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A Priori Estimates for Lagrangian Mean Curvature Flows

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We derive a priori estimates for solutions of Lagrangian mean curvature flows. A Liouville-type result for ancient solutions is also obtained in the course of the arguments.

1 Introduction

In this paper, we derive a priori $C^{2+1,1+\frac{1}{2}}$ estimates for the fully nonlinear parabolic equations

$$u_t = \sum_{i=1}^{n} \arctan \lambda_i. \tag{1}$$

Here, $u(x, t)$ is a scalar function and λ_i's are the eigenvalues of the Hessian D^2u.

Equation (1) is the potential equation for the Lagrangian mean curvature flow. The (Lagrangian) submanifold $(x, Du(x, t)) \subset \mathbb{R}^{2n}$ is deforming according to its mean curvature.

Let $Q_r(x, t) = B_r(x) \times [t - r^2, t] \subset \mathbb{R}^n \times (-\infty, 0]$ and $Q_r := Q_r(0, 0)$. Let $X = (x, t)$. Set the (space–time semi) norm for functions in $Q_r(X)$

$$[f]_{1,\frac{1}{2};Q_r(X)} = \sup_{\substack{X^i \in Q_r(X) \\ X^1 \neq X^2}} \frac{|f(X^1) - f(X^2)|}{\max\{|x^1 - x^2|, |t^1 - t^2|^{1/2}\}}.$$

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Our main $C^{2+1,1+\frac{1}{2}}$ result is the following.

Theorem 1.1. Let $u(x,t)$ be a smooth solution to (1) in $Q_1 \subset \mathbb{R}^n \times (-\infty, 0]$. When $n \geq 4$ we also assume that at least one of the following conditions holds in Q_1

- $\Theta \geq (n-2)\frac{\pi}{2}$ with $\Theta = \sum_{i=1}^n \arctan \lambda_i(x,t)$,
- $3 + \lambda_i^2(x,t) + 2\lambda_i(x,t)\lambda_j(x,t) \geq 0$ for all $1 \leq i, j \leq n$.

Then we have

$$[u_t]_{1,\frac{1}{2};Q_{1/2}} + [D^2 u]_{1,\frac{1}{2};Q_{1/2}} \leq C(||D^2 u||_{L^\infty(Q_1)}). \qquad \square$$

The strategy of the proof is similar to that of the elliptic case in [8], where $C^{2,\alpha}$ a priori estimates for solutions to special Lagrangian equations were established via a Liouville property and a "self-improving" argument. The Liouville property was used in showing the Hessian is in VMO (vanishing mean oscillation) space. However, the execution is different here. Since we start with smooth (actually C^3) solutions, we reach the $C^{2+1,1+\frac{1}{2}}$ (or any finite order) estimates directly by a compactness argument. Note that our estimates are independent of the modulus of continuity of Hessian D^2u. When the modulus of continuity is available, one can get the estimates depending on the modulus from the standard bootstrapping arguments. We do not yet have the same a priori estimates for $C^{1+1,0+1}$ solutions, because the needed Liouville property obtained in Section 2 is not available to us for those less regular solutions to (1).

Our Liouville-type result for ancient solutions (Proposition 2.1) relies on the "new" nondivergence equation (3). One cannot draw the same Liouville-type result from the usual divergence parabolic equation for Θ (under a re-parameterization ξ) $\Theta_t = \frac{1}{\sqrt{g(\xi,t)}}\partial_{\xi_i}(\sqrt{g}g^{ij}\partial_{\xi_j}\Theta)$, as the coefficient $\frac{1}{\sqrt{g(\xi,t)}}$ also depends on time, in turn, Moser's Hölder estimates for divergence parabolic equations would not apply. Our Liouville-type result also provides a quick "PDE" argument for the rigidity of translating solutions to (1); see the appendix.

When equation (1) is concave, for example, if solutions are convex (available when $\Theta \geq (\mathbf{n}-\mathbf{1})\pi/2$), the $C^{2+\alpha,1+\alpha}$ (then higher order) estimates also follow from the well-known result of Krylov [3]. For the initial data $u(x,0)$ with restriction $-(1-\delta)I \leq D^2u \leq (1-\delta)I$ on $\mathbb{R}^n$, or equivalently a new initial Hessian with positive lower and upper bounds, the long time existence and a priori estimates for (1) in $\mathbb{R}^n \times (0,\infty)$ were proved by curvature consideration in [1]. In [11] $C^{2+\alpha,1+\alpha/2}$ estimates were derived for a family of fully nonlinear uniformly parabolic equations, provided they satisfy a Liouville property.

2 Liouville Property for Ancient Solutions

In this section, we will prove the following Liouville-type result.

Proposition 2.1. Let u be a smooth solution of (1) in Q_∞. Suppose $|D^2u| \le K < \infty$ in $Q_\infty = \mathbb{R}^n \times (-\infty, 0]$. When $n \ge 4$ we also assume that at least one of the following conditions holds:

- $\Theta \ge (n-2)\frac{\pi}{2}$ with $\Theta = \sum_{i=1}^n \arctan \lambda_i(x,t)$,
- $3 + \lambda_i^2(x,t) + 2\lambda_i(x,t)\lambda_j(x,t) \ge 0$ for all $1 \le i, j \le n$.

Then u is a quadratic polynomial

$$u(x,t) = u(0,0) + \langle Du(0,0), x\rangle + \frac{1}{2}\langle D^2u(0,0)x, x\rangle + \Theta(0,0)t. \tag{2}$$

□

Proof. Differentiating (1) with respect to time, we see that

$$\Theta_t = \sum_{i,j=12}^n \Theta_{u_{ij}} D_{ij}u_t = \sum_{i,j=1}^n \Theta_{u_{ij}} D_{ij}\Theta = \sum_{i,j=1}^n g^{ij} D_{ij}\Theta, \tag{3}$$

where g^{ij} is the inverse of (the induced metric) $g = I + D^2uD^2u$. Since $|D^2u| \le K < \infty$, the bounded Θ satisfies the above uniformly parabolic nondivergence equation in $Q_\infty = \mathbb{R}^n \times (-\infty, 0]$. From Krylov–Safonov's Hölder inequality [3, p. 133]

$$[\Theta]_{\alpha,\frac{\alpha}{2};Q_r} = \sup_{\substack{X^i \in Q_r \\ X^1 \ne X^2}} \frac{|\Theta(X^1) - \Theta(X^2)|}{\max\{|x^1 - x^2|^\alpha, |t^1 - t^2|^{\alpha/2}\}}$$

$$\le C(n, \|D^2u\|_{L^\infty(Q_\infty)})\frac{1}{r^\alpha}$$

with $\alpha = \alpha(n, \|D^2u\|_{L^\infty(Q_\infty)}) > 0$, it follows that Θ is constant in Q_∞ by letting r go to infinity.

Since $u(x,t)$ is continuous and now $u_t = \Theta(0,0)$ in Q_∞, we see that

$$u(x,t) = \Theta(0,0)t + u(x,0)$$

and

$$\sum_{i=1}^n \arctan \lambda_i(D^2u(x,0)) = \Theta(0,0) \quad \text{in } \mathbb{R}^n. \tag{4}$$

From [9, Theorem 1.3], [10, Lemma 2.1 and Step 3], and [7, Theorem 3.1], it follows that $u(x,0)$ is a quadratic polynomial in terms of x. We thus arrive at the conclusion of Proposition 2.1. ■

Remark. Proposition 2.1 still holds under the strictly super critical phase condition alone $\Theta \geq (n-2)\pi/2 + \delta$ for any $\delta > 0$ in all dimensions $n \geq 1$. Using the transformation described in [10, p. 1356], we can re-represent the corresponding Lagrangian mean curvature flow by a new potential $\bar{u}(\bar{x}, t)$ satisfying (1), $|D^2\bar{u}| \leq K < \infty$, and $\Theta(D^2\bar{u}) \geq (n-2)\pi/2$ in Q_∞. Applying Proposition 2.1 to $\bar{u}$, we obtain the expression for u in Proposition 2.1. □

3 Proof of Theorem

By equation (1), we only need to estimate D^2u. We get the estimate by controlling Du_t and D^3u. We argue by contradiction. Suppose there exist a family of solutions u^k with $k = 1, 2, \ldots$ to (1) such that $\|D^2u^k\|_{L^\infty(Q_1)} \leq K$, but

$$\|Du_t^k\|_{L^\infty(Q_{1/2})} + \|D^3u^k\|_{L^\infty(Q_{1/2})} =: m_k \to \infty \quad \text{as } k \to \infty.$$

Define the distance of $X \in Q_1$ to the parabolic boundary of Q_1

$$d(X) = \sup\{r > 0 : \ Q_r(X) \subset Q_1(0,0)\}.$$

Then

$$\sup_{X \in Q_1} d(X)[|Du_t^k(X)| + |D^3u^k(X)|] =: m_k' \geq \frac{m_k}{2}.$$

We may assume that there exists $X^k = (x^k, t^k) \in Q_1$ such that

$$d(X^k)[|Du_t^k(X^k)| + |D^3u^k(X^k)|] = m_k'$$

or

$$|Du_t^k(X^k)| + |D^3u^k(X^k)| = M_k \quad \text{with } M_k = \frac{m_k'}{d(X^k)} \geq m_k.$$

For $X \in Q_{\frac{1}{2}d(X^k)}(X^k)$, we have

$$d(X)[|Du_t^k(X)| + |D^3u^k(X)|] \leq d(X^k)M_k$$

and then

$$|Du_t^k(X)| + |D^3u^k(X)| \le \frac{d(X^k)}{d(X)} M_k \le 2M_k.$$

Thus we get

$$\|Du_t^k\|_{L^\infty(Q_{\frac{1}{2}d(X^k)}(X^k))} + \|D^3u^k\|_{L^\infty(Q_{\frac{1}{2}d(X^k)}(X^k))} \le 2M_k.$$

We "blow up" u^k by setting

$$v^k(X) = M_k^2\left[u^k\left(\frac{x}{M_k} + x^k, \frac{t}{M_k^2} + t^k\right) - u^k(X^k) - Du^k(X^k)\cdot\frac{x}{M_k}\right]$$

for $X \in Q_{\frac{1}{2}m_k'}$. Those v^ks are solutions to (1) in $Q_{\frac{1}{2}m_k'}$ with uniformly bounded Hessian

$$\|D^2v^k\|_{L^\infty(Q_{\frac{1}{2}m_k'})} \le K.$$

Furthermore,

$$\|Dv_t^k\|_{L^\infty(Q_{\frac{1}{2}m_k'})} + \|D^3v^k\|_{L^\infty(Q_{\frac{1}{2}m_k'})} \le 2$$

and

$$|Dv_t^k(0,0)| + |D^3v^k(0,0)| = 1.$$

By a space–time interpolation lemma (cf. [4, Lemma 3.1], we see that

$$[D^2v^k]_{1,\frac{1}{2};Q_{\frac{1}{2}m_k'}} \le C(2).$$

From (1), we also get

$$[v_t^k]_{1,\frac{1}{2};Q_{\frac{1}{2}m_k'}} \le C(2).$$

Using Arzela–Arscoli, we find a subconvergence sequence of v^k that converges to a solution v to (1) on Q_∞. By Schauder theory for parabolic equation (cf. [2, p. 92]) and boot strapping, we can make the convergence in C^4 norm.

Now v is a C^4 solution to (1) on Q_∞ satisfying the conditions in Proposition 2.1 and

$$|Dv_t(0,0)| + |D^3v(0,0)| = 1.$$

By Proposition 2.1, v is a quadratic polynomial such that $|Dv_t| + |D^3 v| \equiv 0$. This contradiction shows that

$$\|Du_t\|_{L^\infty(Q_{1/2})} + \|D^3 u\|_{L^\infty(Q_{1/2})} \le C\,(\|D^2 u\|_{L^\infty(Q_1)}).$$

Applying the interpolation lemma [4, Lemma 3.1] again, we obtain the desired estimate of Theorem 1.1.

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Appendix

From our Liouville result of Proposition 2.1 for ancient solutions, one immediately sees some rigidity properties for translating solutions to (1) in $\mathbb{R}^n \times \mathbb{R}^1$ and shrinking solutions to (1) in $\mathbb{R}^n \times (-\infty, 0]$. Here, we only record the results for translating solutions. By a translating solution u to the potential equation for Lagrangian mean curvature flows, we mean

$$\{(x, Du(x,t)) :\ x \in \mathbb{R}^n\} = \{(x, Du(x,0)) :\ x \in \mathbb{R}^n\} + t(B, A)$$

for some (possibly vanishing) vector $(B, A) \in \mathbb{R}^n \times \mathbb{R}^n$ and $t \in \mathbb{R}^1$. For such a solution $u(x,t)$, from the differential relation $J\nabla_g \Theta = H = (B, A)^\perp$, a simple calculation (cf. [1, p. 5]) shows that $u(x,0)$ satisfies

$$\sum_{i=1}^n \arctan \lambda_i(D^2 u(x,0)) = \langle A, x\rangle - \langle B, Du(x,0)\rangle + c$$

for some constant c. We state the following consequences of Proposition 2.1.

(i) Any (smooth) translating solution $u(x,t)$ with bounded Hessian to (1) in $\mathbb{R}^n \times \mathbb{R}^1$ must be stationary:

$$\{(x, Du(x,t)) : \in \mathbb{R}^n\} = \{(x, Du(x,0)) :\ x \in \mathbb{R}^n\}.$$

(ii) The translating Lagrangian graph $\{(x, Du(x,0)) : x \in \mathbb{R}^n\}$ must be a plane if the bounded Hessian, in the case $n \geq 4$, also satisfies $\Theta \geq (n-2)\pi/2$ or $3 + \lambda_i^2 + 2\lambda_i\lambda_j \geq 0$ for all $1 \leq i, j \leq n$.

(iii) Moreover, any (smooth) translating solution $u(x,t)$ to (1) with $\Theta \geq (n-2)\pi/2 + \delta$ for any $\delta > 0$ in $\mathbb{R}^n \times \mathbb{R}^1$ $(n \geq 1)$ must be stationary and the corresponding translating Lagrangian graph is a plane.

Indeed, all translating solutions are ancient. As in the proof of Proposition 2.1, any ancient solution with bounded Hessian is stationary in the sense of (4). The above property (i) follows. Note now (B, A) is forced to vanish because the mean curvature of $\{(x, Du(x,0)) : x \in \mathbb{R}^n\}$ is zero and $\{(x, Du(x,0)) : x \in \mathbb{R}^n\}$ does not move at all. From the conclusion of Proposition 2.1, we get property (ii). By the remark on Proposition 2.1 in Section 2, we obtain the above property (iii).

André Neves communicated to us that he could show property (i) by an argument involving the monotonicity formula for mean curvature flows.

Chau et al. proved the second property (ii) under the condition $|\lambda_i| \leq 1 - \delta$ for small $\delta > 0$ by curvature considerations in [1, Theorem 1.2].

Added in proof

It turns out that Theorem 1.1 is still valid for less smooth $C^{1+1,0+1}$ solutions. This is because we can obtain the needed Liouville property Proposition 2.1 for those viscosity solutions to (1) by the Hölder estimates in [5, Theorem 4.8] for $\Theta = u_t$, then, for example, rely on the L^p estimates in [6, Theorem 4.1] to reach the conclusion as in [11].

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