SINGULAR SOLUTIONS TO SPECIAL LAGRANGIAN EQUATIONS WITH SUBCRITICAL PHASES AND MINIMAL SURFACE SYSTEMS

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Abstract. We construct singular solutions to special Lagrangian equations with subcritical phases and minimal surface systems. A priori estimate breaking families of smooth solutions are also produced correspondingly. A priori estimates for special Lagrangian equations with certain convexity are largely known by now.

1. Introduction

In this paper, we construct *singular* solutions to the special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i = \Theta \tag{1.1}$$

with subcritical phase $|\Theta| < (n-2)\pi/2$, where λ_i are the eigenvalues of D^2u, and the minimal surface system for k-vector valued functions of n-variables

$$\triangle_g U = \sum_{i,j=1}^{n} \frac{1}{\sqrt{g}} \partial_{x_i} \left(\sqrt{g} g^{ij} \partial_{x_j} U \right) = 0, \tag{1.2}$$

where the induced metric

$$g = I + (DU)^T DU.$$

Equation (1.1) is the potential equation for (1.2) with solutions $U = Du$. The Lagrangian graph $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ is called special and in fact volume minimizing when the phase or the argument of the complex number $\left(1+\sqrt{-1}\lambda_1\right)\cdots\left(1+\sqrt{-1}\lambda_n\right)$ is constant Θ, or equivalently u satisfies equation (1.1); see the work [HL1, Theorem 2.3, Proposition 2.17] by Harvey and Lawson. The phase $(n-2)\pi/2$ is said critical because the level set $\{\lambda \in \mathbb{R}^n | \, \lambda \text{ satisfying } (1.1)\}$ is convex *only* when $|\Theta| \geq (n-2)\pi/2$ [Y2,

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Lemma 2.1]. In dimension three, when $|\Theta| = \pi/2$ or $|\Theta| = 0,\ \pi$, equation (1.1) also takes the quadratic and cubic algebraic forms respectively

$$\sigma_2\left(D^2u\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1 \tag{1.3}$$

or

$$\triangle u = \det D^2u. \tag{1.4}$$

We state our first main result.

Theorem 1.1. *There exist $C^{1,1/(2m-1)}$ ($m = 2, 3, 4, \cdots$) viscosity solutions u^m to (1.1) with $n = 3$ and each $\Theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, such that $u^m \in C^{1,1/(2m-1)}(B_1) \cap C^\infty(B_1\backslash\{0\})$ for $B_1 \subset \mathbb{R}^3$ but $u^m \notin C^{1,\delta}$ for any $\delta > 1/(2m-1)$.*

Rotating forth and back, we obtain our second ("smooth") result.

Theorem 1.2. *There exist a family of smooth solutions u^ε to (1.1) in $B_1 \subset \mathbb{R}^3$ with $n = 3$ and each fixed $\Theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that*

$$\|Du^\varepsilon\|_{L^\infty(B_1)} \leq C \quad \text{but} \quad \left|D^2u^\varepsilon(0)\right| \to \infty \quad \text{as } \varepsilon \to 0.$$

For each u^ε with small ε fixed in Theorem 1.2, the Hessian $\left|D^2u^\varepsilon(0)\right|$ (in the max eigenvalue norm) is strictly larger that its nearby values in the three dimensional domain of the solution to a now uniformly elliptic equation (1.1). (It can be seen by Property 2.4 in Section 2 and tracing the eigenvalue dependency in Section 4.) This violates the maximum principle. In contrast to the two dimensional fully nonlinear uniformly elliptic equations, it is classically known that the Hessian of any solution enjoys the maximum principle (cf. [GT, p. 301]). To the solutions in the above two theorems, by adding quadratics of extra variables in higher dimensions $n \geq 4$, we immediately get the corresponding counterexamples for (1.1) with all subcritical phases $|\Theta| < (n-2)\pi/2$. Furthermore, we convert our counterexamples to the ones for minimal surface system (1.2).

Theorem 1.3. *There exist a family of weak solutions U^m to (1.2) in $B_1 \subset \mathbb{R}^3$ with $n = 3$, $k = 3$, and $m = 2, 3, 4, \cdots$ such that*

$$U^m \in W^{1,p}(B_1) \quad \text{for any } p < \frac{2m+1}{2m-2} \quad \text{but} \quad U^m \notin W^{1,\frac{2m+1}{2m-2}}(B_1).$$

Furthermore, there exist a family of smooth solutions U^ε to (1.2) in $B_1 \subset \mathbb{R}^3$ with $n = 3$ and $k = 3$ such that

$$\|U^\varepsilon\|_{L^\infty(B_1)} \leq C \quad \text{but} \quad |DU^\varepsilon(0)| \to \infty \quad \text{as } \varepsilon \to 0.$$

The vector valued functions U^m are taken as Du^m with u^m from Theorem 1.1, thus the first part of the theorem gives a negative answer to Nadirashvili's question whether there is an ε improvement of $W^{2,1}$ solutions to special Lagrangian equation (1.1) in general. We are grateful for this question. In terms of minimal surface system (1.2), the question would be whether there is an ε improvement of $W^{1,1}$ solutions.

For special Lagrangian equation (1.1) with critical and supercritical phases $|\Theta| \geq (n-2)\,\pi/2$ in dimension two and three, with very large phase $|\Theta| \geq (n-1)\,\pi/2$ in general dimensions, a priori Hessian and gradient estimates, and consequently, armed with the solvability of the Dirichlet problem with smooth boundary data to the now convex special Lagrangian equation (1.1) in the critical and supercritical phase cases, the regularity of C^0 viscosity solutions were derived in [WY1] [WY2] [WY3] [CWY]. In passing, we also mention that the existence and uniqueness of the C^0 viscosity solution for the Dirichlet problem to strictly elliptic equation (1.1) is known (cf. [CWY, p. 594]). In recent years, there has been a new approach toward the existence and uniqueness of C^0 viscosity or weak solution for the Dirichlet problem to strictly elliptic as well as degenerate elliptic fully nonlinear equations by Harvey and Lawson [HL2] [HL3].

Recently Nadirashvili and Vlăduţ [NV] constructed beautiful $C^{1,1/3}$ singular viscosity solutions to (1.1) with subcritical phases $|\Theta| < \pi/2$ in dimension three, relying on "brutal force" calculations (for the approximate solutions) and a hard and deep topological result in [EL] (for the injectivity of the gradient maps).

For minimal surface equations, namely (1.2) with $k = 1$, the gradient estimate in terms of the height of the minimal surfaces, is the classic result by Bombieri-De Giorgi-Miranda [BDM], from which it follows the regularity of weak or viscosity solutions. For smooth solutions to (1.2) with $n = 2$, Gregori [G] extended Heinz's Jacobian estimate to get a gradient bound in terms of the heights of the two dimensional minimal surfaces with any codimension. For smooth solutions to general minimal surface system (1.2) with certain constraints on the gradients themselves, a gradient estimate was obtained by Wang [W], using an integral method developed for codimension one minimal graphs. Nonetheless, there do exist singular $W^{1,2-}$ weak solutions (in fact Lagrangian) to (1.2) with $n = 2$; see Osserman [O]. Now gradient estimates for (1.2) with $k = 2$ and $n \geq 3$ still remain mysterious and challenging.

Our construction goes as follows. In the first stage, we solve the special Lagrangian equation (1.1) with the critical phase by Cauchy-Kowalevskaya. The approximate solutions or initial data for the relatively "easier" corresponding quadratic equation (1.3) are built up via a *systematic* procedure, which allows us to have the approximation at arbitrarily high order (Property 2.1 and 2.2), and eventually those highly ("oddly" $C^{1,1/(2m-1)}$) singular solutions in Theorem 1.1 and Theorem 1.3. In the second stage, we take an "inversion" $\frac{\pi}{2}$ rotation of the solutions from the first stage to obtain those singular solutions with phase 0 (Proposition 3.1). The singular solutions with other subcritical phases are achieved via a preliminary "horizontal" rotation before the "inversion" $\frac{\pi}{2}$ rotation (Step 1 of Section 3). Some remarks are in order. Those $U(n)$ rotations are "obvious" to produce for the $U(n)$ invariant special Lagrangian equation (1.1). But it is by no means easy to justify that the special Lagrangian submanifold is still a graph in the rotated new coordinate system, thus a valid equation (1.1) to work on. (Earlier development of those $U(n)$ rotations for (1.1) can be found in [Y2] [Y3].) Here our *elementary analytic* justification for the "inversion" $\frac{\pi}{2}$ rotation (Proposition 3.1) avoids a hard and deep topological formula of [EL], which was employed in [NV]. Lastly we point out that the Legendre transformation (usually used for convex functions), is just the "inversion" $\frac{\pi}{2}$ rotation followed by a conjugation for converting "gradient" graph $(x, Du(x))$ to the one $(Du^*(y), y)$ (now with saddle potentials u and u^*). In the third stage, we kick in a little bit extra to the preliminary "horizontal" rotations of Stage 2, then after the same "inversion" $\frac{\pi}{2}$ rotation, we make up a corresponding little bit "backward" rotation to finally generate the desired family of smooth solutions in Theorem 1.2, which break a priori Hessian estimates for special Lagrangian equation (1.1) with subcritical phase. Note that here one cannot produce those a priori estimate breaking family of smooth solutions by the usual way, that is to solve the Dirichlet problem with smooth approximate boundary data of the boundary value of a singular solution, as Theorem 1.1 shows the *non-solvability* of smooth solution to the Dirichlet problem to (1.1) of subcritical phase even with smooth boundary data. The Dirichlet problem to the saddle branch of (1.4) or the equivalent (1.1) with $n = 3$ and $\Theta = 0$ was "invited" by Caffarelli, Nirenberg, and Spruck in [CNS].

In closing, we point out that any further regularity beyond continuity for continuous viscosity solutions to general special Lagrangian equation (1.1) is unknown. We are also curious to know whether there exist other $C^{1,\alpha}$ (no better) singular solutions to (1.1) with, in particular, irrational exponents α between those odd reciprocals $1/(2m-1)$. Meanwhile, we guess that all $C^{1,\alpha}$ for $\alpha > \frac{1}{3}$ solutions to special Lagrangian equation (1.1) with $n = 3$ should be regular (analytic). This regularity for $C^{1,1}$ solutions to (1.1) in dimension three was shown in [Y1]. Earlier on, Urbas [U, Theorem 1.1] proved the regularity for better than Pogorelov solutions, namely all $C^{1,\alpha}$ for $\alpha > 1 - \frac{2}{n}$ (convex) solutions to the (dual) Monge-Ampère equation $\ln\det D^2u = \ln\lambda_1 + \cdots + \ln\lambda_n = c$ are $C^{3,\beta}$ and eventually analytic. Finally recall that the singularities of Pogorelov-like singular solutions to Monge-Ampère equation extend (in fact, must, by Caffarelli [C]) to the boundary of the domain of the solutions; while the singularity of singular solutions so far constructed to special Lagrangian equation is in the interior of the domain.

2. Cauchy-Kowalevskaya with critical phase $\Theta = \frac{\pi}{2}$

As a preparation for the constructions in the next three sections, we solve the following special Lagrangian equation with critical phase in dimension three by Cauchy-Kowalevskaya. The quadratic nature of the equation at the critical phase is easier to work with than the cubic nature of the equations otherwise.

Our approximate solution $P(x)$ to the equation

$$\left\{\begin{array}{l}\sigma_2\left(D^2u\right)=\frac{1}{2}\left[(\triangle u)^2-\left|D^2u\right|^2\right]=1 \text{ or } \sum_{i=1}^3 \arctan\lambda_i=\frac{\pi}{2}\\ u_3\left(x_1,x_2,0\right)=P_3\left(x_1,x_2,0\right)\\ u\left(x_1,x_2,0\right)=P\left(x_1,x_2,0\right)\end{array}\right. \tag{2.1}$$

is a polynomial of degree $2m$

$$P=\frac{1}{2}\left(x_1^2+x_2^2\right)+\operatorname{Re}Z^m x_3+\frac{m^2}{4}\rho^{2m-2}x_3^2+\nu\sum_{j=0}^m a_j x_3^{2m-2j}\rho^{2j},$$

where $Z=x_1+\sqrt{-1}x_2=\rho\exp\left(\sqrt{-1}\theta\right)$, coefficients ν and a_js are to be determined later. We construct this P satisfying the following four properties, so does u then, for $|x|=r\le r_m$ with positive r_m depending only on m.

Property 2.1. $\sigma_2\left(D^2P\right)-1=\left[r^{3m-3}\right]$, here $\left[r^k\right]$ represents an analytic function starting from order k. Then the solution u coincides with P up to order $3m-2$ ($\ge 2m$ for $m\ge 2,3,4,\cdots$).

Property 2.2. The three eigenvalues of D^2P, then also D^2u satisfy

$$\lambda_1=1+\left[r^{m-1}\right]$$
$$\lambda_2=1+\left[r^{m-1}\right]$$
$$-\delta_2(m)\,r^{2m-2}\le\lambda_3\le-\delta_1(m)\,r^{2m-2}$$

Property 2.3. The "gradient" graph

$$(x,Du)=$$
$$\left(\begin{array}{c}x,x_1+O(\rho)\left[r^{m-1}\right]+\left[r^{2m}\right],\ x_2+O(\rho)\left[r^{m-1}\right]+\left[r^{2m}\right],\\ \operatorname{Re}Z^m+\frac{m^2}{2}\rho^{2m-2}x_3-2m\nu x_3^{2m-1}+\nu\rho^2\left[r^{2m-3}\right]+\left[r^{2m}\right]\end{array}\right).$$

Property 2.4. The gradient Du satisfies

$$\delta_3(m)\,r^{2m-1}\le|Du(x)|\le\delta_4(m)\,r.$$

We first find the equation near a quadratic solution. Let

$$u=\frac{1}{2}\left(\mu_1x_1^2+\mu_2x_2^2+\mu_3x_3^2\right)+w(x).$$

Then

$$\sigma_2\left(D^2u\right)-1=\frac{1}{2}\left[(\triangle u)^2-\left|D^2u\right|^2\right]-1$$
$$=\frac{1}{2}\left[\left(\mu_1+\mu_2+\mu_3+\triangle w\right)^2-\sum_{i=1}^3\left(\mu_i+w_{ii}\right)^2-2w_{12}^2-2w_{23}^2-2w_{13}^2\right]-1$$
$$=\mu_1\left(\triangle w-w_{11}\right)+\mu_2\left(\triangle w-w_{22}\right)+\mu_3\left(\triangle w-w_{33}\right)+\frac{1}{2}\left[(\triangle w)^2-\left|D^2w\right|^2\right]$$
$$+\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1-1.$$

Set $\mu_1 = \mu_2 = 1$ and $\mu_3 = 0$, we get

$$\begin{aligned}\sigma_2\left(D^2u\right) - 1 &= w_{11} + w_{22} + 2w_{33} + \frac{1}{2}\left[(\triangle w)^2 - \left|D^2 w\right|^2\right] \\ &= \tilde{\triangle} w + \frac{1}{2}\left[(\triangle w)^2 - \left|D^2 w\right|^2\right],\end{aligned}$$

where $\tilde{\triangle} = \partial_{11} + \partial_{22} + 2\partial_{33}$. To make the right hand side of the above equation vanish at high orders, we choose $w = h + Q + H$, where

$$\begin{aligned}h &= \operatorname{Re} Z^m x_3, \ \text{an ad hoc "harmonic" function;} \\ Q &= \frac{m^2}{4}\rho^{2m-2}x_3^2, \ \text{to match } \sigma_2\left(D^2h\right); \\ H &= \nu\left(-x_3^{2m} + \sum_{j=1}^{m} a_j x_3^{2m-2j}\rho^{2j}\right), \ \text{to make eigenvalue } \lambda_3 \text{ negative.}\end{aligned}$$

Then

$$\sigma_2\left(D^2u\right) - 1 = \underbrace{\tilde{\triangle} h}_{0} + \tilde{\triangle} Q + \tilde{\triangle} H + \frac{1}{2}\left[(\triangle h)^2 - \left|D^2 h\right|^2\right] + \left[r^{3m-3}\right].$$

A simple calculation leads to

$$D^2h = \begin{bmatrix} \operatorname{Re}\left[m(m-1)Z^{m-2}\right]x_3 & -\operatorname{Im}\left[m(m-1)Z^{m-2}\right]x_3 & \operatorname{Re} mZ^{m-1} \\ & -\operatorname{Re}\left[m(m-1)Z^{m-2}\right]x_3 & -\operatorname{Im} mZ^{m-1} \\ & & 0 \end{bmatrix}.$$

It follows that

$$\sigma_2\left(D^2h\right) = -\left[m(m-1)\rho^{m-2}\right]^2 x_3^2 - m^2\rho^{2m-2}.$$

Thus

$$\tilde{\triangle} Q + \sigma_2\left(D^2h\right) = \left[m(m-1)\rho^{m-2}\right]^2 x_3^2 + m^2\rho^{2m-2} + \sigma_2\left(D^2h\right) = 0.$$

Finally we fix the "harmonic" H satisfying $\tilde{\triangle} H = 0$ with

$$\begin{aligned}a_0 &= -1 \\ a_j &= -\frac{2\cdot(2m-2j+2)(2m-2j+1)}{(2j)^2}a_{j-1} \\ &= (-1)^{j+1}\frac{2^j 2m(2m-1)\cdots(2m-2j+1)}{2^2 4^2\cdots(2j)^2} \quad \text{for } j \geq 1,\end{aligned}$$

and ν is still pending. Therefore, $P = \frac{1}{2}\left(x_1^2 + x_2^2\right) + h + Q + H$, satisfies

$$\sigma_2\left(D^2P\right) - 1 = \left[r^{3m-3}\right].$$

Now the analytic solution u to (2.1) with initial data P follows from Cauchy-Kowalevskaya. As in [NV], considering the linear equation for difference $u - P$, the Cauchy-Kowalevskaya procedure implies that the solution u coincides with P up to order $3m-2$ ($\geq 2m$ for $m \geq 2$). Thus Property 2.1 is verified.

We move to Property 2.2. We have

$$D^2u = \begin{bmatrix} 1+\left[r^{m-1}\right] & \left[r^{m-1}\right] & \operatorname{Re} mZ^{m-1}+\left[r^{2m-2}\right] \\ & 1+\left[r^{m-1}\right] & -\operatorname{Im} mZ^{m-1}+\left[r^{2m-2}\right] \\ & & \frac{m^2}{2}\rho^{2m-2}+H_{33}+\left[r^{2m-1}\right] \end{bmatrix}. \tag{2.2}$$

Because the eigenvalues are Lipschitz functions of the matrix entries, we get

$$\lambda_1 = 1+\left[r^{m-1}\right]$$
$$\lambda_2 = 1+\left[r^{m-1}\right].$$

By the quadratic Taylor expansion of the isolated eigenvalue λ_3 in terms of the matrix entries near $D^2u\,(0)$, we obtain

$$\begin{aligned}\lambda_3 &= u_{33}-u_{13}^2-u_{23}^2+\left[r^{3m-3}\right] \\ &= \begin{array}{c}\frac{m^2}{2}\rho^{2m-2}+\nu\sum_{j=0}^m (2m-2j)\,(2m-2j-1)\,a_j x_3^{2m-2j}\rho^{2j} \\ -m^2\rho^{2m-2}+\left[r^{2m-1}\right] \quad \text{for } m\geq 2\end{array} \\ &= \begin{array}{c}\nu\left[-2m\,(2m-1)\,x_3^{2m-2}+\tilde{a}_2 x_3^{2m-4}\rho^2+\cdots+\tilde{a}_{m-1}\rho^{2m-2}\right]-\frac{m^2}{2}\rho^{2m-2} \\ +\left[r^{2m-1}\right]\end{array} \\ &= H_{33}-\frac{m^2}{2}\rho^{2m-2}+\left[r^{2m-1}\right].\end{aligned}$$

The "harmonic" function H_{33} cannot have a definite sign near the origin, but with the help of $-\frac{m^2}{2}\rho^{2m-2}$ and small ν, we make λ_3 negative. Let η be a small positive constant to be chosen shortly.

Case 1: $\eta\,|x_3|\geq\rho$. We have

$$\left[-2m\,(2m-1)\,x_3^{2m-2}+\tilde{a}_2 x_3^{2m-4}\rho^2+\cdots\right] = -\left[2m\,(2m-1)+O\,(1)\,\eta^2\right]x_3^{2m-2}.$$

Note $r/\sqrt{1+\eta^2}\leq |x_3|\leq r$, then

$$-\left\{\begin{array}{c}\nu\left[2m\,(2m-1)+O\,(1)\,\eta^2\right] \\ +\frac{m^2}{2}+o\,(1)\end{array}\right\}r^{2m-2}\leq\lambda_3\leq -\nu\left[\begin{array}{c}\frac{2m(2m-1)+O(1)\eta^2}{\left(\sqrt{1+\eta^2}\right)^{2m-2}} \\ +o\,(1)\end{array}\right]r^{2m-2}.$$

Case 2: $\eta\,|x_3|<\rho$. Note $r\eta/\sqrt{1+\eta^2}\leq\rho\leq r$, we have

$$\left[-2m\,(2m-1)\,x_3^{2m-2}+\tilde{a}_2 x_3^{2m-4}\rho^2+\cdots\right] = \frac{O\,(1)}{\eta^{2m-2}}\rho^{2m-2},$$

then

$$\lambda_3 = -\left[\frac{m^2}{2}-\frac{\nu O\,(1)}{\eta^{2m-2}}\right]\rho^{2m-2}+\left[r^{2m-1}\right]$$

and

$$-\left[\begin{array}{c}\frac{m^2}{2}-\frac{\nu O(1)}{\eta^{2m-2}} \\ +o\,(1)\end{array}\right]r^{2m-2}\leq\lambda_3\leq\left\{\begin{array}{c}-\left[\frac{m^2}{2}-\frac{\nu O(1)}{\eta^{2m-2}}\right]\frac{\eta^{2m-2}}{\left(\sqrt{1+\eta^2}\right)^{2m-2}} \\ +o\,(1)\end{array}\right\}r^{2m-2}.$$

We first choose $\eta = \eta(m) > 0$ small, next $\nu = \nu(\eta, m) > 0$ smaller, then there exist $\delta_1 = \delta(\eta, m) > 0$ and $\delta_2 = \delta_2(m) > 0$ such that

$$-\delta_2 r^{2m-2} \leq \lambda_3 \leq -\delta_1 r^{2m-2}$$

for $r \leq r_m$. Here r_m is within the valid radius for the Cauchy-Kowalevskaya solution u.

Property 2.3 follows from $u = P + \left[r^{3m-2}\right]$.

Finally we prove Property 2.4. The upper bound is straightforward. For the lower bound, from Property 2.3, we have

$$\begin{aligned}|Du(x)|^2 &= (x_1 + [r^m])^2 + (x_2 + [r^m])^2)^2 \\ &\quad + (ReZ^m + \frac{m^2}{2}\rho^{2m-2}x_3 - 2m\nu x_3^{2m-1} + \nu\rho^2[r^{2m-2}] + [r^{2m}])^2.\end{aligned}$$

Case 1: $x_3^2 \geq \rho$. From $r^2 = \rho^2 + x_3^2 \leq \left(x_3^2 + 1\right) x_3^2$, we know

$$|x_3| \geq r.$$

Note that the other terms than $-2m\nu x_3^{2m-1}$ in $u_3(x)$ have the following asymptotic behavior near the origin

$$\begin{aligned}|ReZ^m| &\leq \rho^m = x_3^{2m}, \\ \left|\frac{m^2}{2}\rho^{2m-2}x_3\right| &\leq \frac{m^2}{2}|x_3|^{4m-3}, \\ \nu\rho^2[r^{2m-2}] &= O(x_3^{2m+2}), \\ [r^{2m}] &= O(x_3^{2m}).\end{aligned}$$

It follows that

$$\begin{aligned}|Du(x)|^2 \geq |u_3(x)|^2 &= \left[-2m\nu x_3^{2m-1} + O(x_3^{2m})\right]^2 \\ &\geq \delta_3(m)\, x_3^{2(2m-1)} \geq \delta_3(m)\, r^{2(2m-1)}\end{aligned}$$

for $|x| \leq r_m$ with positive r_m and $\delta_3(m)$ to be fixed shortly.

Case 2: $x_3^2 < \rho$. From $r^2 = \rho^2 + x_3^2 \leq (\rho + 1)\rho$, we know

$$\rho > r^2.$$

Then

$$\begin{aligned}|Du(x)|^2 \geq u_1^2(x) + u_2^2(x) &= \rho^2 + 2x_1[r^m] + 2x_2[r^m] + 2[r^m]^2 \\ &= \rho^2 + O(\rho^{\frac{m+1}{2}}) \\ &\geq \frac{1}{2}\rho^2 \geq \frac{1}{2}r^4 \geq r^{2(2m-1)}\end{aligned}$$

for $\rho \leq r \leq r_m$ with the positive r_m to be fixed next.

Now we choose positive $\delta_3(m)$ small and the small positive r_m within the valid radius for Cauchy-Kowalevskaya solution u and Property 2.2, Property 2.4 is then completely justified.

Since $u(r_m x)/r_m^2$ is still a solution to $\sigma_2\left(D^2u\right)=1$ in $B_1\subset R^3$. We may assume the above constructed solution is already defined in $B_1\subset\mathbb{R}^3$. Note that $D\left[u(r_m x)/r_m^2\right]=Du(r_m x)/r_m$ and $D^2\left[u(r_m x)/r_m^2\right]=D^2u(r_m x)$, we see that Property 2.2 and Property 2.4 are still valid in B_1 with $\delta_1,\ \delta_2,\ \delta_3$ replaced by $r_m^{2m-2}\delta_1,\ r_m^{2m-2}\delta_2,\ r_m^{2m-2}\delta_2$ respectively, and δ_4 unchanged.

3. Rotate to subcritical phases $|\Theta|<\frac{\pi}{2}$: proof of Theorem 1.1

In this section, we carry out the construction of the singular solutions in Theorem 1.1 by "horizontally" and $\pi/2$ rotating the Cauchy-Kowalevskaya solutions from Section 2. The latter rotation, Proposition 3.1 is pivotal.

Step 1. Let $\alpha\in[0,\pi/4)$. We will take $\alpha=\Theta/2$ for $\Theta\in[0,\pi/2)$ in Step 3 of this section. We make a $U(3)$ rotation in $\mathbb{C}^3:\tilde z'=e^{\alpha\sqrt{-1}}z'$ and $\tilde z_3=z_3$ with $\tilde z=(\tilde z',\tilde z_3)=(\tilde x',\tilde x_3)+\sqrt{-1}(\tilde y',\tilde y_3)$ and $z=(z',z_3)=(x',x_3)+\sqrt{-1}(y',y_3)$. Because $U(3)$ rotations preserve the length and complex structure, $\mathfrak{M}=(x,Dv(x))$ for $x\in B_1$ is still a special Lagrangian submanifold in the new coordinate system with parameterization

$$\left\{\begin{array}{c}\tilde x=(x_1\cos\alpha+u_1(x)\sin\alpha,\ x_2\cos\alpha+u_2(x)\sin\alpha,\ x_3)\\ \tilde y=(-x_1\sin\alpha+u_1(x)\cos\alpha,\ -x_2\sin\alpha+u_2(x)\cos\alpha,\ u_3(x))\end{array}\right. . \tag{3.1}$$

We show that $\mathfrak{M}$ is also a "gradient" graph over $\tilde x$ space. From Property 2.2, we know that $u(x',x_3)$ is a convex function in terms of x' for $|x|\le 1$, or if necessary $|x|\le r_m$ with r_m depending only on m. From (2.2) we also assume $|D'u_3(x)|=|(u_{13},u_{23})(x)|\le 1/2$ for $|x|\le r_m$. Then we have

$$\delta_5(m)|x-x^*|^2\ge|\tilde x(x)-\tilde x(x^*)|^2 \tag{3.2}$$

$$=\left|\left(x'-x^{*\prime}\right)\cos\alpha+\left[\begin{array}{c}D'u(x',x_3)-D'u(x',x_3^*)\\+\underline{D'u(x',x_3^*)-D'u(x^{*\prime},x_3^*)}\end{array}\right]\sin\alpha\right|^2+|x_3-x_3^*|^2$$

$$\ge\left[\begin{array}{c}\frac12\left|(x'-x^{*\prime})\cos\alpha+\left(\underline{D'u(x',x_3^*)-D'u(x^{*\prime},x_3^*)}\right)\sin\alpha\right|^2\\-|(D'u(x',x_3)-D'u(x',x_3^*))\sin\alpha|^2+|x_3-x_3^*|^2\end{array}\right]$$

$$\ge\left[\begin{array}{c}\frac{\cos^2\alpha}{2}|x'-x^{*\prime}|^2+\cos\alpha\sin\alpha\underbrace{\left\langle x'-x^{*\prime},D'u\left(x',x_3^*\right)-D'u\left(x^{*\prime},x_3^*\right)\right\rangle}_{\ge0}\\-\sin^2\alpha\underbrace{2\left\|D'u_3\right\|_{L^\infty(B_{r_m})}}_{\le1}|x_3-x_3^*|^2+|x_3-x_3^*|^2\end{array}\right]$$

$$\ge\frac{\cos^2\alpha}{2}\left|x'-x^{*\prime}\right|^2+\left(1-\sin^2\alpha\right)|x_3-x_3^*|^2$$

$$\ge\frac14|x-x^*|^2. \tag{3.3}$$

It follows that $\mathfrak{M}$ is a special Lagrangian graph $(\tilde x,D\tilde u(\tilde x))$ over a domain containing a ball of radius $1/\sqrt2$ in $\tilde x$ space. The Hessian of the potential function $\tilde{u}$ satisfies

$$D^2\tilde{u} = \frac{\partial \tilde{y}}{\partial \tilde{x}} = \frac{\partial \tilde{y}}{\partial x}\left(\frac{\partial \tilde{x}}{\partial x}\right)^{-1}$$

$$= \begin{bmatrix} -\sin\alpha + u_{11}\cos\alpha & u_{12}\cos\alpha & u_{13}\cos\alpha \\ u_{12}\cos\alpha & -\sin\alpha + u_{22}\cos\alpha & u_{23}\cos\alpha \\ u_{13} & u_{23} & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} \cos\alpha + u_{11}\sin\alpha & u_{12}\sin\alpha & u_{13}\sin\alpha \\ u_{12}\sin\alpha & \cos\alpha + u_{22}\sin\alpha & u_{23}\sin\alpha \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \tan\left(\frac{\pi}{4}-\alpha\right) & & \\ & \tan\left(\frac{\pi}{4}-\alpha\right) & \\ & & 0 \end{bmatrix} + \left[r^{m-1}\right] \tag{3.4}$$

and

$$\det D^2\tilde{u} = \tan\left(\frac{\pi}{4}-\alpha\right)\left[-\frac{m^2}{2}\rho^{2m-2} + \tan\left(\frac{\pi}{4}-\alpha\right)H_{33}\right] - \left[r^{2m-1}\right], \tag{3.5}$$

where the above abused notation $\left[r^{m-1}\right]$ also represents a matrix whose entries are all analytic functions starting from order $m-1$, and (3.4) (3.5) follow from a simple calculation and the asymptotic behavior of D^2u, (2.2). We verify the following properties for $D^2\tilde{u}$. There exists a positive number $\tilde{r}_{m,\alpha}$ depending only on m and $\alpha \in [0, \pi/4)$ such that for $|\tilde{x}| \le \tilde{r}_{m,\alpha}$ we have:
Property 3.1. The determinant $\det D^2\tilde{u}(\tilde{x})$ is negative for small $\tilde{x} \neq 0$, indeed

$$\det D^2\tilde{u}(\tilde{x}) \approx -\tan\left(\frac{\pi}{4}-\alpha\right)|\tilde{x}|^{2m-2};$$

Property 3.2. The upper left 2×2 principle minor of the Hessian $D^2\tilde{u}$,

$$2\tan\left(\frac{\pi}{4}-\alpha\right)I \ge \left(D^2\tilde{u}\right)' \ge \frac{\tan\left(\frac{\pi}{4}-\alpha\right)}{2}I;$$

Property 3.3. The three eigenvalues $\tilde{\lambda}_i$ of the Hessian $D^2\tilde{u}$ satisfy

$$\begin{cases} \tilde{\theta}_1 = \arctan\tilde{\lambda}_1 = \left(\frac{\pi}{4}-\alpha\right)\left[1 + O\left(|\tilde{x}|^{m-1}\right)\right] \\ \tilde{\theta}_2 = \arctan\tilde{\lambda}_2 = \left(\frac{\pi}{4}-\alpha\right)\left[1 + O\left(|\tilde{x}|^{m-1}\right)\right] \\ \tilde{\theta}_3 = \arctan\tilde{\lambda}_3 \approx -\frac{1}{\tan\left(\frac{\pi}{4}-\alpha\right)}|\tilde{x}|^{2m-2}\left[1 + O\left(|\tilde{x}|^{m-1}\right)\right] \end{cases};$$

where " $\approx$" means two quantities are comparable up to a multiple of constant depending only on m and α. Relying on (3.5), repeating the arguments for the estimate of λ_3 in Section 2, using (3.3) and (3.1), we obtain Property 3.1. Property 3.2 follows from (3.4). From (3.3) (3.4) and the Lipschitz continuity of eigenvalues in terms of matrix entries, we derive the estimates for the first two eigenvalues in Property 3.3. In turn, noticing $\tilde{\lambda}_3 = \det D^2\tilde{u}/\left(\tilde{\lambda}_1\tilde{\lambda}_2\right)$, relying on both (3.2) and (3.3) we get two sided estimates of the last eigenvalue.

Step 2. We proceed with the following proposition.

Proposition 3.1. *Let $\mathcal{L}=(x,Df)$ be a Lagrangian surface in $\mathbb{C}^3=\mathbb{R}^3\times\mathbb{R}^3$ with the smooth potential f over $B_\rho\subset\mathbb{R}^3$, satisfying:*

$$Df(0)=0,$$
$$\det D^2 f(x)<0 \quad \text{for } x\neq 0,$$
$$\left\{\begin{array}{c}\kappa^{-1}I\geq\left[\begin{array}{cc}f_{11}(x) & f_{12}(x)\\ f_{21}(x) & f_{22}(x)\end{array}\right]\geq\kappa I\\ |D'f_3(x)|=|(f_{13},f_{23})(x)|\leq\frac{1}{2},\ \text{say}\end{array}\right\}\ \text{for } x\in B_\rho,\ \kappa>0. \tag{3.6}$$

Then $\mathcal{L}$ can be re-represented as a graph $(\tilde{x},\tilde{y})=\left(\tilde{x},D\tilde{f}(\tilde{x})\right)$ over the open set $\Omega=Df\left(B_{\frac{1}{2}\kappa^2\rho}\right)$ with $\tilde{x}+\sqrt{-1}\tilde{y}=e^{-\frac{\pi}{2}\sqrt{-1}}\left(x+\sqrt{-1}y\right)$ and $\tilde{f}\in C^1(\Omega)\cap C^\infty(\Omega\backslash\{0\})$.

Proof of Proposition 3.1. Note that the $U(3)$ rotation by $\pi/2$ is $(\tilde{x},\tilde{y})=(y,-x)$. This proposition really says that the map Df has a (unique) continuous inverse $\Phi=-D\tilde{f}$.

Step 2.1. We first prove Df is one-to-one on $B_{\kappa^2\rho}$. Consider a coordinate change given by $t=\Psi(x)=(f_1(x),f_2(x),x_3)$. Then the Jacobian of Ψ is

$$\det D_x\Psi(x)=\det\left[\begin{array}{ccc}f_{11} & f_{12} & f_{13}\\ f_{21} & f_{22} & f_{23}\\ 0 & 0 & 1\end{array}\right](x)=\det\left[\begin{array}{cc}f_{11} & f_{12}\\ f_{21} & f_{22}\end{array}\right](x)>0. \tag{3.7}$$

Hence Ψ is a local diffeomorphism on B_ρ. Note that Ψ is actually a distance expansion map. We have for all $x,\ x^{\#}$ in B_ρ

$$\left|\Psi(x)-\Psi\left(x^{\#}\right)\right|^2=\left|\begin{array}{c}D'f(x',x_3)-D'f\left(x^{\#\prime},x_3\right)\\ +D'f\left(x^{\#\prime},x_3\right)-D'f\left(x^{\#\prime},x_3^{\#}\right)\end{array}\right|^2+\left|x_3-x_3^{\#}\right|^2$$
$$\geq\left[\begin{array}{c}\frac{1}{2}\left|D'f(x',x_3)-D'f\left(x^{\#\prime},x_3\right)\right|^2-\left|D'f\left(x^{\#\prime},x_3\right)-D'f\left(x^{\#\prime},x_3^{\#}\right)\right|^2\\ +\left|x_3-x_3^{\#}\right|^2\end{array}\right]$$
$$=\left[\begin{array}{c}\frac{1}{2}\left|D'f(x',x_3)-D'f\left(x^{\#\prime},x_3\right)-\kappa\left(x'-x^{\#\prime}\right)+\kappa\left(x'-x^{\#\prime}\right)\right|^2\\ -\left|D'f\left(x^{\#\prime},x_3\right)-D'f\left(x^{\#\prime},x_3^{\#}\right)\right|^2+\left|x_3-x_3^{\#}\right|^2\end{array}\right]$$
$$\geq\left[\begin{array}{c}\underbrace{\left\langle D'f(x',x_3)-D'f\left(x^{\#\prime},x_3\right)-\kappa\left(x'-x^{\#\prime}\right),\kappa\left(x'-x^{\#\prime}\right)\right\rangle}_{\geq 0}\\ +\frac{1}{2}\left|\kappa\left(x'-x^{\#\prime}\right)\right|^2-2\left\|D'f\right\|^2_{L^\infty(B_\rho)}\left|x_3-x_3^{\#}\right|^2+\left|x_3-x_3^{\#}\right|^2\end{array}\right]$$
$$\geq\frac{\kappa^2}{2}\left|x'-x^{\#\prime}\right|^2+\left(1-2\left\|D'f\right\|^2_{L^\infty(B_\rho)}\right)\left|x_3-x_3^{\#}\right|^2$$
$$\geq\frac{\kappa^2}{2}\left|x-x^{\#}\right|^2, \tag{3.8}$$

where we used (3.6). Thus Ψ is a "global" diffeomorphism on B_ρ.

We claim that the Ψ-image of B_ρ, $\Psi(B_\rho) \supset B^t_{\frac{\kappa}{\sqrt{2}}\rho}$. Otherwise, let $t^\#$ be a boundary point of $\Psi(B_\rho)$ in $\mathring{B}^t_{\frac{\kappa}{\sqrt{2}}\rho}$. We know there exist a sequence of points $x_i \in B_\rho$ such that $\Psi(x_i)$ goes to $t^\#$ and x_i goes to $x^\# \in \bar{B}_\rho$ as i goes to infinity. If $x^\# \in \mathring{B}_\rho$, then $\Psi\left(x^\#\right) = t^\#$ by the continuity of Ψ. But this is impossible because $\Psi\left(x^\#\right)$ is an interior point of $\Psi(B_\rho)$ under the diffeomorphism of Ψ. If $x^\# \in \partial B_\rho$, from (3.8), we have

$$|t^\#| = \lim_{i\to\infty} |\Psi(x_i) - 0| \ \geq \lim_{i\to\infty} \frac{\kappa}{\sqrt{2}}|x_i - 0| = \frac{\kappa}{\sqrt{2}}|x^\#| = \frac{\kappa}{\sqrt{2}}\rho.$$

This contradicts $t^\# \in \mathring{B}^t_{\frac{\kappa}{\sqrt{2}}\rho}$.

From (3.6) (only the upper bound), then

$$|\Psi(x) - \Psi(x^\#)| \leq ||DD'f||_{L^\infty(B_\rho)}\ |x - x^\#| \leq \kappa^{-1}|x - x^\#|,$$

if follows that Ψ^{-1} is also a distance expansion map with a factor κ. Apply the arguments above we get $\Psi^{-1}(B^t_{\frac{1}{2}\kappa\rho}) \supset B_{\frac{1}{2}\kappa^2\rho}$ or $B^t_{\frac{1}{2}\kappa\rho} \supset \Psi(B_{\frac{1}{2}\kappa^2\rho})$.

Now for the injectivity of Df on $B_{\frac{1}{2}\kappa^2\rho}$, it suffices to show that

$$y\,(t) = Df \circ \Psi^{-1}\,(t) = (t_1, t_2,\ f_3(x\,(t))$$

is one-to-one in $B^t_{\frac{1}{2}\kappa\rho}$. Suppose that $y\,(t) = y\left(t^\#\right)$, then

$$t_1^\# = t_1,\ t_2^\# = t_2, \quad y_3(t^\#) = y_3(t).$$

Note that

$$\begin{aligned}\frac{\partial y_3\,(t_1, t_2, \xi)}{\partial t_3} &= \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & \frac{\partial y_3(t_1,t_2,\xi)}{\partial t_3} \end{bmatrix} \\ &= \det D_t\left(Df \circ \Psi^{-1}\right) = \det\left(D^2 f\right)\big|_{\Psi^{-1}(t_1,t_2,\xi)} \cdot \det D_t\Psi^{-1}\big|_{(t_1,t_2,\xi)} < 0\end{aligned}$$

for $(t_1, t_2, \xi) \neq 0$, where we used (3.7) and $\det D^2 f\,(x) < 0$ for $x \neq 0$. It follows that the function $y_3(t_1, t_2, \xi)$ is strictly decreasing in ξ. Now $y_3(t_1, t_2, t_3^\#) = y_3(t_1, t_2, t_3)$ implies $t_3^\# = t_3$. This shows that $y = Df \circ \Psi^{-1}$ is one-to-one.

So far we have obtained the inverse function $\Phi = (Df)^{-1}$ on $\Omega = Df\left(B_{\frac{1}{2}\kappa^2\rho}\right)$.

Step 2.2. We prove Df is an open map from B_ρ to $\mathbb{R}^3$. Since the Jacobian $\det D^2 f\,(x) \neq 0$ for $x \neq 0$, Df is already a local diffeomorphism for $x \neq 0$. It suffices to show that the image of an open neighborhood of 0 in B_ρ, under Df, contains an open neighborhood of 0 in $\mathbb{R}^3$. Since Ψ is a diffeomorphism, we only need to show this property for $Df \circ \Psi^{-1}$. Indeed we only need to consider the image of the ball $B^t_{2\eta}$ of radius 2η centered at $t = 0$ for a small $\eta > 0$. According to Step 2.1, $y\,(t_1, t_2, \cdot)$ is strictly decreasing in the third variable. So $2h_- = y_3(0, 0, \eta) < 0$ and $2h_+ = y_3(0, 0, -\eta) > 0$. By continuity of $y = Df \circ \Psi^{-1}$, there exists $\eta' \in (0, \eta)$ such that $y_3(t_1, t_2, \eta) <$

$h_- < 0$ and $y_3(t_1, t_2, -\eta) > h_+ > 0$ for $|(t_1, t_2)| \le \eta'$. Then by intermediate value theorem (for function $y_3(t_1, t_2, \cdot)$), the open set

$$\begin{aligned}\{|(y_1, y_2)| < \eta'\} \times \{h_- < y_3 < h_+\} &\subset Df \circ \Psi^{-1}(\{|(t_1, t_2)| \le \eta'\} \times \{|t_3| \le \eta\}) \\ &\subset Df \circ \Psi^{-1}(B^t_{2\eta}).\end{aligned}$$

Thus Df is an open map.

Step 2.3. Now $\Omega = Df(B_{\kappa^2\rho})$ is an open neighborhood of $y = 0$, and Φ is continuous on Ω by the openness of Df. Lastly we find a potential for the Lagrangian submanifold $\mathcal{L}$ now represented as $(\tilde{x}, -\Phi(\tilde{x}))$. Let

$$\tilde{f}(\tilde{x}) = \int_0^{\tilde{x}} -\Phi^1(s)ds_1 - \Phi^2(s)ds_2 - \Phi^3(s)ds_3.$$

Because $D_{\tilde{x}}(-\Phi(\tilde{x})) = -\left(D^2 f\right)^{-1}$ is symmetric when $\tilde{x} \ne 0$ and Φ is bounded, $\tilde{f}(\tilde{x})$ is well-defined on Ω. Further we know $\tilde{f} \in C^1(\Omega) \cap C^\infty(\Omega \setminus \{0\})$.

The proof for Proposition 3.1 is complete. □

Remark. For the purpose of Theorem 1.1, we can replace (3.6) by a weaker condition (3.7) $\det\left[D^2 f\right]' > 0$. Consequently we have no estimate on the size of the existing neighborhood supporting the solution $\tilde{\tilde{u}}$ in this section, then u^m for each single m and Θ. The stronger assumption (3.6) is designed for Theorem 1.2 where we need a uniform control with respect to ε on the valid radius for the solutions $\tilde{\tilde{u}}^\varepsilon$, then u^ε. Lastly there is another argument for the openness of the particular map $D\tilde{u} = Df$, relying on the uniqueness of the pre-image of $y = 0$ (which can be also derived from the distance expansion property at the origin (4.3)), instead of using the strict monotonicity property in Step 2.2.

Step 3. Equipped with Property 3.1, 3.2, and (3.4), we apply Proposition 3.1 to our function $\tilde{u}(\tilde{x})$ with $\tilde{x}$ replaced by $\tilde{\tilde{x}}$, x replaced by $\tilde{x}$, $\rho = \tilde{r}_{m,\alpha}$, and $\kappa = \tan\left(\frac{\pi}{4} - \alpha\right)/2$. Then we get a new C^1 function $\tilde{\tilde{u}}\left(\tilde{\tilde{x}}\right)$, defined on an open neighborhood of $\tilde{\tilde{x}} = 0$. By Property 3.3, the three eigen-angles $\tilde{\tilde{\theta}}_i = \arctan \tilde{\tilde{\lambda}}_i$ of $D^2\tilde{\tilde{u}}\left(\tilde{\tilde{x}}\right)$ away from the origin satisfy

$$\left\{\begin{array}{l} \tilde{\tilde{\theta}}_1 = \tilde{\theta}_1 - \frac{\pi}{2} = -\frac{\pi}{4} - \alpha + o(1) \\ \tilde{\tilde{\theta}}_2 = \tilde{\theta}_2 - \frac{\pi}{2} = -\frac{\pi}{4} - \alpha + o(1) \\ \tilde{\tilde{\theta}}_3 = \tilde{\theta}_3 - \frac{\pi}{2} + \pi \\ \quad = \frac{\pi}{2} - \frac{\delta_{m,\alpha}(x)}{\tan\left(\frac{\pi}{4} - \alpha\right)} \left|D\tilde{\tilde{u}}\left(\tilde{\tilde{x}}\right)\right|^{2m-2} \left[1 + O\left(\left|D\tilde{\tilde{u}}\left(\tilde{\tilde{x}}\right)\right|^{m-1}\right)\right] \end{array}\right. , \tag{3.9}$$

where the positive number $\delta_{m,\alpha}(x)$ is bounded from both below and above uniformly with respect to $\tilde{\tilde{x}}$, and

$$\sum_{i=1}^{3} \tilde{\tilde{\theta}}_i = -2\alpha.$$

We verify $\tilde{\tilde{u}}$ is still a viscosity solution to (1.1) with $\Theta = -2\alpha$ across the origin. For any quadratic Q touching $\tilde{\tilde{u}}$ at the origin from below, we have under the "diagonalized" coordinate system for $D^2\tilde{\tilde{u}}\,(0)$

$$D^2 Q \leq \begin{bmatrix} \tan\left(-\frac{\pi}{4}-\alpha\right) & & \\ & \tan\left(-\frac{\pi}{4}-\alpha\right) & \\ & & \infty \end{bmatrix}.$$

It follows that the eigenvalues λ_i^* of D^2Q must satisfy

$$\arctan\lambda_1^* \leq -\frac{\pi}{4}-\alpha, \ \arctan\lambda_2^* \leq -\frac{\pi}{4}-\alpha, \ \text{and} \ \arctan\lambda_3^* < \frac{\pi}{2}.$$

Then the quadratic satisfies

$$\sum_{i=1}^{3}\arctan\lambda_i^* < -2\alpha.$$

Observe that we can never arrange any quadratic touching $\tilde{\tilde{u}}$ from above at the origin. Then there is nothing to check. When those testing quadratics touch the smooth $\tilde{\tilde{u}}$ away from the origin, the verification according to the definition of viscosity solutions is straightforward. Thus $\tilde{\tilde{u}}$ is a viscosity solution to (1.1) with $\Theta = -2\alpha$ in a neighborhood of the origin.

Step 4. Lastly we verify that the solution $\tilde{\tilde{u}}$ is in fact $C^{1,1/(2m-1)}$ but not $C^{1,\delta}$ for any $\delta > 1/(2m-1)$ in a neighborhood of the origin. The latter is easy. From Property 2.3 and (3.3), we see that

$$(0,0,\tilde{x}_3, D\tilde{u}\,(0,0,\tilde{x}_3)) = \left(0,0,\tilde{x}_3, \left[\tilde{x}_3^{2m}\right], \left[\tilde{x}_3^{2m}\right], -2m\varepsilon\tilde{x}_3^{2m-1} + \left[\tilde{x}_3^{2m}\right]\right).$$

It follows that

$$\frac{|\tilde{x}_3 - 0|}{|D\tilde{u}\,(0,0,\tilde{x}_3) - D\tilde{u}\,(0)|^{\delta}} = \frac{|\tilde{x}_3|}{(2m\varepsilon + [\tilde{x}_3])^{\delta}\,|\tilde{x}_3|^{(2m-1)\delta}} \to \infty$$

as $\tilde{x}_3 \to 0$ for any $\delta > \frac{1}{2m-1}$. This shows that $\tilde{\tilde{u}}$ is not $C^{1,\delta}$ for any $\delta > 1/(2m-1)$.

Next we prove that $\tilde{\tilde{u}}$ is $C^{1,1/(2m-1)}$ by the argument in [NV]. Observe that for $i=1,2,3$

$$\begin{aligned}
&\left[\frac{\left|\tilde{\tilde{u}}_i\left(\tilde{\tilde{x}}\right)-\tilde{\tilde{u}}_i\left(\tilde{\tilde{x}}^*\right)\right|}{\left|\tilde{\tilde{x}}-\tilde{\tilde{x}}^*\right|^{1/(2m-1)}}\right]^{2m-1}\\
&=\left[\frac{\left|\tilde{\tilde{u}}_i\left(\tilde{\tilde{x}}\right)-\tilde{\tilde{u}}_i\left(\tilde{\tilde{x}}^*\right)\right|}{\left|\tilde{\tilde{u}}_i^{2m-1}\left(\tilde{\tilde{x}}\right)-\tilde{\tilde{u}}_i^{2m-1}\left(\tilde{\tilde{x}}^*\right)\right|^{1/(2m-1)}}\right]^{2m-1}\frac{\left|\tilde{\tilde{u}}_i^{2m-1}\left(\tilde{\tilde{x}}\right)-\tilde{\tilde{u}}_i^{2m-1}\left(\tilde{\tilde{x}}^*\right)\right|}{\left|\tilde{\tilde{x}}-\tilde{\tilde{x}}^*\right|}\\
&\leq C\left(m\right)\left(2m-1\right)\sup_{\tilde{\tilde{x}}}\left|\tilde{\tilde{u}}_i\left(\tilde{\tilde{x}}\right)\right|^{2m-2}\left|D\tilde{\tilde{u}}_i\left(\tilde{\tilde{x}}\right)\right|\\
&\leq C\left(m\right)\sup_{\tilde{\tilde{x}}}\left|D\tilde{\tilde{u}}\left(\tilde{\tilde{x}}\right)\right|^{2m-2}\left|D^2\tilde{\tilde{u}}\left(\tilde{\tilde{x}}\right)\right|\\
&\leq C\left(m\right)\sup_{\tilde{\tilde{x}}}\left|D\tilde{\tilde{u}}\left(\tilde{\tilde{x}}\right)\right|^{2m-2}\frac{1}{\left|D\tilde{\tilde{u}}\left(\tilde{\tilde{x}}\right)\right|^{2m-2}}\\
&\leq C\left(m\right),
\end{aligned}$$

where we used the fact that the scalar function $t^{1/(2m-1)}$ is $C^{1/(2m-1)}\left(\mathbb{R}^1\right)$ for the first inequality, and (3.9) for the third inequality.

Finally by scaling $u^m\left(x\right)=\tilde{\tilde{u}}\left(\tau x\right)/\tau^2$ with valid radius τ implicitly depending on m and the $\tilde{r}_{m,\alpha}$ in Step 1 (We need to make this dependence explicit and then to have a uniform control with respect to ε on the valid radius for the solutions u^ε in Section 4. Our guaranteed valid radius goes to zero as m goes to infinity.), the desired solutions in Theorem 1.1 with each fixed $\Theta\in(-\frac{\pi}{2},0]$ are achieved. By symmetry, $-u^m$ are the sought solutions with phase $\Theta\in[0,\frac{\pi}{2})$.

4. Rotate to smooth solutions: proof of Theorem 1.2

In this section, we create the desired family of solutions by another corresponding families of $U\left(3\right)$ rotations in $\mathbb{C}^3$ on top those two in Section 3. For any fixed $\Theta\in[0,\frac{\pi}{2})$, let $4\gamma=\frac{\pi}{2}-\Theta>0$. We start the construction by taking small positive numbers $\varepsilon\in(0,\gamma)$ and solution u with fixed m in Section 1.

Step 1. We take the $U\left(3\right)$ rotation in Step 1 of Section 3 with $\alpha=\frac{\Theta}{2}-\frac{3\varepsilon}{2}$. The valid radius of the rotation and the estimates of the Hessian $D^2\tilde{u}$ are still valid. To prepare the final rotations in the last Step of this section, we require the following estimates of $D^2\tilde{u}$ with eigenvalues $\tilde{\lambda}_i$ by shrinking the radius for $\tilde{x}$ or $|x|\leq r_\Theta$:

$$\left\{\begin{array}{l}
\tilde{\theta}_1^\varepsilon=\arctan\tilde{\lambda}_1^\varepsilon=\frac{\pi}{4}-\frac{\Theta}{2}+\frac{3\varepsilon}{2}+o\left(1\right)\geq\gamma\\
\tilde{\theta}_2^\varepsilon=\arctan\tilde{\lambda}_2^\varepsilon=\frac{\pi}{4}-\frac{\Theta}{2}+\frac{3\varepsilon}{2}+o\left(1\right)\geq\gamma\\
\tilde{\theta}_3^\varepsilon=\arctan\tilde{\lambda}_3^\varepsilon=-\frac{\delta_{m,\alpha}(\tilde{x})}{\tan\left(\frac{\pi}{4}-\frac{\Theta}{2}+\frac{3\varepsilon}{2}\right)}\left|\tilde{x}\right|^{2m-2}\left[1+O\left(\left|\tilde{x}\right|^{m-1}\right)\right]
\end{array}\right.,\tag{4.1}$$

where again the positive $\delta_{m,\alpha}(\tilde{x})$ is bounded from both below and above uniformly with respect to $\tilde{x}$ and ε, further the above estimates and r_Θ are both uniform with respect to ε.

Step 2. Exactly as in Step 3 of Section 3, we apply Proposition 3.1 with $\rho = r_\Theta$ and $\kappa = \tan\left(\frac{\pi}{4} - \alpha\right)/2$ to $(\tilde{x}, D\tilde{u})$ to get the potential $\tilde{\tilde{u}}^\varepsilon$ with $\left(\tilde{\tilde{x}}, D\tilde{\tilde{u}}^\varepsilon\right)$ for $\tilde{\tilde{x}} \in D\tilde{u}\left(B_{\frac{1}{2}\kappa^2 r_\Theta}\right)$. It follows from (4.1) and (3.9) that

$$\left\{\begin{array}{l} \tilde{\tilde{\theta}}_1 = -\frac{\pi}{4} - \frac{\Theta}{2} + \frac{3\varepsilon}{2} + o(1) \geq \gamma - \frac{\pi}{2} \\ \tilde{\tilde{\theta}}_2 = -\frac{\pi}{4} - \frac{\Theta}{2} + \frac{3\varepsilon}{2} + o(1) \geq \gamma - \frac{\pi}{2} \\ \tilde{\tilde{\theta}}_3 = \frac{\pi}{2} - |o(1)| \end{array}\right. \tag{4.2}$$

for $\left|\tilde{\tilde{x}}\right| = |D\tilde{u}(\tilde{x})| \leq \tilde{\tilde{r}}_\Theta$. We need to show this $\tilde{\tilde{r}}_\Theta$ and the above $o(1)$ terms are still uniform with respect to ε, and also the $|o(1)|$ term for $\tilde{\tilde{\theta}}_3$ never vanishes when the input $\tilde{\tilde{x}}$ does not vanish (actually this $|o(1)|$ can be made explicit enough by (4.3)). All these can be seen from the following inequalities

$$\delta_6(m)\,|\tilde{x}| \geq \left|\tilde{\tilde{x}}(\tilde{x})\right| = |D\tilde{u}(\tilde{x})| \geq \delta_7(m)\,|\tilde{x}|^{2m-1}. \tag{4.3}$$

Indeed we see the first inequality by recalling (3.1)

$$D\tilde{u}(\tilde{x}(x)) = \tilde{y}(x) = \left(\cos\alpha\ D'u(x) - \sin\alpha\ x', u_3(x)\right),$$
$$\left(D'u, u_3\right)\left(x', x_3\right) = Du(x) \in C^1,$$

and (3.2). We have to work a little harder for the second inequality. Because of (2.2) and $\alpha \in \left(-\frac{3}{2}\gamma, \frac{\pi}{4} - 4\gamma\right)$, the following convexity for function

$$u'_{x_3}\left(x'\right) = \cos\alpha\ u\left(x', x_3\right) - \frac{\sin\alpha}{2}\left|x'\right|^2$$

is available

$$\cos\alpha\ \left[D^2u(x)\right]' - \sin\alpha\begin{bmatrix} 1 & \\ & 1\end{bmatrix} \geq \frac{\cos\alpha - \sin\alpha}{2} I > 0$$

for $|(x', x_3)| \leq r_\Theta$, where we shrink r_Θ if necessary. Then we get

$$\begin{aligned} |\tilde{y}(x)|^2 &= \left|\left(Du'_{x_3}\left(x'\right), u_3(x)\right)\right|^2 = \left|\left(Du'_{x_3}\left(x'\right) - bx' + bx', u_3(x)\right)\right|^2 \\ &\geq \left|bx'\right|^2 + 2\underbrace{\left\langle Du'_{x_3}\left(x'\right) - bx', bx'\right\rangle}_{\geq 0} + |u_3(x)|^2 \\ &\geq b^2\left|x'\right|^2 + |u_3(x)|^2, \end{aligned}$$

where we set $b = (\cos\alpha - \sin\alpha)/2$ for simplicity of notation. In order to bound $|x'|^2$ from below, we use Property 2.3 to obtain

$$\begin{aligned}\left|D'u\left(x',x_3\right)\right|^2 &\leq 2\left|D'u\left(x',x_3\right) - D'u\left(0,x_3\right)\right|^2 + 2\left|D'u\left(0,x_3\right)\right|^2\\ &\leq C_m\left|x'\right|^2 + \left|\left[r^{2m}\right]\right|^2, \quad \text{or}\\ \left|x'\right|^2 &\geq \frac{1}{C_m}\left|D'u\left(x\right)\right|^2 - \left|\left[r^{2m}\right]\right|^2.\end{aligned}$$

Hence

$$|\tilde{y}(x)|^2 \geq \frac{b^2}{C_m}|Du(x)|^2 - \left|\left[r^{2m}\right]\right|^2,$$

where we assumed the positive $b^2/C_m \leq 1$ without loss of generality. By virtue of Property 2.4, we get

$$\begin{aligned}|\tilde{y}(x)|^2 &\geq \frac{b^2}{C_m}\left|r^{2m-1}\right|^2 - \left|\left[r^{2m}\right]\right|^2\\ &\geq \frac{(\delta_7(m))^2}{\delta_5(m)}\left(|x|^{2m-1}\right)^2\end{aligned}$$

for $|x| \leq r_\Theta$ and small positive $\delta_7(m,\alpha)$, where again we shrink r_Θ if necessary. By (3.2) we arrive at the second inequality of (4.3).

Step 3. We make a final family of $U(3)$ rotations in $\mathbb{C}^3 : \tilde{\tilde{z}} = e^{\varepsilon\sqrt{-1}}\tilde{z}$. Again because $U(3)$ rotation preserves the length and complex structure, $\mathfrak{M} = \left(\tilde{x}, D\tilde{u}^\varepsilon\right)$ for $\left|\tilde{r}\right| \leq \tilde{r}_\Theta$ still a smooth special Lagrangian submanifold with parameterization

$$\left\{\begin{array}{l}\tilde{\tilde{x}} = \tilde{x}\cos\varepsilon + D\tilde{u}^\varepsilon\left(\tilde{x}\right)\sin\varepsilon\\ \tilde{\tilde{y}} = -\tilde{x}\sin\varepsilon + D\tilde{u}^\varepsilon\left(\tilde{x}\right)\cos\varepsilon\end{array}\right..$$

We show that $\mathfrak{M}$ is still a "gradient" graph over $\tilde{\tilde{x}}$ space. From (4.2) we know that the function $\tilde{u}^\varepsilon\left(\tilde{x}\right) + \frac{1}{2}\tan\left(\frac{\pi}{2} - \gamma\right)\left|\tilde{x}\right|^2$ is convex. We then have

$$\begin{aligned}\left|\tilde{\tilde{x}}\left(\tilde{x}\right) - \tilde{\tilde{x}}\left(\tilde{x}^*\right)\right|^2 &= \left|\left(\tilde{x} - \tilde{x}^*\right)\cos\varepsilon + \left(D\tilde{u}^\varepsilon\left(\tilde{x}\right) - D\tilde{u}^\varepsilon\left(\tilde{x}^*\right)\right)\sin\varepsilon\right|^2\\ &= \left|\begin{array}{c}\left(\tilde{x} - \tilde{x}^*\right)\left[\cos\varepsilon - \tan\left(\frac{\pi}{2} - \gamma\right)\sin\varepsilon\right] +\\ + \left[\left(D\tilde{u}^\varepsilon\left(\tilde{x}\right) - D\tilde{u}^\varepsilon\left(\tilde{x}^*\right)\right) + \left(\tilde{x} - \tilde{x}^*\right)\tan\left(\frac{\pi}{2} - \gamma\right)\right]\sin\varepsilon\end{array}\right|^2\\ &\geq \left\{\begin{array}{c}\left|\left(\tilde{x} - \tilde{x}^*\right)\left(\cos\varepsilon - \tan\left(\frac{\pi}{2} - \gamma\right)\sin\varepsilon\right)\right|^2 +\\ +2\left[\begin{array}{c}\cos\varepsilon -\\ \tan\left(\frac{\pi}{2} - \gamma\right)\sin\varepsilon\end{array}\right]\sin\varepsilon\underbrace{\left\langle \tilde{x} - \tilde{x}^*, \left[\begin{array}{c}\left(D\tilde{u}^\varepsilon\left(\tilde{x}\right) - D\tilde{u}^\varepsilon\left(\tilde{x}^*\right)\right) +\\ \left(\tilde{x} - \tilde{x}^*\right)\tan\left(\frac{\pi}{2} - \gamma\right)\end{array}\right]\right\rangle}_{\geq 0}\end{array}\right\}\\ &= \left|\tilde{x} - \tilde{x}^*\right|^2\cos^2\varepsilon\left(1 - \frac{\tan\varepsilon}{\tan\gamma}\right)^2 \geq \frac{1}{4}\left|\tilde{x} - \tilde{x}^*\right|^2\end{aligned}$$

provided we take $\varepsilon \in (0,\gamma)$ even smaller. It follows that the smooth $\mathfrak{M}$ is a special Lagrangian graph $\left(\tilde{\tilde{x}}, D\tilde{\tilde{u}}^\varepsilon\left(\tilde{\tilde{x}}\right)\right)$ over a domain containing a ball of radius $\frac{1}{2}\tilde{\tilde{r}}_\Theta$ in $\tilde{\tilde{x}}$ space. The eigenvalues $\tilde{\tilde{\lambda}}_i^\varepsilon$ of the Hessian $D^2\tilde{\tilde{u}}^\varepsilon$ satisfy

$$\left\{\begin{array}{l}\tilde{\tilde{\theta}}_1^\varepsilon = \arctan \tilde{\tilde{\lambda}}_1^\varepsilon = -\frac{\pi}{4} - \frac{\Theta}{2} + \frac{3\varepsilon}{2} - \varepsilon + o\left(1\right) \\ \tilde{\tilde{\theta}}_2^\varepsilon = \arctan \tilde{\tilde{\lambda}}_2^\varepsilon = -\frac{\pi}{4} - \frac{\Theta}{2} + \frac{3\varepsilon}{2} - \varepsilon + o\left(1\right) \\ \tilde{\tilde{\theta}}_3^\varepsilon = \arctan \tilde{\tilde{\lambda}}_3^\varepsilon = \frac{\pi}{2} - \varepsilon - \left|o\left(1\right)\right| \end{array}\right. . \tag{4.4}$$

It follows that $\tilde{\tilde{u}}^\varepsilon$ is smooth and satisfies

$$\arctan \tilde{\tilde{\lambda}}_1^\varepsilon + \arctan \tilde{\tilde{\lambda}}_2^\varepsilon + \arctan \tilde{\tilde{\lambda}}_3^\varepsilon = -\Theta \quad \text{in} \quad B_{\frac{1}{2}\tilde{\tilde{r}}_\Theta}.$$

Finally set

$$u^\varepsilon\left(x\right) = -\frac{\tilde{\tilde{u}}^\varepsilon\left(\frac{1}{2}\tilde{\tilde{r}}_\Theta\ x\right)}{\left(\frac{1}{2}\tilde{\tilde{r}}_\Theta\right)^2}.$$

Observe that the gradients of the potential functions, or the heights of the special Lagrangian graphs are kept uniformly bounded with respect to ε under the above three families of $U\left(3\right)$ rotations. Combined with (4.4), we obtain the desired family of smooth solutions to (1.1) with $n = 3$ and fixed $\Theta \in [0, \pi/2)$. By symmetry, $-u^\varepsilon$ are the other family of solutions to (1.1) with $n = 3$ and fixed $\Theta \in (-\pi/2, 0]$.

5. Minimal surface system: proof of Theorem 1.3

In this section, we prove Theorem 1.3. Take the singular solutions u^m from Theorem 1.1 with $\Theta = 0$ and $m = 2, 3, 4, \cdots$. Let

$$U^m = Du^m.$$

From Property 2.2 and Proposition 3.1, we see that

$$\left|DU^m\left(y\right)\right| = \left|D^2u^m\left(y\right)\right| \approx \frac{1}{\left|Du^m\left(y\right)\right|^{2m-2}}.$$

Here " $\approx$ " means two quantities are equivalent up to a multiple of constant depending only on the dimension and m. Then we have

$$\begin{aligned}\int_{B_1}\left|DU^m\left(y\right)\right|^p dy &\approx \int_{B_1}\frac{1}{\left|Du^m\left(y\right)\right|^{(2m-2)p}}dy \\ &= \int_{Du^m(B_1)}\frac{1}{\left|x\right|^{(2m-2)p}}\left|\det\left[D^2u^m\left(y\right)\right]^{-1}\right| dx \\ &\approx \int_{Du^m(B_1)}\frac{1}{\left|x\right|^{(2m-2)p}}\left|x\right|^{2m-2} dx_1dx_2dx_3.\end{aligned}$$

It follows that

$$U^m \in W^{1,p}\left(B_1\right) \quad \text{for any } p < \frac{2m+1}{2m-2} \quad \text{but} \quad U^m \notin W^{1,\frac{2m+1}{2m-2}}\left(B_1\right).$$

We next show that U^m satisfies (1.2) in the integral sense, namely

$$\int_{B_1} \sum_{i,j=1}^{3} \sqrt{g} g^{ij} \left\langle \partial_{x_i} U, \partial_{x_j} \Phi \right\rangle dx = 0$$

for all $\Phi \in C_0^\infty \left(B_1, \mathbb{R}^3\right)$. This is because the integrand is 0 everywhere except at the origin and we have the following bound on the integrand near the origin. Diagonalizing $D^2 u^m$, we see that

$$\left| \sum_{i,j=1}^{3} \sqrt{g} g^{ij} \left\langle \partial_{x_i} U, \partial_{x_j} \Phi \right\rangle \right| = \left| \sum_{i=1}^{3} \sqrt{\left(1+\lambda_1^2\right) \cdots \left(1+\lambda_3^2\right)} \frac{\lambda_i}{1+\lambda_i^2} \partial_i \Phi^i \right|$$
$$\leq C\left(3, m\right) \left| D^2 u^m \right| \left| D\Phi \right| = C\left(3, m\right) \left| DU^m \right| \left| D\Phi \right| \in L^1,$$

where we used again the fact (3.9) that two of the eigenvalues of $D^2 u^m$ are bounded. The first part of the Theorem 1.3 is proved.

The second part of Theorem 1.3 is straightforward if we take $U^\varepsilon = Du^\varepsilon$ with smooth solutions u^ε in Theorem 1.2 for any fixed $\Theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

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