimates, we immediately know all C^0 viscosity solutions to (2.1) are smooth, and even analytic. For comparison, in the 1980s Caffarelli–Nirenberg–Spruck [3] obtained the interior regularity for solutions with C^4 smooth boundary data to the special Lagrangian equation (1.2) with $|\Theta| = [\frac{n-1}{2}]\pi$. Another direct consequence is that every entire solution with quadratic growth to critical phase special Lagrangian equation

$$\arctan D^2u = (n-2)\frac{\pi}{2}$$

is quadratic.

We briefly explain the possible reason and the idea in obtaining the Hessian estimates. Heuristically, the Hessian of any solution to (2.1) in certain norm is strongly subharmonic, such that its reciprocal is superharmonic. Thus, if this superharmonic quantity is zero somewhere, then it is zero everywhere. That is, if the Hessian is unbounded at one point, then it must be unbounded everywhere. Roughly, this contradicts the graphical picture of the corresponding "gradient" graph (x, Du). A key point in the argument is to show

$$\Delta_g \frac{1}{\sqrt{1 + \lambda_{\max}^2}} \leq 0,$$

where $\lambda_{\max}$ is the maximal eigenvalue of $D^2 u$, and Δ_g is the Laplace operator with respect to the induced metric of the Lagrangian submanifold. The above superharmonicity inequality is equivalent to the Jacobi inequality

$$\Delta_g \ln \sqrt{1 + \lambda_{\max}^2} \geq |\nabla_g \ln \sqrt{1 + \lambda_{\max}^2}|^2.$$

The outline of the argument is to start from the mean value inequality on the minimal Lagrangian graph, relying on the Sobolev inequality, Jacobi inequality, and the divergence structure of $\sigma_k(D^2 u)$, then to control the integral average of the logarithm of the maximal eigenvalue $\ln \sqrt{1 + \lambda_{\max}^2}$ in terms of the gradient of the solution. The process can be viewed as an arduous nonlinearization of the mean value equality proof for the *a priori* estimate of the Hessian in terms of the gradient of a harmonic function.

2.5. Singular solutions to special Lagrangian equation with subcritical phase

For the special Lagrangian equation with subcritical phase $|\Theta| < (n - 2)\frac{\pi}{2}$, the above *a priori* Hessian estimates are not valid. Nadirashvili–Vladuct [17] first constructed $C^{1,\frac{1}{3}}$ singular solutions to three-dimensional special Lagrangian equation

$$\sum_{i=1}^{3} \arctan \lambda_i = 0.$$

For the three-dimensional special Lagrangian equation with arbitrary subcritical phase $|\Theta| \in (-\frac{\pi}{2}, \frac{\pi}{2})$, Wang–Yuan [25] constructed $C^{1,r}$ singular solutions, where $r = \frac{1}{2m-1} \in (0, \frac{1}{3}], m = 2, 3, \ldots$. To produce higher-dimensional singular solutions to subcritical special Lagrangian equation, we only need to add quadratics in terms of the extra variables to those three-dimensional singular solutions. The main new tool in [25] is a partial $U(n)$ coordinate rotation, the difficulty lies in proving that, after the rotation of preliminary solutions, the special Lagrangian submanifold is still a graph. The concrete construction goes as follows: first consider critical phase special Lagrangian equation $|\Theta| = \frac{\pi}{2}$; its algebraic equivalent form is the σ_2-equation

$$\sigma_2(D^2 u) = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 = 1. \tag{2.4}$$

We construct a family of approximate polynomials P of order $2m$ such that the dihedral angles between the tangent plane of the corresponding "gradient" graph and the x coordinate plane are roughly $(0^-, \frac{\pi}{4}, \frac{\pi}{4})$ (Figure 6). Then taking this fam-
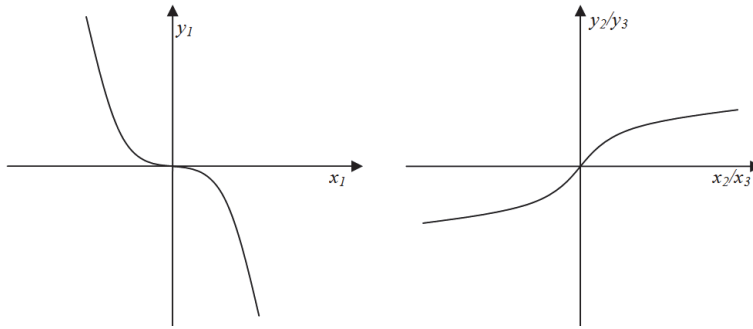
FIGURE 6. Construction of dihedral angles

ily of approximate solutions as initial data, we obtain a family of exact solutions u to equation (2.4) by Cauchy–Kowalevskaya. Next, we make a $U(3)$ coordinate rotation of $-\frac{\pi}{2}$, namely the Legendre transformation of u, to get singular $\tilde{u}$ with roughly the dihedral angles $(\frac{\pi}{2}^-, -\frac{\pi}{4}, -\frac{\pi}{4})$ satisfying the special Lagrangian equation with zero phase. Finally using a "horizontal" rotation which keeps the z_1 plane invariant, we can adjust the phase of $\tilde{u}$ to any subcritical one, to obtain the desired singular solutions.

3. Curvature flows with potential

3.1. Lagrangian mean curvature flow in Euclidean space

Under mean curvature flow, a submanifold is being deformed according to its mean curvature in the ambient space. The (effective) equation is

$$\partial_t X = \mathbf{H} = \Delta_g X,$$

where $X(\cdot, t)$ is a family of immersed submanifolds with time parameter, $\mathbf{H}$ is the mean curvature, and g is the induced metric from the ambient space. A known fact is that the Lagrangian structure of Lagrangian submanifolds is preserved under the mean curvature flow Smoczyk [20].

Meanwhile, we consider the following fully nonlinear parabolic equation satisfied by potential $u(x, t)$

$$\partial_t u = \arctan D^2 u. \tag{3.1}$$

Differentiating both sides of the equation with respect to space variables, we have

$$\partial_t(x, Du) = \sum_{i,j=1}^{n} g^{ij} \partial_{ij}(x, Du), \tag{3.2}$$

where parabolic coefficients g^{ij} are the inverse of the induced metric $g = I + D^2 u D^2 u$ of the "gradient" graph (x, Du) in Euclidean space $(\mathbb{R}^n \times \mathbb{R}^n, dx^2 + dy^2)$. The normal projection of the right-hand side of this equation (3.2) is the mean

curvature, thus the effective part of the deformation of the "gradient" graph is indeed equal to its mean curvature. In dimension one, (3.1) and (3.2) respectively simplify to

$$\partial_t u = \arctan u_{xx} \;\; \text{and} \;\; \partial_t u_x = \frac{u_{xxx}}{1 + u_{xx}^2}.$$

For the initial value problem for the potential equation (3.1) of the Lagrangian mean curvature flow, in the periodic case, namely the gradient Du_0 of initial data u_0 is a lift to $\mathbb{R}^n$ of a map from $\mathbb{T}^n$ into itself, applying Krylov's theory for fully nonlinear uniformly parabolic equation with concavity, Smoczyk–Wang [21] showed, under the "uniform" convexity assumption $0 \le D^2 u_0 \le C$ or equivalently

$$-(1 - \delta)I_n \le D^2 u_0 \le (1 - \delta)I_n, \;\; \delta > 0,$$

on the initial data, the long time existence of solutions to equation (3.1). Chau–Chen–He [5] removed the periodicity assumption on Du_0; their *a priori* estimates deteriorate as $\delta \to 0$. For weak solutions to equation (3.1) with continuous initial data on $\mathbb{R}^n$, Chen–Pang [8] proved the long time existence and uniqueness of continuous viscosity solutions. For the standard heat equation $u_t = \Delta u$, it is worth noting here that there are the Tikhonov nonuniqueness example and the finite time blow-up solution $u(x, t) = \frac{1}{\sqrt{1-t}} \exp\left(\frac{x^2}{4(1-t)}\right)$. The contrasting phenomena can be explained by the heat conduction coefficient being uniform for the standard heat equation, but degenerate for fully nonlinear parabolic equation (3.1) when the spatial Hessian becomes unbounded. Moreover, saddle solutions to (3.1) could blow up in finite time at the second spatial derivative level.

Here, we explain a result on long time existence of smooth solutions with almost convexity by Chau–Chen–Yuan [6]. If initial potential u_0 satisfies

$$-(1 + \eta)I \le D^2 u_0 \le (1 + \eta)I, \tag{3.3}$$

where $\eta = \eta(n)$ is a small-dimensional positive constant, then the potential equation (3.1) of the Lagrangian mean curvature flow has a unique long time solution $u(x, t) : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^1$ such that u is smooth for $t > 0$; and moreover

1) $-\sqrt{3}I \le D^2 u(x, t) \le \sqrt{3}I$ for any $t > 0$;
2) $\|D^l u\|_{L^\infty(\mathbb{R}^n)} \le C_l t^{2-l}$, for any $t > 0, l \ge 3$;
3) $Du(x, t)$ is $C^{\frac{1}{2}}$ with respect to t at $t = 0$.

Relying on this result, via the $U(n)$ coordinate rotation technique described in the above, we immediately obtain long time existence of smooth solutions and related estimates to equation (3.1) with locally $C^{1,1}$ convex initial data or initial data u_0 with a large phase $\arctan D^2 u_0 \ge (n - 1)\frac{\pi}{2}$.

We point out that one cannot apply Krylov's theory for fully nonlinear uniformly parabolic equation with convexity here under the almost convexity (3.3), as the convexity condition fails. To overcome the difficulty, Chau–Chen–Yuan used approximation and the compactness of the solution space. The key tools are the uniqueness of solutions by Chen–Pang and the parabolic Schauder estimate for the potential equation (3.1) of the Lagrangian mean curvature flow with certain

convexity but not the "full" convexity condition. And not surprisingly, the *a priori* estimates by Nguyen–Yuan [18] are based on the Bernstein–Liouville type results for the corresponding elliptic special Lagrangian equation.

3.2. Lagrangian mean curvature flow in pseudo-Euclidean space and Kähler–Ricci flow on Kähler manifold

We have introduced the parabolic version of the special Lagrangian equation

$$\partial_t v = \arctan D^2 v. \tag{3.4}$$

For the Monge–Ampère equation, we can consider its parabolic version too

$$\partial_t v = \ln \det D^2 v. \tag{3.5}$$

Again differentiating the equation with respect to spatial variables, we have

$$\partial_t (x, Dv) = \sum_{i,j=1}^{n} g^{ij} \partial_{ij} (x, Dv),$$

where parabolic coefficients g^{ij} are the inverse of the induced metric $g = D^2 v$ of the spacelike "gradient" graph in pseudo-Euclidean space $(\mathbb{R}^n \times \mathbb{R}^n, dxdy)$. Similarly, the normal projection of the right-hand side of this equation is the mean curvature; thus, the effective part of the deformation of the "gradient" graph is indeed equal to its mean curvature. We can also consider the parabolic complex Monge–Ampère equation, which is satisfied by a real-valued scalar function v on complex space $\mathbb{C}^m$

$$\partial_t v = \ln \det \partial \bar{\partial} v. \tag{3.6}$$

Differentiating the equation with respect to spatial variables twice, we have

$$\partial_t g_{i\bar{k}} = -R_{i\bar{k}}$$

where $g_{i\bar{k}} = v_{i\bar{k}}$ is the Kähler metric and $R_{i\bar{k}} = -\partial_i \bar{\partial}_k \ln \det \partial \bar{\partial} v$ is the Kähler–Ricci curvature. Thus the second-order parabolic potential equation (3.6) actually corresponds to the Kähler–Ricci flow in geometric analysis.

We investigate a class of self-similar solutions to the above three parabolic equations, that is, shrinking solitons in the form

$$v(x, t) = -tu \left(\frac{x}{\sqrt{-t}} \right).$$

If the above-defined v satisfies the three parabolic equations (3.4), (3.5), and (3.6) respectively, then the profile u respectively satisfies the following three elliptic equations:

$$\arctan D^2 u = \frac{1}{2} x \cdot Du(x) - u(x), \tag{3.7}$$

$$\ln \det D^2 u = \frac{1}{2} x \cdot Du(x) - u(x), \tag{3.8}$$

$$\ln \det \partial \bar{\partial} u = \frac{1}{2} x \cdot Du(x) - u(x). \tag{3.9}$$

For shrinking solitons, Chau–Chen–Yuan [7] proved the following rigidity result:

1) If u is an entire smooth solution to equation (3.7) on $\mathbb{R}^n$, then $u(x) = u(0) + \frac{1}{2}\left\langle D^2 u(0)x, x\right\rangle$.

2) If u is an entire convex smooth solution to equation (3.8) on $\mathbb{R}^n$, and satisfies $D^2 u(x) \geq \frac{2(n-1)}{|x|^2}$ near ∞, then u is quadratic.

3) If u is an entire complex convex (pluri-subharmonic, $\partial\bar{\partial} u \geq 0$) smooth solution to equation (3.9) on $\mathbb{C}^m$, and satisfies $\partial\bar{\partial} u(x) \geq \frac{2m-1}{2|x|^2}$ near ∞, then u is quadratic.

In fact, after differentiating the parabolic equations with respect to the time variable, Chau–Chen–Yuan observed that the phase function corresponding to shrinking solitons satisfies a second-order elliptic equation with an "amplifying" force term on the whole space. In dimension one, this elliptic equation can be interpreted in terms of acceleration being proportional to velocity. Hence, the changing rate of the phase function cannot be non-zero; in turn, the phase is constant. Further, notice that the right-hand side of the self-similar equation is the "excess of the potential from being quadratic" so we see that the smooth potential must be quadratic.

Let us explain more the argument for the above result by Chau–Chen–Yuan, using the first case as an example. Let $\Theta = \arctan D^2 u$. Simple calculation shows that given solution u to equation (3.7), the phase function Θ satisfies

$$\sum_{i,j=1}^{n} g^{ij}\partial_{ij}\Theta(x) = \frac{1}{2}x \cdot D\Theta(x), \qquad (3.10)$$

Where, g^{ij} being the inverse of the induced metric $g = I + D^2 u D^2 u$, has an upper bound. The above second-order elliptic equation with the "amplifying" force term allows us to construct a suitable barrier, so that we can prove that Θ attains its minimum at a finite point. Then the strong minimum principle forces Θ to be a constant. Finally, Euler's theorem on homogeneous functions, applied to equation (3.7), leads to the desired quadratic conclusion of u.

As a matter of fact, in the above case of Monge–Ampère, the inverse square lower bound on the induced metrics is a concrete condition for the metric being complete. Now if we assume the metric is complete (abstractly), then the above rigidity result for the shrinking solitons in the Monge–Ampère case (complex as well as real) is also true. This is contained in Drugan–Lu–Yuan [12]. The further observation is that the radial derivative of the phase is the negative of the scalar curvature of the corresponding Kähler metric (3.10). On the other hand, the scalar curvature for self-shrinking solitons is nonnegative. In turn, the phase function attains its maximum at the origin. Similarly we arrive at the rigidity conclusion by applying the strong maximum principle. Heuristically, the non-negativity of scalar curvature R can be seen from its equation

$$\Delta_g R \leq \frac{1}{2}rR_r + R - \frac{1}{m}R^2.$$

If R attains its minimum somewhere, then $0 \leq R_{\min} - R_{\min}^2/m$. It follows that $R \geq 0$. The proof can actually be realized when the metric is complete.

Ding–Xin [11] and Wang [26] respectively proved a Bernstein type result for self-similar real Monge–Ampère equation (3.8) and one-dimensional complex Monge–Ampère equation (3.9); namely every entire solution is quadratic.

4. Problems

Problem 1. Can one find a pointwise argument for the *a priori* Hessian estimates to the special Lagrangian equation? Our proof is in integral form. If possible, it would represent a push-forward for a long time open problem on Hessian estimates for the quadratic symmetric Hessian equation $\sigma_2(D^2 u) = 1$. The desire for such a pointwise way is because so far, we have not seen any structure in high dimensions ($n \geq 4$), as in the low-dimensional case ($n \leq 3$) for this equation, resulting in an effective mean value inequality to be employed. Recall for codimension one minimal surface equation $\mathrm{div}\left(Df/\sqrt{1 + |Df|^2}\right) = 0$, one has the classic gradient estimates for solutions

$$|Df(0)| \leq C(n) \exp\left[C(n) \|f\|_{L^\infty(B_1)}\right].$$

The proof by Bombieri–De Giorgi–Miranda in the 1960s and its simplification by Trudinger in the 1970s are both in integral form. In the 1980s, Korevaar found a strikingly simple pointwise argument. They are all based on the Jacobi inequality

$$\Delta_g \ln \sqrt{1 + |Df|^2} \geq |\nabla_g \ln \sqrt{1 + |Df|^2}|^2.$$

Problem 2. Construction of nontrivial entire solutions to the special Lagrangian equation with critical phase $\arctan D^2 u = (n-2)\pi/2$ in high dimensions ($n \geq 3$). The construction in dimension three by Warren is through separating variables with adjustment. The key for a systematic method is to search for nontrivial super and sub solutions. This is because we already have the follow-up tool to finish, namely the Hessian estimates in term of the solutions. A more urgent problem is the existence or nonexistence of nontrivial homogeneous-order two solutions to the special Lagrangian equation with subcritical phase in high dimension ($n \geq 5$). The rigidity and regularity for general special Lagrangian equation hinge on it.

Problem 3. Is every entire smooth solution to self-similar complex Monge–Ampère equation $\ln \det \partial \bar{\partial} u = \frac{1}{2} x \cdot Du(x) - u(x)$ quadratic? As mentioned above, it is indeed so in complex dimension one. Now there is known quite a lot of nontrivial entire solutions with corresponding Kähler metric being complete and non-flat to the complex Monge–Ampère equation $\ln \det \partial \bar{\partial} u = 0$, but the self-similar term on the right-hand side of the self-similar equation should still have a strong effect to force entire solutions to be trivial. Just as in the cases of self-similar codimension one minimal surface equation and self-similar special Lagrangian equation, rigidity is available, because of the self-similarity. Once self-similarity is removed, nontrivial entire solutions do exist in both cases.

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