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A LIOUVILLE PROBLEM FOR THE SIGMA-2 EQUATION

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Dedicated to Louis Nirenberg on his 85th birthday

Abstract. We show that any global convex solution to the Sigma-2 equation must be quadratic.

1. **Introduction.**

In this note, we show that any global convex solution in $\mathbb{R}^n$ to the Hessian equation

$$\sigma_k\left(D^2u\right)=\sigma_k\left(\lambda\right)=\sum_{1\le i_1<\cdots<i_k\le n}\lambda_{i_1}\cdots\lambda_{i_k}=1$$

with $k=2$ must be quadratic. Here $\lambda_i s$ are the eigenvalues of the Hessian D^2u. Classically any global convex solution in $\mathbb{R}^n$ to the Laplace equation $\sigma_1\left(D^2u\right)=\triangle u=1$ or the Monge-Ampère equation $\sigma_n\left(D^2u\right)=\det D^2u=1$ must be quadratic.

Theorem 1.1. *Let u be any smooth solution in $\mathbb{R}^n$ to $\sigma_2\left(D^2u\right)=1$ with $D^2u\ge\left[\delta-\sqrt{\frac{2}{n(n-1)}}\right]\ I$ for any $\delta>0$. Then u is quadratic.*

The lower bound $D^2u\ge(\delta-K)\ I$ with $K=\sqrt{2/n\left(n-1\right)}$ forces the Hessian, or the eigenvalues λ on the positive branch of $\sigma_2\left(\lambda\right)=1$. (Because $\sigma_1\left(\lambda\right)\le-nK$ for λ on the negative branch of $\sigma_2\left(\lambda\right)=1$.) We really need this particular bound $K=\sqrt{2/n\left(n-1\right)}$ in our argument for the convexity of a corresponding new equation. The solution u satisfies the above elliptic equation $\sigma_2\left(D^2u\right)=1$ with convex level set $\left\{\lambda\mid\sigma_2\left(\lambda\right)=1\right\}$. However the ellipticity is not uniform even under the strict convexity assumption on u, $D^2u>0$. The standard Evans-Krylov-Safonov theory does not apply. To apply this theory, we make a Legendre-Lewy type transformation of the solution u so that the new function has bounded Hessian (Step 1); the new corresponding equation is convex (only under the particular assumption $D^2u\ge\left[\delta-\sqrt{2/n\left(n-1\right)}\right]\ I$, Step 2.1); and the new equation is uniformly elliptic (Step

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2.2). The standard theory on Hölder estimates for the Hessian leads to our theorem (Step 3).

We guess that Theorem 1.1 should still be true under the semiconvexity assumption $D^2u \geq -KI$ with arbitrarily large K, even for general equation $\sigma_k\left(D^2u\right)=1$ with $2 \leq k \leq n-1$. At least this is the case when $n=3$ and $k=2$; see [4, Theorem 1.3] where a different transformation and the geometric measure theory were employed.

2. **Proof.** Step 1. We first make a (Legendre-Lewy type) transformation of the function u so that the Hessian of the new function $\tilde{u}$ is bounded from both sides. The negative $\tilde{u}$ is the Lewy type rotation of u, which in turn is nothing but the Legendre transformation of the function $w(x)=u(x)+\frac{1}{2}K|x|^2$; see [1]. Geometrically the Legendre transformation is to re-present the "gradient" graph $G: y=Dw(x)$, or $(x, Dw(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ over y-space (that is, to switch x and y coordinates) as another "gradient" graph in $\mathbb{R}^{2n}$. Any tangent vector to G takes the form

$$\left(e, D^2w\ e\right) \quad \text{or} \quad \left(\left(D^2w\right)^{-1}\ \bar{e},\ \bar{e}\right),$$

where vector e is in x-space and $\bar{e}$ is in y-space. Note that the (canonical) angles between the tangent planes of G and x-space are

$$\arctan\left(\lambda_i+K\right) \in [\arctan\delta, \frac{\pi}{2})$$

by the semiconvexity assumption $\lambda_i \geq \delta-K$. From this angle condition and the symmetry of $\left(D^2w\right)^{-1}$, it follows that G can still be represented as a "gradient" graph $\bar{x}=D\bar{w}(y)$, or $(D\bar{w}(y), y)$ over the whole y-space; further

$$\arctan\bar{\lambda}_i=\frac{\pi}{2}-\arctan\left(\lambda_i+K\right) \in (0, \frac{\pi}{2}-\arctan\delta],$$

where $\bar{\lambda}_i s$ are the eigenvalues of the Hessian $D^2\bar{w}$.

Therefore, the entire function $\tilde{u}(y)=-\bar{w}(y)$ satisfies

$$-\frac{1}{\delta}I \leq D^2\tilde{u}=-\left(D^2u+K\right)^{-1}<0$$

or

$$\tilde{\lambda}_i=-\frac{1}{\lambda_i+K} \in [-\frac{1}{\delta}, 0), \quad \text{equivalently } \lambda_i=-\frac{1}{\tilde{\lambda}_i}-K \geq \delta-K,$$

where $\tilde{\lambda}_i s$ are the eigenvalues of the Hessian $D^2\tilde{u}$.

Remark. In the "gradient" graph space $\mathbb{R}^n \times \mathbb{R}^n=\mathbb{C}^n$, the Legendre transformation is a $\pi/2$-$U(n)$ rotation followed by a conjugation. The transformations described in [4] are $U(n)$ rotations with arbitrary angles.

Step 2.1. We next show that the Hessian $D^2\tilde{u}$ is on a convex hyper surface in the symmetric matrix space. By calculating the double derivatives with the chain rule, or writing symmetric convex functions as maxima of linear functions with certain properties, we only need to verify the eigenvalues of $D^2\tilde{u}$ sit on a convex level set in the $\tilde{\lambda}$ space.

Let

$$g\left(\tilde{\lambda}\right)=\sigma_2\left(-\frac{1}{\tilde{\lambda}_1}-K, \cdots, -\frac{1}{\tilde{\lambda}_n}-K\right)=\sigma_2(\lambda)=f(\lambda).$$

Note

$$\sigma_2\left(-\frac{1}{\tilde{\lambda}_1}-K,\cdots,-\frac{1}{\tilde{\lambda}_n}-K\right)=\frac{\sigma_{n-2}\left(\tilde{\lambda}\right)}{\sigma_n\left(\tilde{\lambda}\right)}+(n-1)\,K\frac{\sigma_{n-1}\left(\tilde{\lambda}\right)}{\sigma_n\left(\tilde{\lambda}\right)}+\left(\begin{array}{c}n\\2\end{array}\right)K^2$$
$$=\frac{\sigma_{n-2}\left(\tilde{\lambda}\right)}{\sigma_n\left(\tilde{\lambda}\right)}+(n-1)\,K\frac{\sigma_{n-1}\left(\tilde{\lambda}\right)}{\sigma_n\left(\tilde{\lambda}\right)}+1,$$

where we used $K=\sqrt{2/n\,(n-1)}$, then the level set

$$\Gamma=\left\{\tilde{\lambda}\mid g\left(\tilde{\lambda}\right)=1\right\}=\left\{\tilde{\lambda}\mid \sigma_{n-2}\left(\tilde{\lambda}\right)+(n-1)\,K\sigma_{n-1}\left(\tilde{\lambda}\right)=0,\ \tilde{\lambda}_i<0\right\}.$$

It follows from an old result (cf. [2, Theorem 15.16]) that Γ is convex.

Remark. The level set Γ is saddle for large K and $n\geq 3$ in general.

Step 2.2. We now show that $\tilde{u}$ satisfies a uniformly elliptic equation. We only need to demonstrate that the normal to the level set Γ is uniformly inside the positive cone $\left\{\tilde{\lambda}\mid\tilde{\lambda}_i>0\ \text{ for } i=1,\cdots,n\right\}$, when $-\delta^{-1}\leq\tilde{\lambda}_i<0$ or $\lambda_i\geq\delta-K$. To achieve this, we multiply the gradient Dg by a (conformal) factor and show the resulting vector is uniformly inside the positive cone.

The gradient Dg has components

$$g_{\tilde{\lambda}_i}\left(\tilde{\lambda}\right)=f_{\lambda_i}\frac{1}{\tilde{\lambda}_i^2}=\left[\sigma_1\left(\lambda\right)-\lambda_i\right]\left(\lambda_i+K\right)^2.$$

Let

$$N=\frac{1}{\sqrt{(1+\lambda_1^2)\ldots(1+\lambda_n^2)}}Dg$$
$$=\frac{\left(\left(\sigma_1-\lambda_1\right)\left(\lambda_1+K\right)^2,\ldots,\left(\sigma_1-\lambda_n\right)\left(\lambda_n+K\right)^2\right)}{\sqrt{(1+\lambda_1^2)\ldots(1+\lambda_n^2)}}.$$

Remark. For $n=3$ still with $\sigma_2\left(\lambda\right)=1$, N also takes the form

$$N=\left(\frac{\left(\lambda_1+K\right)^2}{1+\lambda_1^2},\frac{\left(\lambda_2+K\right)^2}{1+\lambda_2^2},\frac{\left(\lambda_3+K\right)^2}{1+\lambda_3^2}\right).$$

We proceed with the following simple algebraic lemma.

Lemma 2.1. *Assume* $F\left(D^2u\right)=\sigma_2\left(\lambda\right)=1$ *and* λ *is on the positive branch* $\left\{\lambda\mid\sigma_2\left(\lambda\right)=1,\ \sigma_1\left(\lambda\right)>0\right\}$. *Then*

$$\left(F_{u_{ij}}\right)=\begin{bmatrix}\sigma_1-\lambda_1&&&\\&\sigma_1-\lambda_2&&\\&&\cdots&\\&&&\sigma_1-\lambda_n\end{bmatrix}\geq c\left(n\right)\begin{bmatrix}\frac{1}{\lambda_1}&&&\\&\lambda_1&&\\&&\cdots&\\&&&\lambda_1\end{bmatrix},$$

when D^2u *is diagonalized with eigenvalues* $\lambda_1\geq\cdots\geq\lambda_n$.

Proof. This lemma follows from the argument for Theorem 1 in [3]. For completeness, we include a proof.

The matrix equality is straight forward. We prove the inequalities. Denote

$$\sigma_{k;i_1i_2}\left(\lambda\right)=\sigma_k\left(\lambda\right)|_{\lambda_{i_1}=\lambda_{i_2}=0}.$$

We go with $\sigma_1 - \lambda_1 = \sigma_{1;1}$ first. Note

$$\sigma_{2;1} + \lambda_1 \sigma_{1;1} = \sigma_2 = 1$$

and

$$\sigma_{2;1} = \frac{(\sigma_{1;1})^2 - |(0, \lambda_2, \cdots, \lambda_n)|^2}{2} < \frac{1}{2} (\sigma_{1;1})^2 ,$$

then

$$\frac{1}{2} (\sigma_{1;1})^2 + \lambda_1 \sigma_{1;1} - 1 > 0.$$

By the assumption that λ is on the positive branch of $\sigma_2 (\lambda) = 1$, it is standard that

$$\sigma_{1;1} = \frac{\partial \sigma_2}{\partial \lambda_1} > 0.$$

Hence we get

$$\sigma_{1;1} > -\lambda_1 + \sqrt{\lambda_1^2 + 2} = \frac{2}{\lambda_1 + \sqrt{\lambda_1^2 + 2}} \geq \frac{c(n)}{\lambda_1}.$$

The last inequality is from the fact that

$$\lambda_1 \geq \frac{\sigma_1}{n} = \frac{1}{n} \sqrt{2\sigma_2 + |D^2 u|^2} \geq c(n).$$

For $i \geq 2$, the lower bound of $\sigma_1 - \lambda_i = \sigma_{1;i}$ is estimated as follows. We have

$$\sigma_{1;1i} + \lambda_1 = \sigma_{1;i}$$

and

$$\sigma_{2;1i} + \lambda_1 \sigma_{1;1i} = \sigma_{2;i} = \sigma_2 - \lambda_i \sigma_{1;i} > -\lambda_i \sigma_{1;i}.$$

These inequalities imply

$$\sigma_{2;1i} - (\sigma_{1;1i})^2 > -\lambda_i \sigma_{1;i} - \sigma_{1;1i} \sigma_{1;i} = -\sigma_{1;1} \sigma_{1;i} \geq -(\sigma_{1;i})^2 .$$

where we used $0 < \sigma_{1;1} \leq \sigma_{1;i}$. Again noting

$$\sigma_{2;1i} \leq \frac{(\sigma_{1;1i})^2}{2},$$

we get

$$(\sigma_{1;i})^2 > \frac{(\sigma_{1;1i})^2}{2}$$

and consequently

$$|\sigma_{1;1i}| < \sqrt{2} \sigma_{1;i}.$$

Thus

$$\lambda_1 = \sigma_{1;i} - \sigma_{1;1i} < \left(1 + \sqrt{2}\right) \sigma_{1;i},$$

that is

$$\sigma_{1;i} > \left(\sqrt{2} - 1\right) \lambda_1.$$

□

Now we show that each component of N has positive lower and upper bounds. We first need to show that λ_i is also bounded from above for $i \geq 2$. We have

$$\begin{aligned} 1 = \sigma_2 &= \lambda_i (\sigma_1 - \lambda_i) + \sigma_{2;i} \\ &= \lambda_i (\sigma_1 - \lambda_i) + \lambda^+ \cdot \lambda^- + \lambda^+ \cdot \lambda^+ + \lambda^- \cdot \lambda^- \\ &\geq \lambda_i (\sigma_1 - \lambda_i) - \lambda^+ \cdot \lambda^-, \end{aligned}$$

where $\lambda^+ \cdot \lambda^-$, $\lambda^+ \cdot \lambda^+$, and $\lambda^- \cdot \lambda^-$ represent the sum of products of all pair eigenvalues in $\sigma_{2;i}$ with the opposite signs, the same positive signs, and the same negative signs respectively. From the above lemma and the assumption $\lambda_i \geq \delta - K$, we obtain

$$1 \geq \lambda_i \left(\sqrt{2} - 1\right) \lambda_1 - C(n) K \lambda_1,$$

consequently

$$\lambda_i \leq \frac{1 + C(n) K \lambda_1}{\left(\sqrt{2} - 1\right) \lambda_1} \leq C(n) K,$$

where we used $\lambda_1 > c(n)$ again. Further for $i \geq 2$

$$\frac{1}{\sqrt{1 + \lambda_i^2}} \geq c(n, K).$$

We are ready to show a "lower bound" for N. From the above bounds for λ and the lemma, we get

$$N_1 \geq c(n, K) \frac{(\lambda_1 + K)^2}{\lambda_1 \sqrt{(1 + \lambda_1^2)}} > c(n, K) > 0$$

and

$$N_i \geq c(n, K) \frac{\lambda_1}{\sqrt{1 + \lambda_1^2}} \delta^2 \geq c(n, K, \delta) > 0 \quad \text{for } i \geq 2.$$

We next show an "upper bound" for N. From

$$\lambda_1 (\sigma_1 - \lambda_1) + \sigma_{2;1} = 1$$

it follows that

$$\lambda_1 (\sigma_1 - \lambda_1) = 1 - \sigma_{2;1} < C(n, K).$$

Then we get

$$\begin{aligned} N_1 &< (\sigma_1 - \lambda_1) \frac{(\lambda_1 + K)^2}{\sqrt{1 + \lambda_1^2}} \\ &= (\sigma_1 - \lambda_1) \lambda_1 \frac{(\lambda_1 + K)}{\lambda_1} \frac{(\lambda_1 + K)}{\sqrt{1 + \lambda_1^2}} < C(n, K). \end{aligned}$$

For $i > 1$

$$N_i < \frac{(\sigma_1 - \lambda_i)}{\sqrt{(1 + \lambda_1^2)}} (\lambda_i + K)^2 < C(n, K),$$

where we used the fact $\lambda_i \leq C(n) K$ for $i \geq 2$.

Finally the inequalities

$$0 < c(n, K, \delta) \leq N_i \leq C(n, K) \quad \text{for all } i = 1, \cdots, n$$

immediately show that $N = Dg / \sqrt{(1 + \lambda_1^2) \dots (1 + \lambda_n^2)}$, and consequently the normal to the level set Γ, $N/|N|$ is uniformly inside the positive cone.

Remark. Unlike the convexity, the uniform ellipticity is valid for large K in general.

Step 3. The closing argument is standard. We now have a global solution $\tilde{u}$ with bounded Hessian satisfying a convex and uniformly elliptic equation. By the Evans-Krylov-Safonov theory, we obtain

$$\left[D^2 \tilde{u}\right]_{C^\alpha(B_R)} \leq \frac{C(n, \delta)}{R^\alpha} \left\|D^2 \tilde{u}\right\|_{L^\infty(B_{2R})} \leq \frac{C(n, \delta)}{R^\alpha} \to 0 \quad \text{as } R \to \infty,$$

where $\alpha = \alpha(n, \delta) > 0$. We conclude that $D^2\tilde{u}$ is a constant matrix, and consequently D^2u is also a constant matrix.

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REFERENCES

[1] Hans Lewy, *A priori limitations for solutions of Monge-Ampère equations,* II. Trans. Amer. Math. Soc., **41** (1937), 365–374.
[2] Gary M. Lieberman, "Second Order Parabolic Differential Equations," World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
[3] Mi Lin and Neil S. Trudinger, *On some inequalities for elementary symmetric functions,* Bull. Austral. Math. Soc., **50** (1994), 317–326.
[4] Yu Yuan, *A Bernstein problem for special Lagrangian equations,* Invent. Math., **150** (2002), 117–125.

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