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A Bernstein problem for special Lagrangian equations in exterior domains ☆

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ABSTRACT

We establish quadratic asymptotics for solutions to special Lagrangian equations with supercritical phases or with semiconvexity on solutions in exterior domains. The method for "convex" case is based on an "exterior" Evans-Krylov or exterior Liouville type result for general fully nonlinear elliptic equations toward constant asymptotics of bounded Hessian, and also certain rotation arguments toward Hessian bound. Our unified approach also leads to quadratic asymptotics for convex solutions to Monge-Ampère equations (previously known), quadratic Hessian equations, and inverse harmonic Hessian equations over exterior domains. The semiconvex case is based on Allard-Almgren's uniqueness of tangent cones.

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1. Introduction

In this paper, we establish exterior Bernstein type results for special Lagrangian equations with supercritical phases or with semiconvex solutions: every exterior solution is asymptotic to a quadratic polynomial at infinity.

Theorem 1.1. *Let u be a smooth solution of the special Lagrangian equation*

$$\sum_{i=1}^{n} \arctan \lambda_i(D^2 u) = \Theta \quad in \quad \mathbb{R}^n \setminus \bar{\Omega}, \tag{1.1}$$

where constant Θ satisfies $|\Theta| > (n-2)\pi/2$, Ω is a bounded domain, and $\lambda_i(D^2u)$'s denote the eigenvalues of the Hessian D^2u. Then there exists a unique quadratic polynomial $Q(x)$ such that when $n \geq 3$,

$$u(x) = Q(x) + O_k(|x|^{2-n}) \quad as \quad |x| \to \infty \tag{1.2}$$

for all $k \in \mathbb{N}$, and when $n = 2$,

$$u(x) = Q(x) + \frac{d}{2} \log x^T (I + (D^2 Q)^2) x + O_k(|x|^{-1}) \quad as \quad |x| \to \infty \tag{1.3}$$

for all $k \in \mathbb{N}$, where

$$d = \frac{1}{2\pi} \left(\int_{\partial\Omega} \cos\Theta\, u_\nu + \sin\Theta\, u_1 (u_{22}, -u_{12}) \cdot \nu \, ds - \sin\Theta\, |\Omega| \right),$$

ν is the unit outward normal of the boundary $\partial\Omega$, and the notation $\varphi(x) = O_k(|x|^m)$ means that $|D^k \varphi(x)| = O(|x|^{m-k})$.

Theorem 1.2. *Let u be a smooth solution of the special Lagrangian equation (1.1) where Ω is still a bounded domain. Suppose that all the eigenvalues $\lambda_i(D^2u)$ satisfy $\lambda_i \geq -K$ for $n \leq 4$ and $\lambda_i \geq -\frac{1}{\sqrt{3}} - \epsilon(n)$ for $n \geq 5$, where K is an arbitrary large constant and $\epsilon(n)$ is a small dimensional constant. Then the same conclusion of Theorem 1.1 holds.*

Special Lagrangian equation (1.1) is the potential equation for the minimal Lagrangian or "gradient" graph $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$, in calibrated geometry [18]. When $n = 2$, the trigonometric equation (1.1) also takes the algebraic form $\cos\Theta\, \Delta u + \sin\Theta \det D^2 u = \sin\Theta$; while for $n = 3$, and $|\Theta| = \pi$ or $\pi/2$, equation (1.1) is equivalent to $\Delta u = \det D^2 u$ or $\sigma_2(D^2u) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1$ respectively. The phase or Lagrangian angle $(n-2)\pi/2$ is called *critical*, since the level set $\{\lambda \in \mathbb{R}^n \mid \lambda \text{ satisfying } (1.1)\}$ is convex *only* when $|\Theta| \geq (n-2)\pi/2$ [36, Lemma 2.1]. Simple solution $\sin x_1 e^{x_2}$ and precious one $(x_1^2 + x_2^2)e^{x_3} - e^{x_3} + e^{-x_3}/4$ [28] to (1.1) with $\Theta = (n-2)\pi/2$, $n = 2$ and $n = 3$

respectively show that the "critical" phase condition in Theorem 1.1 as well as the lower bounds on Hessian of the solutions in Theorem 1.2 are indeed necessary. The "entire" Bernstein-Liouville type problem has been well-studied, see for instance [4,14,2,35,36,29]. However, whether one can lower the $-1/\sqrt{3}$ bound in Theorem 1.2 arbitrarily in general dimension still remains an issue to us.

Corresponding to minimal surface equations over exterior domains, there are the well-known exterior Bernstein type results only in low dimensions [3] (for $n = 2$) [24] (for $3 \leq n \leq 7$), which assert that all solutions approach linear functions asymptotically near infinity. The same linear asymptotics continue to hold in all higher dimensions, if certain necessary conditions such as the boundedness of the gradient of solutions are assumed (cf. [24]). For Monge-Ampère equations in exterior domains, there are exterior Jörgens-Calabi-Pogorelov type results [11] (for $n = 2$) and [7] [5] (for $n \geq 2$), which state that all (convex) solutions are asymptotic to quadratic polynomials near infinity.

Heuristically the plane asymptotic behavior for minimal surfaces in exterior domains (quadratic for special Lagrangian equations and linear for minimal surface systems) is "clear"—seen through the monotonicity formula—once the tangent cone at infinity is flat. As one tries to employ De Giorgi-Allard ε-regularity to locate the "flat" plane, those approximated planes over larger and larger annuli could potentially keep changing. The possible changing planes can be in fact fixed by Allard-Almgren through their uniqueness of tangent cones [1]; see also Simon [23], Lin-Wang [20]. This enables us to see the quadratic asymptotics for solutions to general special Lagrangian equations in Theorem 1.1 and, in particular Theorem 1.2, as well as linear asymptotics for solutions to minimal surface systems in exterior domains, where entire rigidity results are available, or tangent cones at infinity are flat, for example [35,36,29,21] and [19,12,33].

If we try to have a "pure" PDE approach to the exterior problems, suited for more general fully nonlinear elliptic equations such as quadratic Hessian equation and inverse harmonic Hessian equation where rich geometric structures as on minimal surfaces are not available, then we take advantage of the fully nonlinear elliptic equation with convexity compensation satisfied by the single potential in the case of Theorem 1.1. The key is to show that the Hessian of the solutions has a finite limit at infinity, corresponding to the uniqueness of tangent cones in minimal surface situation. Still unlike in the case of codimension one minimal surface equation, the gradient of the solution enjoys Moser's Harnack inequality, then the limit of the bounded gradient can be quickly drawn at infinity. Fortunately, the pure second derivatives of the solutions are supersolutions to the linearized elliptic equation, then satisfy Krylov-Safonov's weak Harnack inequality (over annuli) and Evans-Krylov's Hessian estimates. From here, the limit of the Hessian at infinity can be achieved. This is the content of Section 2, where an "exterior" Evans-Krylov, then a finer exterior Liouville theorem for general fully nonlinear uniformly elliptic concave equations with bounded Hessian (Theorem 2.1)—along with an exterior Liouville theorem for positive solutions to linear elliptic equations in nondivergence form (Theorem 2.2)—is established.

Note that our "exterior" Evans-Krylov Lemma 2.1 is essential in drawing the limit of the superharmonic Hessian at ∞ and a weak Harnack alone is not enough, as indicated by simple superharmonic function $f(x_1, x_2, x_3, x_4) = (1 + x_1^2 + x_2^2 + x_3^2)^{-1/2}$, which has no limit at ∞. Even with the $C^{2,\alpha}$, then C^3 (or curvature) estimates in [34] when the corresponding tangent cones at ∞ are flat for the solutions in Theorem 1.1 and Theorem 1.2, the third order derivatives of the exterior solutions can only be shown to decay inverse linearly. This inadequate decay cannot lead to limits of the Hessians of the exterior solutions at ∞.

There is still another hurdle in making all the above work: we need the Hessian of solutions to be bounded and the fully nonlinear concave equation to be uniformly elliptic in the case of Theorem 1.1, and the Hessian of solutions to be bounded with a "convex" restriction as well as the fully nonlinear saddle equation to be uniformly elliptic in the case of Theorem 1.2. This is done via a rotation device developed in [35,36]; see the proof of Theorem 1.1 and Theorem 1.2 in Section 3.

In passing, we make the following remarks. All solutions to special Lagrangian equation (1.1) with critical phase and with quadratic growth near infinity must have the same quadratic asymptotic behavior, if one combines the a priori gradient and Hessian estimates in [30,32,26] with our general exterior Liouville Theorem 2.1. This exterior Liouville type result and the exterior Bernstein Theorem 1.1 also hold true for continuous viscosity solutions, in light of the regularity for solutions of special Lagrangian equations [30–32,8,26].

As alluded in the above, our arguments toward Theorem 1.1 also lead to quadratic asymptotics for convex solutions to quadratic Hessian equations and inverse harmonic Hessian equations in exterior domains, for which Chang and the third author [9] and Flanders [13] obtained entire Liouville type results respectively. Our unified approach also gives a different proof for the Jörgens-Calabi-Pogorelov type result for Monge-Ampère equations in exterior domains, which, as mentioned above, have been done earlier by Ferrer-Martínez-Milán [11] and Caffarelli-Li [7]; see Section 4.

We close this introduction by the following remarks. Turning from infinity back to the origin, our unified argument also gives removable isolated singularity result for special Lagrangian equations with supercritical phase, and quadratic Hessian equations and inverse harmonic Hessian equation with convexity assumption on the solutions, if one more assumption—the gradient has a limit at the isolated singularity—is added on the solutions. Singular radial solutions to all those three equations show the last necessary assumption is indeed needed. Assuming the existence of smooth solutions to a family of fully nonlinear elliptic equations including the above three, Wang and Zhu [27] showed that the same removable singularity result with certain more general singular set.

2. Exterior Liouville theorems

In this section, we establish the following Liouville type theorem for general fully nonlinear elliptic equations with bounded Hessian in exterior domains.

Theorem 2.1. *Let u be a smooth solution of*

$$F(D^2u) = 0 \tag{2.1}$$

in the exterior domain $\mathbb{R}^n \setminus \bar{B}_1$, where $n \geq 3$, F is uniformly elliptic with the ellipticity constants λ and Λ, and also, F is either convex, or concave, or the level set $\{M \mid F(M) = 0\}$ is convex. Suppose

$$\left\|D^2u\right\|_{L^\infty(\mathbb{R}^n \setminus \bar{B}_1)} \leq K < +\infty.$$

Then there exists a unique quadratic polynomial

$$Q(x) = \frac{1}{2}x^TAx + b^Tx + c$$

such that

$$u(x) = Q(x) + O(|x|^{2-n}) \quad as \quad |x| \to \infty.$$

Furthermore, if F is infinitely smooth, then we have

$$u(x) = Q(x) + O_k(|x|^{2-n}) \quad as \quad |x| \to \infty$$

for all $k \in \mathbb{N}$.

By symmetry, we discuss only the case that F is concave, which implies that the pure second derivative u_{ee}, for any fixed direction $e \in \partial B_1$, is a subsolution of the linearized equation $F_{M_{ij}}(D^2u(x))v_{ij} = 0$ of (2.1). Before going further, we first collect here some preliminary results, for their proofs one may consult [6] and [16].

(1) (Krylov-Safonov's weak Harnack inequality in annulus) Let v be a nonnegative supersolution of $a^{ij}(x)v_{ij} = 0$ in $B_{(1+3\gamma)R} \setminus \bar{B}_{(1-3\gamma)R}$, where $a^{ij}(x)$ is uniformly elliptic with the ellipticity constants λ and Λ, and $0 < \gamma < 1/3$ is a constant. Then

$$\left(\frac{1}{|B_{\gamma R}|}\int\limits_{B_{(1+\gamma)R}\setminus\bar{B}_{(1-\gamma)R}} v^\delta\right)^{1/\delta} \leq C \inf_{B_{(1+\gamma)R}\setminus\bar{B}_{(1-\gamma)R}} v,$$

where $\delta = \delta(n, \lambda, \Lambda) > 0$ and $C = C(n, \lambda, \Lambda, \gamma) > 0$.

(2) (Evans-Krylov estimate) Let u be a solution of (2.1) and u_{ee} be its pure second derivative in any fixed direction $e \in \partial B_1$. Then there exist $C = C(n, \lambda, \Lambda) > 0$ and $\alpha = \alpha(n, \lambda, \Lambda) > 0$, such that

$$\operatorname*{osc}_{B_r(z)} u_{ee} \leq \operatorname*{osc}_{B_r(z)} D^2u \leq C\left(\frac{r}{R}\right)^\alpha \operatorname*{osc}_{B_R(z)} D^2u \leq 2CK\left(\frac{r}{R}\right)^\alpha$$

for any $0 < r < R$ and any $B_R(z) \subset \mathbb{R}^n \setminus \bar{B}_1$.

2.1. Limit of the Hessian

The key step toward Theorem 2.1 is the following lemma.

Lemma 2.1. *Let u be as in Theorem 2.1. Then there exists a symmetric matrix A such that*

$$D^2u(x) \to A \quad as \quad |x| \to \infty.$$

To prove this, we need only to show that, for any fixed $e \in \partial B_1$, the pure second derivative u_{ee} tends to some constant number at infinity.

Proof of Lemma 2.1. Set $w(x) = u_{ee}(x)$, $\bar{w} = \overline{\lim\limits_{|x|\to\infty}} w(x)$ and $\underline{w} = \underline{\lim\limits_{|x|\to\infty}} w(x)$ for convenience. It suffices to prove that

$$\bar{w} = \underline{w}.$$

If it were not, we would have $\bar{w} - \underline{w} =: 5d > 0$. Clearly, for any $0 < \varepsilon < d$, there exists a large constant $R = R(\varepsilon) > 1$ such that $\underline{w} - \varepsilon \le w(x) \le \bar{w} + \varepsilon$ for all $x \in B^{\mathcal{C}}_{R/2}$, and also there exists a sequence of $\underline{x}_k$ in $B^{\mathcal{C}}_{R/2}$, tending to ∞, such that

$$w(\underline{x}_k) \le \underline{w} + \varepsilon$$

for all $k \in \mathbb{Z}^+$. Then there exists a point $\bar{x}$ on the sphere $\partial B_{|\underline{x}|}$ for at least one $\underline{x} \in \{\underline{x}_k\}$, such that

$$w(\bar{x}) \ge \bar{w} - \varepsilon.$$

Otherwise, as a subsolution, $w < \bar{w} - \varepsilon$ on the spheres $\partial B_{|\underline{x}_k|}$ for all $k \in \mathbb{Z}^+$, by comparison principle, we would have $w(x) < \bar{w} - \varepsilon$ for all $x \in B^{\mathcal{C}}_{|\underline{x}_1|}$, which leads to $\bar{w} < \bar{w} - \varepsilon$, a contradiction.

Applying the Evans-Krylov estimate to $w = u_{ee}$ in $B_{|\underline{x}|-1}(\underline{x})$, we obtain

$$\operatorname*{osc}_{B_{\gamma|\underline{x}|}(\underline{x})} u_{ee} \le C \left(\frac{\gamma|\underline{x}|}{|\underline{x}| - 1} \right)^{\alpha} \operatorname*{osc}_{B_{|\underline{x}|-1}(\underline{x})} D^2u \le 4CK\gamma^{\alpha} \le d,$$

where $\gamma = \gamma(n, \lambda, \Lambda, K, d) =: \min\left\{1/6, (d/(4CK))^{1/\alpha}\right\}$. Thus we deduce that

$$w(x) \le \underline{w} + \varepsilon + d \le \bar{w} - 3d \quad \text{or} \quad \bar{w} - w(x) \ge 3d \quad \text{for} \quad x \in B_{\gamma|\underline{x}|}(\underline{x}).$$

Employing the weak Harnack inequality to the nonnegative supersolution $v(x) = \bar{w} + \varepsilon - w(x)$ in the annulus $B_{(1+3\gamma)|\underline{x}|} \setminus \bar{B}_{(1-3\gamma)|\underline{x}|}$, we obtain

$$\left(\frac{1}{|B_{\gamma|\underline{x}|}|}\int_{B_{(1+\gamma)|\underline{x}|}\setminus\bar{B}_{(1-\gamma)|\underline{x}|}} v^\delta\right)^{1/\delta} \le C \inf_{B_{(1+\gamma)|\underline{x}|}\setminus\bar{B}_{(1-\gamma)|\underline{x}|}} v \le Cv(\bar{x}) \le 2C\varepsilon.$$

Then $3d \le 2C\varepsilon$, where C is independent of ε. Letting $\varepsilon \to 0$, we get $d = 0$, a contradiction. $\square$

2.2. Finer asymptotic behavior

Once the second order derivatives of u in Theorem 2.1 have limits at infinity, we can get the asymptotic behavior for all other order derivatives of u. To this end, we first note that auxiliary functions $|x|^{-n}$, and $|x|^{-1/2}$ as well as $|x|^{2-n} - |x|^{2-n-\varepsilon}$ are indeed subsolution and supersolutions, respectively, to the linearized equations of $F(D^2u) = 0$, which now are close to constant coefficient ones, say the Laplace equation, near infinity.

Next we prove an exterior Liouville theorem for positive solutions to linear elliptic equations in nondivergence form.

Theorem 2.2. *Let v be a positive solution of $a^{ij}(x)v_{ij} = 0$ in $\mathbb{R}^n \setminus \bar{B}_1$, where $n \ge 3$, $a^{ij}(x)$ is uniformly elliptic and $a^{ij}(x) \to a^{ij}_\infty$ as $|x| \to \infty$. Then there exists a constant v_∞ such that*

$$v(x) = v_\infty + o\left(\frac{1}{|x|^{n-2-\delta}}\right) \text{ as } |x| \to \infty, \text{ for all } \delta > 0. \tag{2.2}$$

Furthermore, if we have in addition

$$|a^{ij}(x) - a^{ij}_\infty| \le \frac{C}{|x|^\alpha} \quad (x \in \mathbb{R}^n \setminus B_1) \tag{2.3}$$

for some positive constants C and α, then

$$v(x) = v_\infty + O\left(\frac{1}{|x|^{n-2}}\right) \text{ as } |x| \to \infty. \tag{2.4}$$

Proof. Without loss of generality and for simplicity of notations, we assume that $a^{ij}_\infty = \delta_{ij}$ and, say $|a^{ij}(x) - \delta_{ij}| \le 1/4$ for $|x| \ge 1$. Note also that the constants C's in the following steps might be different from line to line.

Step 1. We prove $\lim_{|x|\to\infty} v(x)$ exists and is finite in this step. Let $\bar{v} = \overline{\lim}_{|x|\to\infty} v(x)$ and $\underline{v} = \underline{\lim}_{|x|\to\infty} v(x)$. Clearly, $\bar{v} \ge \underline{v} \ge 0$.

We first prove that $\underline{v} < +\infty$. Otherwise, we would have

$$v(x) \to +\infty \quad \text{as} \quad |x| \to \infty. \tag{2.5}$$

Relying on the equation $a^{ij}(x)w_{ij} = 0$, let us take this solution $v(x)$ and supersolution $\xi(x) = 2|x|^{-1/2} - 1$ to bound subsolution $\eta(x) = |x|^{-n}$. For any $\varepsilon > 0$, according to (2.5), there exists $R_\varepsilon > 16$ such that $\varepsilon v(x) > 2$ for all x with $|x| \geq R_\varepsilon$. Then $\eta \leq \xi + \varepsilon v$ on $\partial B_{R_\varepsilon} \cup \partial B_1$. By the comparison principle, we obtain $\eta \leq \xi + \varepsilon v$ in $B_{R_\varepsilon} \setminus B_1$. In particular, at $x^* = (16, 0, ..., 0)$,

$$0 < \eta(x^*) \leq \xi(x^*) + \varepsilon v(x^*) = -1/2 + \varepsilon v(x^*).$$

Letting $\varepsilon \to 0+$, we get $0 \leq -1/2$, a contradiction.

Now we prove that $\bar{v} \leq \underline{v}$. For any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $\widetilde{v}(x) = v(x) - \underline{v} + \varepsilon > 0$ for all $x \in B^{\mathcal{C}}_{R_\varepsilon}$. Since $\varliminf\limits_{|x|\to\infty} \widetilde{v}(x) = \varepsilon$, there exist $\{x_k\}_{k=1}^{\infty}$ such that $2R_\varepsilon \leq r_k = |x_k| \to +\infty$, $r_k < r_{k+1}$ and $\widetilde{v}(x_k) \leq 2\varepsilon$. Applying the Krylov-Safonov's Harnack inequality to $\widetilde{v}$, we obtain $\widetilde{v}(x) \leq C\widetilde{v}(x_k) \leq 2C\varepsilon$ for all $x \in \partial B_{r_k}$ and all $k \in \mathbb{Z}^+$. By the comparison principle, we have $\widetilde{v}(x) \leq 2C\varepsilon$ for all $x \in B^{\mathcal{C}}_{r_1}$. By letting $|x| \to \infty$ and taking limit superior, we get $\bar{v} - \underline{v} + \varepsilon \leq 2C\varepsilon$ for any $\varepsilon > 0$. Letting $\varepsilon \to 0$, we obtain $\bar{v} \leq \underline{v}$.

Therefore, $v(x)$ tends to some finite constant v_∞ as $|x| \to \infty$.

To obtain the finer asymptotic behavior (2.2) and (2.4), we follow the arguments of [15, pp. 324–325] in the rest of the proof.

Step 2. Let $\widetilde{\delta} = \min\{\delta/2, (n-2)/2\}$. Consider the supersolution $\phi(x) = |x|^{2-n+\widetilde{\delta}}$ to $a^{ij}(x)\phi_{ij} = 0$ in $\bar{B}^{\mathcal{C}}_{R_\delta}$. Since, for any $\varepsilon > 0$, there exists $R_\varepsilon > 1$, depending on ε and v, such that

$$|v(x) - v_\infty| < \varepsilon/2, \ x \in B^{\mathcal{C}}_{R_\varepsilon},$$

we conclude that there exists $C > 0$, depending on v, but independent of ε, such that

$$v(x) - v_\infty \leq C\phi + \varepsilon, \ x \in \partial B_R \cup \partial B_{R_\delta}, \ R > \max\{R_\delta, R_\varepsilon\}.$$

Applying the comparison principle, we get

$$v(x) - v_\infty \leq C\phi + \varepsilon, \ x \in B_R \setminus B_{R_\delta}.$$

By letting first $R \to +\infty$ and then $\varepsilon \to 0+$, we deduce that

$$v(x) - v_\infty \leq C\phi \leq C|x|^{2-n+\widetilde{\delta}}, \ x \in B^{\mathcal{C}}_{R_\delta}.$$

Similarly, by considering $v_\infty - v(x)$, we have

$$v(x) - v_\infty \geq -C\phi \geq -C|x|^{2-n+\widetilde{\delta}},\ x \in B^{\mathcal{C}}_{R_\delta}.$$

In summary, the assertion (2.2) is proved.

Step 3. In light of the Hölder continuity condition (2.3) for the coefficient $a^{ij}(x)$ at infinity, as noted in the beginning, function $\widetilde{\phi}(x) = |x|^{2-n} - |x|^{2-n-\alpha/2}$ is a supersolution of $a^{ij}(x)\widetilde{\phi}_{ij} = 0$ in $\bar{B}^{\mathcal{C}}_{R_\alpha}$. By taking $\widetilde{\phi}$ in place of ϕ and following the same lines as in *Step 2*, we conclude that

$$|v(x) - v_\infty| \leq C\widetilde{\phi} \leq C|x|^{2-n},\ x \in B^{\mathcal{C}}_{R_\alpha},$$

the optimal asymptotic behavior (2.4). This finishes the proof of the lemma. □

Corollary 2.1. *Let v be a smooth solution of $a^{ij}(x)v_{ij} = 0$ in $\mathbb{R}^n \setminus \bar{B}_1$, where $n \geq 3$, $a^{ij}(x)$ is uniformly elliptic and $a^{ij}(x) \to a^{ij}_\infty$ as $|x| \to \infty$. Suppose $|Dv(x)| = O(|x|^{-1})$ as $|x| \to \infty$. Then there exists a constant v_∞ such that*

$$v(x) = v_\infty + o\left(\frac{1}{|x|^{n-2-\delta}}\right) \text{ as } |x| \to \infty,\ \text{for all } \delta > 0. \tag{2.6}$$

Furthermore, if we have in addition

$$|a^{ij}(x) - a^{ij}_\infty| \leq \frac{C}{|x|^\alpha} \quad (x \in \mathbb{R}^n \setminus B_1)$$

for some positive constants C and α, then

$$v(x) = v_\infty + O\left(\frac{1}{|x|^{n-2}}\right) \text{ as } |x| \to \infty. \tag{2.7}$$

Proof. In virtue of Theorem 2.2, we need only to show that v is bounded at least on one side.

We show this by contradiction and by following the same way as in the first part of the proof of [15, Theorem 4]. Indeed, if v were unbounded on both sides, there would exist a sequence $\{x_k\}_{k=1}^\infty$, such that $1 < |x_k| < |x_{k+1}| \to +\infty$ and $v(x_k) = 0$ for all $k \in \mathbb{Z}^+$. Then, it follows from $|Dv(x)| \leq C/|x|$ (for all $x \in B_1^{\mathcal{C}}$) that, for any $k \in \mathbb{Z}^+$ and any $x \in \partial B_{|x_k|}$, we have

$$|v(x)| = \left|\int_\gamma \frac{dv}{ds} ds\right| \leq \frac{C}{|x_k|} \cdot 2\pi|x_k| = 2C\pi,$$

where the integration path γ is the minor arc connecting x_k and x in the great circle of the sphere $\partial B_{|x_k|}$. By the maximum principle, we thus conclude that $|v(x)| \leq 2C\pi$

on $\bar{B}_{|x_{k+1}|} \setminus B_{|x_k|}$ for all $k \in \mathbb{Z}^+$. Therefore, $|v(x)| \leq 2C\pi$ on $B^{\mathcal{C}}_{|x_1|}$, contradicts the unboundedness assumption. □

Remark 2.1. Corollary 2.1 is slightly different from [15, Theorem 4] which only asserts that the limit $\lim_{|x|\to\infty} v(x)$ exists, but does not state that the limit can not be infinity. We also note that [22, Lemma 11] says that, for solution to the linear elliptic equation in divergence form on the exterior domain, if the limit exists and the dimension $n \geq 3$, then the limit must be finite. This result does not need the coefficient $a^{ij}(x)$ converges, that is "close to the Laplacian", but needs the divergence structure of the equation. For nondivergence equations, if the coefficient $a^{ij}(x)$ does not converge, there are counterexamples for the finiteness of the limit; for example, function $v(x) = \log|x|$ satisfies the nondivergence uniformly elliptic equation $(\delta_{ij} + (n-2)x_ix_j|x|^{-2})v_{ij} = 0$ in $\mathbb{R}^n \setminus \{0\}$. Furthermore, $v(x) = \log|x|$ also satisfies a fully nonlinear concave elliptic equation $F(D^2v) = (n-2)\lambda_{\min}(D^2v) + \Delta v = 0$, which shows that neither $F(D^2v) = 0$ nor its linearized equations have divergence structure for $n \geq 3$.

Now we proceed with the proof of Theorem 2.1. Note that the constants $C > 0$ appeared in the following proof might be different from line to line.

Proof of Theorem 2.1. *Step 1.* Let

$$v(x) = u(x) - \frac{1}{2}x^TAx,$$

where A comes from Lemma 2.1 and satisfies $F(A) = 0$. Then we have

$$F(D^2v + A) = F(D^2u) = 0 = F(A),$$

$$\bar{a}^{ij}v_{ij} = \int_0^1 F_{M_{ij}}(tD^2v(x) + A)dt \cdot \partial v_{ij}(x) = F(D^2v + A) - F(A) = 0,$$

$$\widehat{a}^{ij}(v_e)_{ij} = F_{M_{ij}}(D^2v(x) + A)(v_e)_{ij} = 0,$$

and

$$\begin{aligned}\widehat{a}^{ij}(v_{ee})_{ij} &= F_{M_{ij}}(D^2v(x) + A)(v_{ee})_{ij}\\ &= -F_{M_{ij},M_{kl}}(D^2v(x) + A)(v_e)_{ij}(v_e)_{kl} \geq 0,\end{aligned}$$

for all $e \in \partial B_1$, where

$$\bar{a}^{ij}(x) = \int_0^1 F_{M_{ij}}(tD^2v(x) + A)dt \quad \text{and} \quad \widehat{a}^{ij}(x) = F_{M_{ij}}(D^2v(x) + A)$$

are uniformly elliptic with the corresponding ellipticity constants depending only on n, λ and Λ.

It is clear that

$$\bar{a}^{ij}(x) \to \bar{a}^{ij}_\infty = F_{M_{ij}}(A) \quad \text{and} \quad \widehat{a}^{ij}(x) \to \widehat{a}^{ij}_\infty = F_{M_{ij}}(A),$$

since $D^2 v(x) \to 0$ $(|x| \to \infty)$ according to Lemma 2.1. Thus, by assuming without loss of generality that $\widehat{a}^{ij}_\infty = \delta_{ij}$, we have the supersolution

$$\varphi(x) = |x|^{-1/2}$$

of $\widehat{a}^{ij}(x) w_{ij} = 0$ in $\bar{B}^{\mathcal{C}}_{R_0}$ for some large $R_0 > 1$. Since, for any $e \in \partial B_1$,

$$\widehat{a}^{ij}(v_{ee})_{ij} \geq 0 \quad \text{and} \quad v_{ee}(x) \to 0 \ (|x| \to \infty), \tag{2.8}$$

we can use φ as a barrier function, as in *Step 2* of the proof of Theorem 2.2, to conclude that

$$v_{ee}(x) \leq C\varphi(x) \leq C|x|^{-1/2},$$

for some constant $C > 0$. Let $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ denote the maximal and minimal eigenvalue of the matrix M, respectively. Then we have

$$\lambda_{\max}(D^2 v)(x) \leq C|x|^{-1/2}.$$

On the other hand, since $\bar{a}^{ij}(x) v_{ij} = 0$ and $\bar{a}^{ij}(x)$ is uniformly elliptic, we get

$$\lambda_{\min}(D^2 v)(x) \geq -C\lambda_{\max}(D^2 v)(x) \geq -C|x|^{-1/2}.$$

Hence we conclude that

$$\left|D^2 u(x) - A\right| = \left|D^2 v(x)\right| \leq C|x|^{-1/2},$$

which in turn implies

$$\left|\widehat{a}^{ij}(x) - \widehat{a}^{ij}_\infty\right| \leq C|x|^{-1/2}, \ x \in B^{\mathcal{C}}_1.$$

Thus the function

$$\widetilde{\varphi}(x) = 2|x|^{2-n} - |x|^{2-n-1/4}$$

is a supersolution of $\widehat{a}^{ij}(x) w_{ij} = 0$ in $\bar{B}^{\mathcal{C}}_{R_0}$ for some other large $R_0 > 1$. Recalling (2.8) and using $\widetilde{\varphi}$ as a barrier function, as in *Step 2* or rather *Step 3* of the proof of Theorem 2.2, we conclude that

$$v_{ee}(x) \le C\widetilde{\varphi}(x) \le C|x|^{2-n}.$$

Repeating the argument above, we get

$$\lambda_{\max}(D^2 v)(x) \le C|x|^{2-n},$$
$$\lambda_{\min}(D^2 v)(x) \ge -C\lambda_{\max}(D^2 v)(x) \ge -C|x|^{2-n},$$

and hence

$$\left|D^2 u(x) - A\right| = \left|D^2 v(x)\right| \le C|x|^{2-n}. \tag{2.9}$$

Therefore,

$$\left|\bar{a}^{ij}(x) - \bar{a}^{ij}_{\infty}\right| \le C|x|^{2-n}, \ x \in B_1^{\mathcal{C}}, \tag{2.10}$$

and

$$\left|\widehat{a}^{ij}(x) - \widehat{a}^{ij}_{\infty}\right| \le C|x|^{2-n}, \ x \in B_1^{\mathcal{C}}. \tag{2.11}$$

Step 2. We follow the argument in [7, p. 567] to capture the linear and constant terms in the asymptotic quadratic polynomial $Q(x)$ for the solution $u(x)$. For any $e \in \partial B_1$, it follows from (2.9) that

$$|Dv_e| \le \left|D^2 v(x)\right| \le C|x|^{2-n} \le C|x|^{-1}.$$

Since $\widehat{a}^{ij}(v_e)_{ij} = 0$, by (2.11) and Corollary 2.1, we conclude that there exists a constant b_e such that

$$v_e(x) = b_e + O(|x|^{2-n}) \ (|x| \to \infty).$$

Let $b = (b_{e_1}, ..., b_{e_n})^T$ with $e_1, ..., e_n$ being the coordinate unit vector in $\mathbb{R}^n$ and

$$\bar{v}(x) = v(x) - b^T x = u(x) - \left(\frac{1}{2}x^T A x + b^T x\right).$$

Then

$$|Du(x) - (Ax + b)| = |D\bar{v}(x)| = O(|x|^{2-n}) \ (|x| \to \infty). \tag{2.12}$$

In particular,

$$|D\bar{v}(x)| \le C|x|^{-1}.$$

Since $\bar{a}^{ij}\bar{v}_{ij} = \bar{a}^{ij}v_{ij} = 0$, by (2.10) and Corollary 2.1, we thus deduce that there exists a constant c such that

$$\bar{v}(x) = c + O(|x|^{2-n}) \ (|x| \to \infty).$$

Let

$$Q(x) = \frac{1}{2}x^T Ax + b^T x + c.$$

Then

$$|u(x) - Q(x)| = |\bar{v}(x) - c| = O(|x|^{2-n}) \ (|x| \to \infty).$$

Step 3. For any fixed x with $|x|$ sufficiently large, let

$$E(y) = \left(\frac{2}{|x|}\right)^2 (u - Q)\left(x + \frac{|x|}{2}y\right).$$

Then

$$\underline{a}^{ij}(y)E_{ij}(y) = F(A + D^2E(y)) - F(A) = 0, \ y \in B_1,$$

where

$$\underline{a}^{ij}(y) = \int_0^1 F_{M_{ij}}(A + tD^2E(y))dt.$$

By the Evans-Krylov estimate (fully nonlinear Schauder estimate) and the Schauder estimate, we have

$$|D^kE(0)| \le C_k \|E\|_{L^\infty(B_1)} \le C_k|x|^{-n}, \text{ for all } k \in \mathbb{N},$$

and hence

$$|D^k(u - Q)(x)| \le C_k|x|^{2-n-k}, \text{ for all } k \in \mathbb{N}.$$

Step 4. The uniqueness of the quadratic polynomial $Q(x)$ can be traced from the above argument. Another way is the following. Given the asymptotic behavior of $u(x)$ to $Q(x)$ near infinity, the difference between any two quadratic asymptotics of the solution $u(x)$ is zero at infinity, and in turn, they must be the same. □

3. Proof of Theorem 1.1 and Theorem 1.2

3.0. Rotation in supercritical phase case

As in [36,35], we first make a transformation of the solution, or a $U(n)$ rotation of the ambient space $\mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n \supset \{(x, Du(x))\}$, so that the Hessian of the new potential function is bounded. By symmetry, we only consider the case $\Theta > (n-2)\pi/2$. Let $\sum_{i=1}^{n} \theta_i = (n-2)\pi/2 + n\vartheta$ with $\theta_i = \arctan \lambda_i$ and $\vartheta \in (0, \pi/n)$. Observe that

$$-\frac{\pi}{2} + n\vartheta < \theta_i < \frac{\pi}{2}.$$

The first inequality follows from $(n-2)\pi/2 + n\vartheta < \theta_i + (n-1)\pi/2$, and it enables us to extend u smoothly over $\bar{\Omega}$ such that

$$D^2 u > -\cot(2\vartheta) I \geq (1 - \cot\vartheta) I. \tag{3.1}$$

We rotate the $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ coordinate system to $(\widetilde{x}, \widetilde{y})$ by ϑ, $\widetilde{z} = e^{-\sqrt{-1}\vartheta} z$, namely $\widetilde{x} = \mathfrak{c}x + \mathfrak{s}y$ and $\widetilde{y} = -\mathfrak{s}x + \mathfrak{c}y$ with $(\mathfrak{c}, \mathfrak{s}) = (\cos\vartheta, \sin\vartheta)$. Then $(x, Du(x))$ has a new parametrization

$$\begin{cases} \widetilde{x} = \mathfrak{c}x + \mathfrak{s}Du(x), \\ \widetilde{y} = -\mathfrak{s}x + \mathfrak{c}Du(x). \end{cases} \tag{3.2}$$

Given the uniform convexity of $w(x) = u(x) + \cot\vartheta\, |x|^2/2$ from (3.1), which results in a quadratic growth of $u(x)$ from below for large x, we can assume that Ω is already enlarged to a bounded domain if necessary, such that $w(x) \geq \max_{\partial\Omega} w$ in $\mathbb{R}^n \setminus \Omega$ and $w(x) = \max_{\partial\Omega} w$ on $\partial\Omega$, and then extend w or u smoothly to $\mathbb{R}^n$ such that $D^2 w \geq I$ in $\mathbb{R}^n$. This gives the distance increasing property

$$|\widetilde{x} - \widetilde{x}^*|^2 = \sin^2\vartheta \left| \cot\vartheta\, x + Du(x) - \cot\vartheta\, x^* - Du(x^*) \right|^2 \geq \sin^2\vartheta\, |x - x^*|^2. \tag{3.3}$$

We deduce that $x \mapsto \widetilde{x}$ is a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $\widetilde{\Omega} = \widetilde{x}(\Omega)$ is a bounded domain (for more details, see *Step 1* of proofs for Theorem 4.2 and Theorem 4.3).

Next we define the new potential

$$\begin{aligned} \widetilde{u}(\widetilde{x}) &= \int^{\widetilde{x}} \widetilde{y} \cdot d\widetilde{x} = \int^{x(\widetilde{x})} \langle -\mathfrak{s}x + \mathfrak{c}Du(x), \mathfrak{c}dx + \mathfrak{s}dDu(x) \rangle \\ &= \frac{1}{2}\mathfrak{c}\mathfrak{s} \left(|Du(x(\widetilde{x}))|^2 - |x(\widetilde{x})|^2 \right) - \mathfrak{s}^2 Du(x(\widetilde{x})) \cdot x(\widetilde{x}) + u(x(\widetilde{x})), \end{aligned} \tag{3.4}$$

where we integrated by parts for the last equality. Note that the above two equivalent integrals are well-defined for diffeomorphism $x \mapsto \widetilde{x} = \mathfrak{c}x + \mathfrak{s}Du(x)$. It follows that $D\widetilde{u}(\widetilde{x}) = \widetilde{y} = -\mathfrak{s}x + \mathfrak{c}Du(x)$, and by the chain rule

$$\begin{aligned} D^2\widetilde{u} &= (-\mathfrak{s}I + \mathfrak{c}D^2u)(\mathfrak{c}I + \mathfrak{s}D^2u)^{-1} \\ &= \begin{pmatrix} \tan(\theta_1 - \vartheta) & & \\ & \ddots & \\ & & \tan(\theta_n - \vartheta) \end{pmatrix} \quad \text{when } D^2u \text{ is diagonalized.} \end{aligned}$$

Therefore $\widetilde{u}$ satisfies

$$\sum_{i=1}^{n} \arctan \lambda_i(D^2\widetilde{u}) = \frac{(n-2)\pi}{2} \quad \text{and} \quad |D^2\widetilde{u}| < \cot\vartheta \quad \text{in} \quad \mathbb{R}^n \setminus \widetilde{\Omega}.$$

3.0. Rotation in semiconvex case

Repeat the above argument by taking $n\vartheta = \pi/2 - \arctan K$ and $\vartheta = \pi/6$ respectively for $D^2u \geq -K$ and $D^2u \geq -1/\sqrt{3} - \epsilon(n) = -\tan(\pi/6 + \beta(n))$, we have new potential $\widetilde{u}$ satisfies respectively

$$\sum_{i=1}^{n} \arctan \lambda_i(D^2\widetilde{u}) = \Theta - \frac{\pi}{2} + \arctan K \quad \text{and} \quad |D^2\widetilde{u}| < \cot\vartheta \quad \text{in} \quad \mathbb{R}^n \setminus \widetilde{\Omega},$$

and

$$\sum_{i=1}^{n} \arctan \lambda_i(D^2\widetilde{u}) = \Theta - n\frac{\pi}{6} \quad \text{and} \quad |D^2\widetilde{u}| < \cot(\frac{\pi}{3} + \beta(n)) \quad \text{in} \quad \mathbb{R}^n \setminus \widetilde{\Omega}.$$

The only difference is that the distance increasing property for $D^2u \geq -(\frac{1}{\sqrt{3}} + \beta(n))$ becomes

$$|\widetilde{x} - \widetilde{x}^*|^2 \geq \frac{1}{2}(\sqrt{3} - \frac{1}{\sqrt{3}} - \epsilon(n))^2\, |x - x^*|^2.$$

3.1. Proof of Theorem 1.1 ($n \geq 3$)

Step 1. Now that $\widetilde{u}$ satisfies a uniformly elliptic fully nonlinear equation, which is also concave by the convexity observation of the level set

$$\{\lambda \in \mathbb{R}^n \,|\, \lambda \text{ satisfying (1.1) with } \Theta = (n-2)\pi/2\}$$

[36, Lemma 2.1]. Applying Theorem 2.1 or Lemma 2.1 to $\widetilde{u}$, we obtain

$$D^2\widetilde{u}(\widetilde{x}) \to \widetilde{A} \quad \text{as} \quad |\widetilde{x}| \to \infty$$

for some constant symmetric matrix $\widetilde{A}$.

Step 2. We claim that

$$\lambda_i(\widetilde{A}) < \cot\vartheta, \text{ for all } i = 1, 2, ..., n. \tag{3.5}$$

Otherwise, by rotating the $\widetilde{x}$-space to make $\widetilde{A}$ diagonal, we may assume that $\widetilde{A}_{11} = \cot\vartheta$. Then the rotated graph $\{(\widetilde{x}, D\widetilde{u}(\widetilde{x}))\}$ would have the asymptote

$$\widetilde{y}_1 = \partial_1 \widetilde{u}(\widetilde{x}) = \cot\vartheta\, \widetilde{x}_1 + \widetilde{b}_1 + O(|\widetilde{x}|^{2-n}) = \cot\vartheta\, \widetilde{x}_1 + O(1)$$

in $\{(\widetilde{x}_1, \widetilde{y}_1)\} \cap \mathbb{R}^n \setminus \bar{\widetilde{\Omega}}$, according to the asymptotic behavior of $D\widetilde{u}$ by Theorem 2.1 (see also (2.12)). Thus we infer that

$$\begin{aligned} x_1 &= \widetilde{x}_1 \cos\vartheta - \widetilde{y}_1 \sin\vartheta \\ &= \widetilde{x}_1 \cos\vartheta - \widetilde{x}_1 \cot\vartheta \sin\vartheta + O(1) = O(1), \end{aligned}$$

which states that $\mathbb{R}^n \setminus \bar{\Omega}$ is bounded in the x_1-direction (geometrically, this also means that, by rotating back to the original (x_1, y_1)-space, the "gradient" graph $\{(x, Du(x))\}$ would be inside a vertical strip of width $O(1)$ around the vertical y_1-axis), a contradiction.

It follows from the above claim that the matrix $\cos\vartheta\, I - \sin\vartheta\, \widetilde{A}$ is invertible. By the explicit formula

$$D^2 u(x) = \left(\sin\vartheta\, I + \cos\vartheta\, D^2\widetilde{u}(\widetilde{x})\right) \left(\cos\vartheta\, I - \sin\vartheta\, D^2\widetilde{u}(\widetilde{x})\right)^{-1} \tag{3.6}$$

resulting from (3.2), we conclude that

$$D^2 u(x) \to A \quad (|x| \to \infty)$$

with

$$A = \left(\sin\vartheta\, I + \cos\vartheta\, \widetilde{A}\right) \left(\cos\vartheta\, I - \sin\vartheta\, \widetilde{A}\right)^{-1}.$$

Thus

$$|D^2 u| \le C(n, \Theta, u) < +\infty \quad \text{in} \quad \mathbb{R}^n \setminus \bar{\Omega},$$

and hence the original equation (1.1) is also uniformly elliptic. Applying Theorem 2.1 to u, we complete the proof of (1.2).

3.2. Proof of Theorem 1.1 ($n = 2$)

Step 1. By rotation (3.2), we have a harmonic function $\widetilde{u}$ satisfying

$$\Delta \widetilde{u} = 0 \quad \text{and} \quad |D^2 \widetilde{u}| \le C(\Theta) \quad \text{in} \quad \mathbb{R}^2 \setminus \bar{\widetilde{\Omega}}.$$

Set $z = \widetilde{x}_1 + \sqrt{-1}\, \widetilde{x}_2$. Then the holomorphic function

$$h(z) = \partial_{\widetilde{x}_1} \widetilde{u} - \sqrt{-1}\, \partial_{\widetilde{x}_2} \widetilde{u}$$

has linear growth at infinity. By the Laurent expansion, we obtain

$$h(z) = a_1 z + a_0 + a_{-1} z^{-1} + a_{-2} z^{-2} + \cdots \tag{3.7}$$

for all large z. Since $\operatorname{Re}\int a_{-1}z^{-1}dz = \operatorname{Re}(a_{-1}\log z)$, as a part of $\widetilde{u}$, is well defined in an exterior domain, we see that a_{-1} must be a real number. Thus we have

$$D\widetilde{u}(\widetilde{x}) = D\widetilde{Q}(\widetilde{x}) + a_{-1} D \log|\widetilde{x}| + O(|\widetilde{x}|^{-2}) \quad \text{as} \quad |\widetilde{x}| \to \infty, \tag{3.8}$$

where

$$\widetilde{Q}(\widetilde{x}) = \frac{1}{2}\widetilde{x}^T \widetilde{A}\widetilde{x} + \widetilde{b}^T \widetilde{x}$$

with

$$\widetilde{A} = \begin{pmatrix} \operatorname{Re} a_1 & -\operatorname{Im} a_1 \\ -\operatorname{Im} a_1 & -\operatorname{Re} a_1 \end{pmatrix} \quad \text{and} \quad \widetilde{b} = (\operatorname{Re} a_0, -\operatorname{Im} a_0). \tag{3.9}$$

Since the Laurent series (3.7) for holomorphic function $h(z)$ is allowed to be taken derivatives term by term, it follows from (3.8) that

$$D^2\widetilde{u}(\widetilde{x}) =: \widetilde{A} + O(|\widetilde{x}|^{-2}) \quad \text{as} \quad |\widetilde{x}| \to \infty.$$

Step 2. By (3.8) and the same strip argument in the proof of (3.5), we deduce that

$$|\lambda_i(\widetilde{A})| < \cot\vartheta$$

for $i = 1, 2$, where $\vartheta = \Theta/2$. Thus the matrix $\cos\vartheta\, I - \sin\vartheta\, \widetilde{A}$ is invertible. Recall $(\mathfrak{c}, \mathfrak{s}) = (\cos\vartheta, \sin\vartheta)$. By the explicit formula (3.6), we obtain

$$\begin{aligned} D^2 u(x) &= (\mathfrak{s}I + \mathfrak{c}\widetilde{A} + O(|\widetilde{x}|^{-2}))(\mathfrak{c}I - \mathfrak{s}\widetilde{A} + O(|\widetilde{x}|^{-2}))^{-1} \\ &= (\mathfrak{s}I + \mathfrak{c}\widetilde{A})(\mathfrak{c}I - \mathfrak{s}\widetilde{A})^{-1} + O(|\widetilde{x}|^{-2}) \\ &=: A + O(|x|^{-2}), \end{aligned} \tag{3.10}$$

where in the last equality we used the inequality $C|\widetilde{x}| \geq |x|$ resulting from the distance increasing inequality (3.3).

Substituting the asymptotic behavior (3.8) of $D\widetilde{u}$ into the inverse rotation formula of (3.2), we get

$$\begin{cases} x = \mathfrak{c}\widetilde{x} - \mathfrak{s}D\widetilde{u}(\widetilde{x}) = (\mathfrak{c}I - \mathfrak{s}\widetilde{A} - \mathfrak{s}a_{-1}|\widetilde{x}|^{-2}I)\widetilde{x} - \mathfrak{s}\widetilde{b} + O(|\widetilde{x}|^{-2}), & \text{(a)} \\ Du(x) = \mathfrak{s}\widetilde{x} + \mathfrak{c}D\widetilde{u}(\widetilde{x}) = (\mathfrak{s}I + \mathfrak{c}\widetilde{A} + \mathfrak{c}a_{-1}|\widetilde{x}|^{-2}I)\widetilde{x} + \mathfrak{c}\widetilde{b} + O(|\widetilde{x}|^{-2}). & \text{(b)} \end{cases} \tag{3.11}$$

It follows from (3.11)(a) that

$$\widetilde{x} = (\mathfrak{c}I - \mathfrak{s}\widetilde{A} - \mathfrak{s}a_{-1}|\widetilde{x}|^{-2}I)^{-1}(x + \mathfrak{s}\widetilde{b}) + O(|\widetilde{x}|^{-2}). \tag{3.12}$$

Plugging (3.12) into (3.11)(b), we obtain

$$\begin{aligned}
Du(x) &= (\mathfrak{s}I + \mathfrak{c}\widetilde{A} + \mathfrak{c}a_{-1}|\widetilde{x}|^{-2}I)(\mathfrak{c}I - \mathfrak{s}\widetilde{A} - \mathfrak{s}a_{-1}|\widetilde{x}|^{-2}I)^{-1}(x + \mathfrak{s}\widetilde{b}) + \mathfrak{c}\widetilde{b} + O(|\widetilde{x}|^{-2}) \\
&= \big[(\mathfrak{s}I + \mathfrak{c}\widetilde{A})(\mathfrak{c}I - \mathfrak{s}\widetilde{A})^{-1} + (\mathfrak{s}I + \mathfrak{c}\widetilde{A})(\mathfrak{c}I - \mathfrak{s}\widetilde{A})^{-2}\mathfrak{s}a_{-1}|\widetilde{x}|^{-2} \\
&\qquad + (\mathfrak{c}I - \mathfrak{s}\widetilde{A})^{-1}\mathfrak{c}a_{-1}|\widetilde{x}|^{-2}\big](x + \mathfrak{s}\widetilde{b}) + \mathfrak{c}\widetilde{b} + O(|\widetilde{x}|^{-2}) \\
&= Ax + (\mathfrak{c}I + \mathfrak{s}A)\widetilde{b} + a_{-1}(\mathfrak{c}I + \mathfrak{s}A)(\mathfrak{c}I - \mathfrak{s}\widetilde{A})^{-1}x/|\widetilde{x}|^2 + O(|\widetilde{x}|^{-2}) \\
&= Ax + b + \frac{a_{-1}(I + A^2)x}{x^T(I + A^2)x} + O(|x|^{-2}), \qquad (3.13)
\end{aligned}$$

where we used

$$1/|\widetilde{x}|^2 = |(\mathfrak{c}I + \mathfrak{s}A)x + O(1)|^{-2} = (x^T(\mathfrak{c}I + \mathfrak{s}A)^2x)^{-2} + O(|x|^{-3}),$$

and

$$(\cos\vartheta\, I - \sin\vartheta\, \widetilde{A})^{-1} = \cos\vartheta\, I + \sin\vartheta\, A = \cos\big((\theta_1^* - \theta_2^*)/2\big)\,(I + A^2)^{1/2}$$

with $\theta_i^* = \arctan\lambda_i(A)$ for $i = 1, 2$.

Finally, by integrating (3.13) term by term, we get

$$\begin{aligned}
u(x) &= \frac{1}{2}x^TAx + b^Tx + c + \frac{a_{-1}}{2}\log x^T(I + A^2)x + O(|x|^{-1}) \\
&=: Q(x) + \frac{a_{-1}}{2}\log x^T(I + (D^2Q)^2)x + O(|x|^{-1}) \quad \text{as} \quad |x| \to \infty. \qquad (3.14)
\end{aligned}$$

Step 3. To calculate the coefficient a_{-1} for the logarithmic term

$$\Gamma(x) = \frac{a_{-1}}{2}\log x^T(I + A^2)x,$$

as in [7, p. 570], we integrate the algebraic form of equation (1.1)

$$\cos\Theta\,\Delta u + \sin\Theta\,\det D^2u = \sin\Theta. \qquad (3.15)$$

We have

$$\int_{E_R\setminus\Omega} \mathcal{S}\,dx = \int_{E_R\setminus\Omega} \mathcal{C}\Delta u + \mathcal{S}\det D^2u\,dx = \int_{\partial(E_R\setminus\Omega)} \mathcal{C}u_\nu + \mathcal{S}u_1(u_{22}, -u_{12})\cdot\nu\,ds,$$

where $E_R = \left\{x \in \mathbb{R}^2 \,|\, x^T(I + A^2)x < R^2\right\}$ and $(\mathcal{C}, \mathcal{S}) = (\cos\Theta, \sin\Theta)$. In view of the asymptotic behaviors (3.10) and (3.13), we get

$$
\begin{aligned}
&\int_{\partial\Omega} \mathcal{C}u_\nu + \mathcal{S}u_1(u_{22},-u_{12})\cdot\nu\,ds + \int_{E_R\setminus\Omega} \mathcal{S}\,dx \\
&= \int_{\partial E_R} \mathcal{C}(Q+\Gamma)_\nu + \mathcal{S}(Q_1+\Gamma_1)(Q_{22}+\Gamma_{22},-Q_{12}-\Gamma_{12})\cdot\nu\,ds + O(R^{-1}) \\
&= \int_{\partial E_R} \mathcal{C}Q_\nu + \mathcal{S}Q_1(Q_{22},-Q_{12})\cdot\nu\,ds \\
&\qquad + \int_{\partial E_R} \mathcal{C}\Gamma_\nu + \mathcal{S}(Q_1(\Gamma_{22},-\Gamma_{12}) + \Gamma_1(Q_{22},-Q_{12}))\cdot\nu\,ds + O(R^{-1}) \\
&= \int_{E_R} \mathcal{C}\Delta Q + \mathcal{S}\det D^2Q\,dx + 2\pi a_{-1} + O(R^{-1}).
\end{aligned}
\tag{3.16}
$$

Letting R go to ∞, we obtain

$$
a_{-1} = \frac{1}{2\pi}\left(\int_{\partial\Omega} \mathcal{C}u_\nu + \mathcal{S}u_1(u_{22},-u_{12})\cdot\nu\,ds - \mathcal{S}|\Omega|\right) = d.
$$

We still have to verify the appearance of the $2\pi a_{-1}$ term in (3.16). Instead of going through the direct, but tricky and long calculation for the corresponding boundary integral, we use divergence theorem. Without loss of generality, we assume A is diagonal with eigenvalues (μ_1,μ_2). From the equation $\arctan\mu_1 + \arctan\mu_2 = \Theta$, it follows that

$$
\begin{aligned}
&\cos\Theta\,\Delta\Gamma + \sin\Theta\,(Q_{22}\Gamma_{11} - 2Q_{12}\Gamma_{12} + Q_{11}\Gamma_{22}) \\
&= \sqrt{(1+\mu_1^2)(1+\mu_2^2)}\left(\frac{1}{1+\mu_1^2}\Gamma_{11} + \frac{1}{1+\mu_2^2}\Gamma_{22}\right) \\
&= 2\pi a_{-1}\delta_0 \quad \text{in} \quad \mathbb{R}^2.
\end{aligned}
\tag{3.17}
$$

Hence

$$
\begin{aligned}
&\int_{\partial E_R} \mathcal{C}\Gamma_\nu + \mathcal{S}\big(Q_1(\Gamma_{22},-\Gamma_{12}) + \Gamma_1(Q_{22},-Q_{12})\big)\cdot\nu\,ds \\
&= \int_{E_R} \cos\Theta\,\Delta\Gamma + \sin\Theta\,(Q_{22}\Gamma_{11} - 2Q_{12}\Gamma_{12} + Q_{11}\Gamma_{22})\,dx = 2\pi a_{-1}.
\end{aligned}
$$

Step 4. For any fixed x with $|x|$ sufficiently large, let

$$
E(y) = \left(\frac{2}{|x|}\right)^2 (u - Q - \Gamma)\left(x + \frac{|x|}{2}y\right) \quad \text{and} \quad \bar{\Gamma}(y) = \left(\frac{2}{|x|}\right)^2 \Gamma\left(x + \frac{|x|}{2}y\right).
$$

Then

$$\bar{a}^{ij}(y)(E+\bar{\Gamma})_{ij}(y) = F(A + D^2E(y) + D^2\bar{\Gamma}(y)) - F(A) = 0,\ y \in B_1,$$

where

$$\bar{a}^{ij}(y) = \int_0^1 F_{M_{ij}}(A + t(D^2E(y) + D^2\bar{\Gamma}(y)))dt,$$

and

$$F(M) = \arctan \lambda_1(M) + \arctan \lambda_2(M). \tag{3.18}$$

By the Nirenberg estimate (two dimensional fully nonlinear Schauder estimate) and the Schauder estimate, we have

$$\begin{aligned}|D^kE(0)| &\le C_k(\|E\|_{L^\infty(B_1)} + \left\|\bar{a}^{ij}\bar{\Gamma}_{ij}\right\|_{C^\alpha(B_1)}) \\ &\le C_k(\|E\|_{L^\infty(B_1)} + |x|^{-4}) \le C_k|x|^{-3},\ \text{for all } k \in \mathbb{N}.\end{aligned}$$

Therefore

$$|D^k(u - Q - \Gamma)(x)| \le C_k|x|^{-k-1},\ \text{for all } k \in \mathbb{N}.$$

Step 5. The uniqueness of the quadratic polynomial $Q(x)$ is proved in the same way as in *Step 4* in the proof of Theorem 2.1.

Remark 3.1. Once we reach a fast enough rate for the Hessian of the solution approaching its limit at infinity, we can reveal the asymptotics of the solution through linearized equations. By the asymptotic behavior (3.10) of D^2u, we set $v(x) = u(x) - x^TAx/2$.

(i) From the original trigonometric form equation (1.1), we have

$$\underline{a}^{ij}(x)v_{ij}(x) = F(A + D^2v(x)) - F(A) = 0,\ x \in \bar{B}_1^{\mathcal{C}},$$

where F is the one given in (3.18) and $\underline{a}^{ij}(x) = \int_0^1 F_{M_{ij}}(A+tD^2v(x))dt$. Since (3.10) reads

$$|D^2v(x)| = O(|x|^{-2})\ (|x| \to \infty),$$

it follows that

$$\operatorname{tr}\left((I + A^2)^{-1}D^2v\right) = F_{M_{ij}}(A)v_{ij} = \left(F_{M_{ij}}(A) - \underline{a}^{ij}\right)v_{ij} = O(|x|^{-4})\ (|x| \to \infty).$$

From the Newtonian representation of $v(x)$ as in [7, p. 569], we deduce that

$$u(x) = \frac{1}{2}x^T Ax + v(x) = \frac{1}{2}x^T Ax + b^T x + c + \frac{d}{2}\log x^T(I + A^2)x + O(|x|^{-1}) \ (|x| \to \infty),$$

for some $b \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$.

(ii) Another way is to consider the algebraic form of the special Lagrangian equation (3.15). It follows from (3.10) and the expansion formula of the determinant that

$$\operatorname{tr}\left(\mathcal{M}D^2 v\right) = O(|x|^{-4}) \ (|x| \to \infty),$$

where $\mathcal{M} = \cos\Theta\, I + \sin\Theta\,(\det A)A^{-1}$. Because of (the "conformality")

$$\mathcal{M}^{-1} = (\cos\Theta\, I_2 + \sin\Theta\,(\det A)A^{-1})^{-1} = \frac{1}{\sqrt{\det(I + A^2)}}(I + A^2),$$

we obtain the same linearized equation and the same logarithmic term $\log x^T \mathcal{M}^{-1} x$.

Note that, when $\Theta = \pi/2$, the special Lagrangian equation (3.15) becomes the Monge-Ampère equation $\det D^2 u = 1$. We have $\mathcal{M} = A^{-1}$ and hence the logarithmic term $\log x^T Ax$, which is as same as the one given in [7, Theorem 1.2] (see also Theorem 4.1).

Remark 3.2. Via the rotation argument in Step 1, we actually have a harmonic representation of the potential u to two dimensional special Lagrangian equations, which in turn, also leads to the asymptotics of the solution.

Write $z = \widetilde{x}_1 + \sqrt{-1}\,\widetilde{x}_2$, $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2)$ and $\widetilde{y} = (\widetilde{u}_{\widetilde{x}_1}, \widetilde{u}_{\widetilde{x}_2})$. From (3.7), we have

$$\begin{aligned} u(x) &= \int^x Du \cdot dx = \int^x (\mathfrak{s}\widetilde{x} + \mathfrak{c}\widetilde{y}) \cdot d(\mathfrak{c}\widetilde{x} - \mathfrak{s}\widetilde{y}) \\ &= \int^x \mathfrak{s}\mathfrak{c}(\widetilde{x}d\widetilde{x} - \widetilde{y}d\widetilde{y}) + \mathfrak{c}^2\widetilde{y}d\widetilde{x} - \mathfrak{s}^2\widetilde{x}d\widetilde{y} \\ &= \int^x \frac{1}{2}\mathfrak{s}\mathfrak{c}d(|\widetilde{x}|^2 - |\widetilde{y}|^2) + \mathrm{Re}(\mathfrak{c}^2 h dz - \mathfrak{s}^2 z h_z dz) \\ &= \frac{1}{2}\mathfrak{s}\mathfrak{c}(|z|^2 - |h|^2) + \mathrm{Re}\int^z (\mathfrak{c}^2 h - \mathfrak{s}^2 z h_z)dz \\ &= \frac{1}{2}\mathfrak{s}\mathfrak{c}(|z|^2 - |h|^2) + \frac{1}{2}(\mathfrak{c}^2 - \mathfrak{s}^2)\mathrm{Re}(a_1 z^2) \\ &\qquad + \mathfrak{c}^2\mathrm{Re}(a_0 z) + a_{-1}\log|z| + O(|z|^{-1}). \end{aligned}$$

Since $\widetilde{x} = (\cos\vartheta\, I + \sin\vartheta\, A)x + O(1)$ via the asymptotic behavior (3.13) of Du (the rough version is enough), we obtain

$$\begin{aligned} \log|z|^2 &= \log x^T(\cos\vartheta\, I + \sin\vartheta\, A)^2 x + O(1) \\ &= \log x^T(I + A^2)x + O(1). \end{aligned}$$

On the other hand, since $x = (\mathfrak{c}I - \mathfrak{s}\widetilde{A})\widetilde{x} + O(1)$ via (3.11)(a), and $A = (\mathfrak{s}I + \mathfrak{c}\widetilde{A})(\mathfrak{c}I - \mathfrak{s}\widetilde{A})^{-1}$, by the definition (3.9) of $\widetilde{A}$, it is not hard to verify that the highest degree term of

$$\mathfrak{s}\mathfrak{c}(|z|^2 - |a_1 z|^2) + (\mathfrak{c}^2 - \mathfrak{s}^2)\mathrm{Re}(a_1 z^2)$$

is exactly $x^T Ax$. Thus we conclude also that

$$u(x) = \frac{1}{2}x^T Ax + b^T x + c + \frac{d}{2}\log x^T(I + A^2)x + O(|x|^{-1})\ (|x| \to \infty).$$

For a complex representation of the solution to the two dimensional Monge-Ampère equation via the Legendre-Lewy transformation, see [10,11].

3.3. Proof of Theorem 1.2

Step 1. As in [35], after blowing down the minimal surface $(\widetilde{x}, D\widetilde{u}(\widetilde{x}))$ at ∞, we have a minimal Lagrangian graphical tangent cone or homogeneous order two solution v to $\sum_{i=1}^n \arctan\lambda_i(D^2 v) = \widetilde{\Theta}$ in $\mathbb{R}^n$ with bounded Hessian $|D^2 v| < \cot\vartheta$ and $|D^2 v| \leq \tan(\frac{\pi}{3} + \beta(n))$ respectively for $n \leq 4$ and $n \geq 5$ from Section 3.0. We claim the tangent cone is flat.

For $n = 2$, the only left case in the first place is an exterior semiconvex harmonic function. The proof for Theorem 1.2 is straightforward.

For $n = 3$, it follows from a well-known fact that every three dimensional minimal graphical cone is flat or v is quadratic; see [17, p. 426] for a "quick" PDE proof.

For $n = 4$, by the dimension reduction argument and De Giorgi-Allard ϵ-regularity in the geometric measure theory, the minimal cone $(\widetilde{x}, Dv(\widetilde{x}))$ is in fact smooth away from its vertex (cf. [35]), then the solution v to the analytic special Lagrangian equation is analytic away from the origin. It follows from Nadirashvili-Vladut [21] v is quadratic or the tangent cone at ∞ is flat.

For $n \geq 5$, with $|D^2 v| \leq \tan(\frac{\pi}{3} + \beta(n)) = \sqrt{3} + \epsilon'(n)$, as noted in [29, p. 924], by a similar compactness argument for Proposition 3.1 in [35], v is quadratic. Here the small dimensional constant $\epsilon(n)$ is determined by the small dimensional constant $\epsilon'(n)$ satisfying $\epsilon(n) = \tan(\arctan(\sqrt{3} + \epsilon'(n)) - \pi/6) - 1/\sqrt{3}$.

Step 2. Having the flat tangent cone at ∞, it follows from Allard-Almgren [1, p. 215, p. 217] that the tangent cone $(\widetilde{x}, L(\widetilde{x}))$ with now integrable Jacobi normal vector fields is unique and satisfies effective uniqueness estimate

$$|D\widetilde{u}(\widetilde{x}) - L(\widetilde{x})| + |\widetilde{x}||D^2\widetilde{u}(\widetilde{x}) - DL| \leq O(|\widetilde{x}|^{1-\alpha})$$

with $\alpha \in (0, 1)$; see also Simon [23, p. 269, II Section 6] [24, p. 239].

A similar strip argument as in Step 2 of Subsection 3.1 shows that our original potential u satisfies

$$|D^2u| \le C(n,u) < +\infty \quad \text{in} \quad \mathbb{R}^n \setminus \bar{\Omega},$$

and

$$|D^2u(x) - A| \le O(|x|^{-\alpha})$$

for some constant matrix A.

Step 3. Via the boundedness and asymptotic behavior of Hessian D^2u from Step 2 in hand, we already reach the conclusion of Lemma 2.1, and in turn, the proof of Theorem 2.1 now with uniformly elliptic

$$F\left(D^2u\right) = \sum_{i=1}^{n} \arctan \lambda_i \left(D^2u\right) - \Theta,$$

goes through without the convexity, concavity, or level set convexity assumption. It is because the shape assumption is only used to draw the asymptotic behavior in Lemma 2.1 for the starting proof of Theorem 2.1. Thus the proof of Theorem 1.2 is complete.

4. Further perspectives

In 2003, Caffarelli and Li [7] extended the Jörgens-Calabi-Pogorelov theorem for Monge-Ampère equations (which asserts that every convex global solution of the Monge-Ampère equation must be a quadratic polynomial) to exterior domains as follows.

Theorem 4.1 *(See [7]). Let u be a smooth convex solution of the Monge-Ampère equation $\det D^2u = 1$ in the exterior domain $\mathbb{R}^n \setminus \bar{\Omega}$. Then there exists a unique quadratic polynomial $Q(x)$ such that when $n \ge 3$,*

$$u(x) = Q(x) + O_k(|x|^{2-n}) \quad as \quad |x| \to \infty \tag{4.1}$$

for all $k \in \mathbb{N}$, and when $n = 2$,

$$u(x) = Q(x) + \frac{d}{2} \log x^T D^2Qx + O_k(|x|^{-1}) \quad as \quad |x| \to \infty \tag{4.2}$$

for all $k \in \mathbb{N}$, where

$$d = \frac{1}{2\pi}\left(\int_{\partial\Omega} u_1(u_{22}, -u_{12}) \cdot \nu \, ds - |\Omega|\right).$$

Case $n = 2$ was treated earlier in 1999 by Ferrer, Martínez and Milán [11] using complex analysis method. The method in [7] is, to deduce first that the solution is close to a quadratic polynomial with a sub-quadratic error, by using Caffarelli's theory on Monge-Ampère equation, and then to obtain the Hölder closedness at infinity of the Hessian to some constant matrix, by estimates of Pogorelov, Evans-Krylov, and Schauder.

We remark that, by extending the solution u inside Ω such that the new u is smooth and convex in $\mathbb{R}^n$, which can be done after subtracting a linear function from u if necessary as in [7, pp. 571–572], and then invoking the Pogorelov estimate (see [25, pp. 73–76] or [16, pp. 467–471]), we deduce that

$$\left\|D^2u\right\|_{L^\infty(\mathbb{R}^n\setminus\bar{\Omega})} \leq C(n,u,\Omega) < +\infty,$$

which also implies that the equation is uniformly elliptic. Applying Theorem 2.1, we have a new proof of the above exterior Jörgens-Calabi-Pogorelov type result for the Monge-Ampère equation when $n \geq 3$. Our argument for Theorem 1.1 ($n = 2$) gives yet another proof of Theorem 4.1 ($n = 2$), as the Monge-Ampère equation now is equivalent to the special Lagrangian equation (1.1) with $\Theta = \pi/2$.

In 2010, Chang and the third author [9] proved an entire Liouville theorem for the quadratic Hessian equation, which asserts that every convex solution must be quadratic. The argument is to make a Legenre-Lewy transformation of the solution to a new solution of a new uniformly elliptic and convex equation with bounded Hessian from both sides, so that Evans-Krylov-Safonov theory applies. Combining this idea with our exterior Liouville Theorem 2.1 for general fully nonlinear elliptic and convex equations, we obtain the following exterior Liouville theorem for the quadratic Hessian equation.

Theorem 4.2. *Let $n \geq 3$ and let u be a smooth solution of the quadratic Hessian equation $\sigma_2(\lambda(D^2u)) = 1$ in the exterior domain $\mathbb{R}^n \setminus \bar{\Omega}$. Suppose*

$$D^2u > \left(\delta - \sqrt{2/(n(n-1))}\right) I \quad in \quad \mathbb{R}^n \setminus \bar{\Omega}$$

for any fixed $\delta > 0$. Then there exists a unique quadratic polynomial $Q(x)$ such that

$$u(x) = Q(x) + O_k(|x|^{2-n}) \quad as \quad |x| \to \infty$$

for all $k \in \mathbb{N}$.

Proof. *Step 1.* As in [9], we make a Legendre-Lewy transformation of the solution to a solution of a new uniformly elliptic and convex equation with bounded Hessian from both sides.

Write $K = \sqrt{2/(n(n-1))}$ and let $w(x) = u(x) + K|x|^2/2$ for all $x \in \mathbb{R}^n \setminus \bar{\Omega}$. Then $D^2w > \delta I$ in $\mathbb{R}^n \setminus \bar{\Omega}$. As in Section 3.0, enlarging Ω if necessary and then extending u smoothly to $\mathbb{R}^n$ such that $D^2w > \delta I$ in $\mathbb{R}^n$, we have the distance increasing property

$$|Dw(x) - Dw(x_\star)| = \left|\int_0^1 D^2w(x_\star + t(x - x_\star))(x - x_\star)dt\right| \geq \delta|x - x_\star|$$

for all $x, x_\star \in \mathbb{R}^n$. Thus $x \mapsto y = Dw(x)$ is globally injective. Because the Jacobian of the map $\det D_x y = \det D^2 w(x) \neq 0$, the closed map $Dw(x)$ is also open. Therefore, $Dw(x)$ is surjective, $Dw(\mathbb{R}^n) = \mathbb{R}^n$, $Dw(\Omega) =: \widetilde{\Omega}$ is a bounded domain, and hence $y \mapsto x$, $\mathbb{R}^n \to \mathbb{R}^n$ is also bijective.

Consider the Legendre transform $\bar{w}(y)$ of $w(x)$ given by $\bar{w}(y) = x(y) \cdot y - w(x(y))$. We have $x = D\bar{w}(y)$ and $D^2\bar{w}(y) = (D^2 w(x))^{-1}$. It follows that the function $\widetilde{u}(y) = -\bar{w}(y)$ satisfies

$$-\delta^{-1} I < D^2 \widetilde{u}(y) = -\left(D^2 u(x) + KI\right)^{-1} < 0 \tag{4.3}$$

for all $y \in \mathbb{R}^n$ and

$$g\left(\widetilde{\lambda}(D^2\widetilde{u})\right) = \sigma_2\left((-\widetilde{\lambda}_1^{-1} - K, ..., -\widetilde{\lambda}_n^{-1} - K)\right) = 1 \quad \text{in} \quad \widetilde{\Omega}^{\mathcal{C}}.$$

As proved in [9, pp. 661–663], we have

(i) the level set $\Sigma = \left\{\widetilde{\lambda} \,|\, g(\widetilde{\lambda}) = 1\right\}$ is convex;
(ii) the normal vector Dg of the level set Σ is uniformly inside the positive cone $\Gamma^+ = \left\{\widetilde{\lambda} \,\middle|\, \widetilde{\lambda}_i > 0 \text{ for all } i = 1, 2, ..., n\right\}$ provided $\widetilde{\lambda}_i \in (-\delta^{-1}, 0)$ for all $i = 1, 2, ..., n$.

Thus $\widetilde{u}(y)$ satisfies a uniformly elliptic equation with convexity.

Step 2. In view of (4.3) and applying Theorem 2.1, we obtain

$$\widetilde{u}(y) = \frac{1}{2} y^T \widetilde{A} y + \widetilde{b}^T y + \widetilde{c} + O_k(|y|^{2-n})$$

as $|y| \to \infty$, for all $k \in \mathbb{N}$. In particular,

$$D^2\widetilde{w}(y) \to \widetilde{A} \quad \text{and} \quad x = D\widetilde{w}(y) = -D\widetilde{u}(y) = -\widetilde{A} y + O(1)$$

as $|y| \to \infty$. By the strip argument described in the proof of (3.5) in Subsection 3.1, we see that $\lambda_i(\widetilde{A}) < 0$ for all $i = 1, 2, ..., n$. (Otherwise, $\mathbb{R}^n \setminus \Omega$ is bounded in x_{i_0}-direction for some i_0, a contradiction.) Thus the matrix $\widetilde{A}$ is invertible. Therefore

$$D^2 u(x) = -(D^2\widetilde{u}(y))^{-1} - KI \to -\widetilde{A}^{-1} - KI =: A$$

as $|x| \to \infty$, and $|D^2 u(x)| \leq C$ for all $x \in \Omega^{\mathcal{C}}$, which implies that the original quadratic Hessian equation $\sigma_2(\lambda) = 1$ is uniformly elliptic in $D^2 u(\Omega^{\mathcal{C}})$. Since the level set $\{\lambda \,|\, \sigma_2(\lambda) = 1\}$ is originally convex, by applying Theorem 2.1 again, we thus complete the proof of Theorem 4.2. □

In 1960, Flanders [13] established an entire Liouville theorem for the inverse harmonic Hessian equation

$$\frac{1}{\lambda_1(D^2u)} + \cdots + \frac{1}{\lambda_n(D^2u)} = 1, \tag{4.4}$$

which says that every smooth convex solution u of (4.4) in the whole space $\mathbb{R}^n$ must be a quadratic polynomial. As an application of the same idea in establishing our main Theorem 1.1, we obtain the following exterior Liouville theorem for inverse harmonic Hessian equations.

Theorem 4.3. *Let u be a smooth convex solution of the inverse harmonic Hessian equation (4.4) in the exterior domain $\mathbb{R}^n\setminus\bar{\Omega}$. Then there exists a unique quadratic polynomial $Q(x)$ such that when $n \geq 3$,*

$$u(x) = Q(x) + O_k(|x|^{2-n}) \quad as \quad |x| \to \infty \tag{4.5}$$

for all $k \in \mathbb{N}$, and when $n = 2$,

$$u(x) = Q(x) + \frac{d}{2}\log x^T(D^2Q)^2x + O_k(|x|^{-1}) \quad as \quad |x| \to \infty \tag{4.6}$$

for all $k \in \mathbb{N}$, where

$$d = \frac{1}{2\pi}\int\limits_{\partial\Omega} u_1(u_{22}, -u_{12})\cdot\nu - u_\nu\, ds.$$

Proof. We first make a Legendre transform of the solution u to a new solution $\bar{u}$ to the Laplace equation with $D^2\bar{u}$ being bounded from both sides.

From the equation (4.4), it is clear that $D^2u > I$ in $\bar{\Omega}^{\mathcal{C}}$. As in Section 3.0, enlarging Ω if necessary and then extending u smoothly to $\mathbb{R}^n$ such that $D^2u > I$ in $\mathbb{R}^n$, we have the distance increasing property

$$|Du(x) - Du(x_\star)| = \left|\int\limits_0^1 D^2u(x_\star + t(x - x_\star))(x - x_\star)dt\right| \geq |x - x_\star|$$

for all $x, x_\star \in \mathbb{R}^n$. Thus $x \mapsto y = Du(x)$ is globally injective. Because the Jacobian of the map $\det D_xy = \det D^2u(x) \neq 0$, the closed map $Du(x)$ is also open. Therefore, $Du(x)$ is surjective, $Du(\mathbb{R}^n) = \mathbb{R}^n$, and $Du(\Omega) =: \widetilde{\Omega}$ is a bounded domain. Then

$$\bar{u}(y) = \int^y x\cdot dDu(x) = x\cdot Du(x) - \int^{x(y)} Du(x)\cdot dx = x(y)\cdot y - u(x(y)),$$

leading to the Legendre transform of u. Note that the above two equivalent integrals are well defined for diffeomorphism $x \mapsto y = Du(x)$. It follows that $x = D\bar{u}(y)$, and by the chain rule, $D^2\bar{u}(y) = (D^2u(x))^{-1}$. Thus

$$\Delta \bar{u} = 1 \quad \text{and} \quad 0 < D^2 \bar{u} < I \quad \text{in} \quad \mathbb{R}^n \setminus \widetilde{\Omega}.$$

Case n ≥ 3. Invoking Theorem 2.1 (the proof is much simpler, as the equation now is Laplace), we have

$$\bar{u}(y) = \frac{1}{2} y^T \bar{A} y + \bar{b}^T y + \bar{c} + O_k(|y|^{2-n}) \quad \text{as} \quad |y| \to \infty$$

for all $k \in \mathbb{N}$. In particular,

$$x = D\bar{u}(y) = \bar{A}y + \bar{b} + O_k(|y|^{1-n}) = \bar{A}y + O(1).$$

By the strip argument, as described in the proof of (3.5) in Subsection 3.1, we see that the matrix $\bar{A}$ is invertible. Hence

$$D^2 u(x) = \big(D^2 \bar{u}(y)\big)^{-1} \to \bar{A}^{-1} =: A \quad \text{and} \quad D^2 u(x) = A + O_k(|x|^{-n}) \tag{4.7}$$

as $|x| \to \infty$, and

$$|D^2 u(x)| \le C, \text{ for all } x \in \Omega^{\mathcal{C}}.$$

Applying Theorem 2.1 again, we finally obtain

$$u(x) = Q(x) + O_k(|x|^{2-n}) \quad \text{as} \quad |x| \to \infty$$

for all $k \in \mathbb{N}$.

Remark 4.1. Another way to reach the above asymptotic behavior (4.5) is to adopt a similar, but simpler (without logarithmic term) substitution procedure as in the proof of (1.3) in Subsection 3.2. Indeed, by substituting

$$y = Ax - A\bar{b} + O_k(|x|^{1-n})$$

into

$$u(x) = x \cdot y - \bar{u}(y) = x \cdot y - \bar{Q}(y) + O_k(|y|^{2-n}),$$

we obtain (4.5). Noting that (4.7) reads $D_x y = D^2 u(x) = A + O_k(|x|^{-n})$, by the chain rule we see that the asymptotic behavior $O_k(|x|^{2-n})$ for any k is also preserved.

Case n = 2. Now we are exactly in a similar situation as in Subsection 3.2 for the proof of (1.3) of Theorem 1.1 ($n = 2$). Repeating the complex analysis argument of *Step 1*, the similar, but simpler notation-wise rotation argument of *Step 2* (the Legendre transform is just a $\pi/2$-$U(n)$ rotation followed by a conjugation, namely, (3.4) with

$(\mathfrak{c}, \mathfrak{s}) = (0, 1)$), the same divergence argument of *Step 3*, and the same Schauder argument of *Step 4* in Subsection 3.2, we conclude that

$$u(x) = Q(x) + \frac{d}{2} \log x^T (D^2 Q)^2 x + O_k(|x|^{-1}) \quad \text{as} \quad |x| \to \infty$$

for all $k \in \mathbb{N}$. In particular, the boundary representation for d is similarly calculated via integration of the algebraic form of the inverse Harmonic Hessian equation

$$\Delta u - \det D^2 u = 0$$

and a corresponding "δ-function" argument to (3.17).

The uniqueness of $Q(x)$ is proved in the same way as in *Step 4* in the proof of Theorem 2.1.

Remark 4.2. Let $w(x) = u(x) - |x|^2/2$. Then equation (4.4) is equivalent to the two dimensional Monge-Ampère equation $\det D^2 w = 1$ and also the two dimensional special Lagrangian equation with $\Theta = \pi/2$. From (1.3) in Theorem 1.1 and $(D^2 Q)^2 = (\det D^2 Q)(D^2 Q - I)$, (4.6) follows.

Note that one can also proceed as in Remark 3.1 or Remark 3.2 to obtain (4.6). □

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