Hessian and gradient estimates for three dimensional special Lagrangian equations with large phase

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HESSIAN AND GRADIENT ESTIMATES FOR THREE DIMENSIONAL SPECIAL LAGRANGIAN EQUATIONS WITH LARGE PHASE

By MICAH WARREN and YU YUAN

Abstract. We obtain a priori interior Hessian and gradient estimates for special Lagrangian equations with phase larger than a critical value in dimension three. Gradient estimates are also derived for critical and super critical phases in general dimensions.

1. Introduction. In this paper, we obtain a priori *interior* Hessian and gradient estimates for the special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i = \Theta \tag{1.1}$$

with super critical phases $|\Theta| > \pi/2$ and $n = 3$, where $\lambda = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of the Hessian D^2u. Gradient estimates are also derived for (1.1) with phase at least a critical value $|\Theta| \geq (n-2)\pi/2$ in general dimensions.

Equation (1.1) is from the special Lagrangian geometry [HL]. The Lagrangian graph $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ is called special when the phase or the argument of the complex number $\left(1+\sqrt{-1}\lambda_1\right)\cdots\left(1+\sqrt{-1}\lambda_n\right)$ is a constant Θ, that is, u satisfies equation (1.1), and it is special if and only if $(x, Du(x))$ is a (volume minimizing) minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$ [HL, Theorem 2.3, Proposition 2.17]. The phase $(n-2)\pi/2$ is called critical because the level set $\{\lambda \in \mathbb{R}^n | \lambda \text{ satisfying (1.1)}\}$ is convex *only* when $|\Theta| \geq (n-2)\pi/2$ [Y2, Lemma 2.1]. When $n = 3$ and $|\Theta| = \pi/2$ or π, equation (1.1) also takes the following forms respectively

$$\sigma_2\left(D^2u\right) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1 \tag{1.2}$$

or

$$\triangle u = \det D^2u. \tag{1.3}$$

Aided by the quadratic nature of (1.2), we demonstrated Hessian estimates for (1.1) with critical phase $|\Theta| = \pi/2$ and $n = 3$ in [WY2]. In dimension three,

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the algebraic form of (1.1) is cubic generically. The complete picture for super critical phase (1.1) in dimension three is the following.

THEOREM 1.1. *Let u be a smooth solution to (1.1) with $|\Theta| \geq \pi/2$ and $n = 3$ on $B_R(0) \subset \mathbb{R}^3$. Then we have*

$$|D^2u(0)| \leq C(3)\exp\left[C(3)\left(\cot\frac{|\Theta|-\pi/2}{3}\right)^2 \max_{B_R(0)}|Du|^7/R^7\right],$$

and also

$$|D^2u(0)| \leq C(3)\exp\left\{C(3)\exp\left[C(3)\max_{B_R(0)}|Du|^3/R^3\right]\right\}.$$

In order to link the dependence of Hessian estimates in the above theorems to the potential u itself, we have the following gradient estimate in general dimensions.

THEOREM 1.2. *Let u be a smooth solution to (1.1) with $|\Theta| \geq (n-2)\frac{\pi}{2}$ on $B_{3R}(0) \subset \mathbb{R}^n$. Then we have*

$$\max_{B_R(0)}|Du| \leq C(n)\left[\operatorname*{osc}_{B_{3R}(0)}\frac{u}{R}+1\right]. \tag{1.4}$$

One application of the above estimates is the regularity (analyticity) of the C^0 viscosity solutions to (1.1) with $|\Theta| \geq \pi/2$ and $n = 3$. Another quick consequence is a Liouville type result for global solutions with quadratic growth to (1.1) with $|\Theta| = \pi/2$ and $n = 3$, namely any such a solution must be quadratic (cf. [Y1], [Y2] where other Liouville type results for convex solutions to (1.1) and Bernstein type results for global solutions to (1.1) with $|\Theta| > (n-2)\pi/2$ were obtained).

In the 1950s, Heinz [H] derived a Hessian bound for the two dimensional Monge-Ampère type equation including (1.1) with $n = 2$; see also Pogorelov [P1] for Hessian estimates for these equations including (1.1) with $|\Theta| > \pi/2$ and $n = 2$. In the 1970s Pogorelov [P2] constructed his famous counterexamples, namely irregular solutions to three dimensional Monge-Ampère equations $\sigma_3(D^2u) = \lambda_1\lambda_2\lambda_3 = \det(D^2u) = 1$; those irregular solutions also serve as counterexamples for cubic and higher order symmetric σ_k equations (cf. [U1]). In passing, we also mention Hessian estimates for solutions with certain strict convexity constraints to Monge-Ampère equations and σ_k equation ($k \geq 2$) by Pogorelov [P2] and Chou-Wang [CW] respectively using the Pogorelov technique. Urbas [U2][U3], also Bao-Chen [BC] obtained (pointwise) Hessian estimates in terms of certain integrals of the Hessian, for σ_k equations and special Lagrangian equation (1.1) with $n = 3$, $\Theta = \pi$ respectively.

A Hessian bound for (1.1) with $n = 2$ also follows from an earlier work by Gregori [G], where Heinz's Jacobian estimate was extended to get a gradient bound in terms of the heights of the two dimensional minimal surfaces with any codimension. A gradient estimate for general dimensional and codimensional minimal graphs with certain constraints on the gradients themselves was obtained in [W], using an integral method developed for codimension one minimal graphs. The gradient estimate of Bombieri-De Giorgi-Miranda [BDM] (see also [T1] [BG] [K]) is by now classic.

The Bernstein-Pogorelov-Korevaar pointwise technique was employed to derive Hessian estimates *only* for (1.1) with certain constraints both on the Hessian and gradient of solutions in [WY1]. A slightly sharper Hessian estimate for (1.1) with $n = 2$ was obtained by elementary methods in [WY3]. The Hessian estimate for the quadratic Hessian equation (1.2), or (1.1) with $|\Theta| = \pi/2$ and $n = 3$ was derived by a "less" involved integral argument in [WY2]. Hessian estimates for convex solutions in general dimension are shown in a "smoother" way [CWY].

We bound the Hessian in the following order: first by its integral, next by an integral of its gradient, then by the volume of the minimal surface, lastly by the height of the special Lagrangian graph. In the first step, we estimate the Hessian by its integral via Michael-Simon's mean value inequality [MS] applied to some subharmonic function in terms of the Hessian. Similar to the critical phase case, a decisive choice is the function $b = \ln \sqrt{1 + \lambda_{\max}^2}$. In the second step of bounding the integral of b by that of its gradient, we cannot simply apply the Sobolev inequality for functions with compact support on the minimal surfaces as for $|\Theta| = \pi/2$ in [WY2]. Equation (1.2) is in quadratic form. Consequently the coefficients of the corresponding linearized operator are *linear* in terms of the Hessian, so that it is possible to contain the extra terms involving the cut-off function. This is not true for the super critical phases, so we have to work harder to obtain a more powerful Sobolev inequality for functions without compact support. A Lewy type rotation (Proposition 2.2 developed in [Y1] and [Y2]) is employed to link a relative isoperimetric inequality on the Euclidean space (Proposition 2.3) to the desired Sobolev inequality on the special Lagrangian graph. For a uniform Sobolev inequality as the phase becomes close to the critical one, thus a uniform Hessian bound, the Hessian estimate for $\Theta = \pi/2$ in [WY2] becomes useful. In the third step, because of the special choice of b whose Laplacian bounds its gradient (Proposition 2.1), we control the integral of the gradient of b in terms of the volume of the minimal surface. Lastly, using the usual Sobolev inequality for functions with compact support on the minimal surfaces and taking advantage of the divergence form of the volume element of the minimal Lagrangian graph, we bound the volume in terms of the height of the special Lagrangian graph, which is the gradient of the solution to equation (1.1). For details, see Section 3.

As for the gradient estimates, we adapt Trudinger's method [T2] for σ_k equations to (1.1) with the critical phase $\Theta = (n-2)\pi/2$. However, we are unable to apply the known techniques ([L], [T2], [CW]) to the super critical phases $\Theta > (n-2)\pi/2$, as equation (1.1) in these cases does not satisfy the required structure. Actually, rough gradient estimates for (1.1) with larger phase

$\Theta > (n-2)\pi/2$ are straightforward consequences of the observation that the Hessians of solutions have lower bound depending on the phase Θ. In order to obtain the uniform gradient estimates that do not deteriorate as Θ is close to the critical phase, we further make use of an "integral" version of the Lewy type rotation to link the corresponding estimates to the one in the case of the critical phase. For details, see Section 4.

As one can see, our arguments for the Hessian estimates and gradient estimates resemble, respectively the "isoperimetric" proof and the simplified "pointwise" proof, of the classical gradient estimate for minimal graphs. Now only some technical obstacles remain for Hessian estimates for (1.1) with large phase $|\Theta| \geq (n-2)\pi/2$ and $n \geq 4$. Yet further new ideas are lacking for us to handle both the Hessian and gradient estimates for the special Lagrangian equation (1.1) with general phases in dimension three and higher, including (1.3) corresponding to $\Theta = 0$ and $n = 3$.

Notation. $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$, $u_i = \partial_i u$, $u_{ji} = \partial_{ij} u$, etc., but $\lambda_1, \dots, \lambda_n$ and $\theta_i = \arctan \lambda_i$ do not represent the partial derivatives. Finally $C(n)$ will denote various constants depending only on dimension n.

2. Preliminaries. Taking the gradient of both sides of the special Lagrangian equation (1.1), we have

$$\sum_{i,j=1}^{n} g^{ij} \partial_{ij} \left(x, Du\left(x\right)\right) = 0, \tag{2.1}$$

where $\left(g^{ij}\right)$ is the inverse of the induced metric $g = \left(g_{ij}\right) = I + D^2 u D^2 u$ on the surface $\left(x, Du\left(x\right)\right) \subset \mathbb{R}^n \times \mathbb{R}^n$. Simple geometric manipulation of (2.1) yields the divergence form of the minimal surface equation

$$\triangle_g \left(x, Du\left(x\right)\right) = 0,$$

where the Laplace-Beltrami operator of the metric g is given by

$$\triangle_g = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \partial_i \left(\sqrt{\det g}\, g^{ij} \partial_j\right).$$

Because we are using harmonic coordinates $\triangle_g x = 0$, we see that $\triangle_g$ also equals the linearized operator of the special Lagrangian equation (1.1) at u,

$$\triangle_g = \sum_{i,j=1}^{n} g^{ij} \partial_{ij}.$$

The gradient and inner product with respect to the metric g are

$$\nabla_g v = \left(\sum_{k=1}^n g^{1k} v_k, \dots, \sum_{k=1}^n g^{nk} v_k\right),$$

$$\langle \nabla_g v, \nabla_g w\rangle_g = \sum_{i,j=1}^n g^{ij} v_i w_j, \quad \text{in particular} \quad |\nabla_g v|^2 = \langle \nabla_g v, \nabla_g v\rangle_g .$$

2.1. Jacobi inequality. The following Jacobi inequality was obtained in [WY2] for $\Theta = \pi/2$. The hard proof there works for phase $\Theta > \pi/2$, simply by replacing $\pi/6$ with $\Theta/3$ in Proposition 2.4 of [WY2]. (A simpler proof is desired in dealing with possible singularities of the Lipschitz function b. A simpler, smooth auxiliary function for Hessian estimates of convex solutions to (1.1) is the volume element [CWY].)

PROPOSITION 2.1. *Let u be a smooth solution to the special Lagrangian equation (1.1) with $n = 3$ and $\Theta \geq \pi/2$ on B_R. Set*

$$b = \max\left\{\ln\sqrt{1+\lambda_{\max}^2},\ K\right\},$$

with $K = 1 + \ln\sqrt{1+\tan^2\left(\frac{\Theta}{3}\right)}$. Then b satisfies the integral Jacobi inequality

$$\int_{B_R} -\langle \nabla_g \varphi, \nabla_g b\rangle_g \, dv_g \geq \frac{1}{3}\int_{B_R} \varphi\, |\nabla_g b|^2 \, dv_g \tag{2.2}$$

for all nonnegative $\varphi \in C_0^\infty(B_R)$.

2.2. Lewy type rotation. The next is the second main result of this section. Our proofs of Theorems 1.1 and 1.2 rely on this new representation of the original special Lagrangian graph.

PROPOSITION 2.2. *Let u be a smooth solution to (1.1) with $\Theta = (n-2)\pi/2 + \delta$ on $B_R(0) \subset \mathbb{R}^n$. Then the special Lagrangian surface $\mathfrak{M} = (x, Du(x))$ can be represented as a gradient graph $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ of the new potential $\bar{u}$ satisfying (1.1) with phase $\Theta = (n-2)\pi/2$ in a domain containing a ball of radius*

$$\bar{R} \geq \frac{R}{2\cos(\delta/n)}.$$

Proof. To obtain the new representation, we use a Lewy type rotation (developed in [Y1], [Y2, p. 1356]). Take a $U(n)$ rotation of $\mathbb{C}^n \cong \mathbb{R}^n \times \mathbb{R}^n$: $\bar{z} = e^{-\sqrt{-1}\delta/n} z$ with $z = x + \sqrt{-1}y$ and $\bar{z} = \bar{x} + \sqrt{-1}\bar{y}$. Because $U(n)$ rotation preserves the length and complex structure, $\mathfrak{M}$ is still a special Lagrangian submanifold with the parametrization

$$\begin{cases} \bar{x} = x \cos \frac{\delta}{n} + Du(x) \sin \frac{\delta}{n} \\ D\bar{u} = -x \sin \frac{\delta}{n} + Du(x) \cos \frac{\delta}{n}. \end{cases}$$

In order to show that this parameterization is that of a gradient graph over $\bar{x}$, we must first show that $\bar{x}(x)$ is a diffeomorphism onto its image. This is accomplished by showing that

$$\left|\bar{x}(x^{\alpha}) - \bar{x}(x^{\beta})\right| \geq \frac{1}{2\cos\delta/n}\left|x^{\alpha} - x^{\beta}\right| \tag{2.3}$$

for any x^{α}, x^{β}. We assume by translation that $x^{\beta} = 0$ and $Du\left(x^{\beta}\right) = 0$. Now $0 < \delta < \pi$, and $\theta_i > \delta - \frac{\pi}{2}$, so $u + \frac{1}{2}\cot\delta\,|x|^2$ is convex, and we have

$$\begin{aligned}
\left|\bar{x}\left(x^{\alpha}\right) - \bar{x}\left(x^{\beta}\right)\right|^2 &= \left|\bar{x}\left(x^{\alpha}\right)\right|^2 = \left|x^{\alpha}\cos\frac{\delta}{n} + Du\left(x^{\alpha}\right)\sin\frac{\delta}{n}\right|^2 \\
&= \left|x^{\alpha}\left(\cos\frac{\delta}{n} - \cot\delta\sin\frac{\delta}{n}\right) + \left[Du\left(x^{\alpha}\right) + x^{\alpha}\cot\delta\right]\sin\frac{\delta}{n}\right|^2 \\
&= \left|x^{\alpha}\right|^2\left[\frac{\sin\frac{(n-1)\delta}{n}}{\sin\delta}\right]^2 + \left|Du\left(x^{\alpha}\right) + x^{\alpha}\cot\delta\right|^2\sin^2\frac{\delta}{n} \\
&\quad + 2\frac{\sin\frac{(n-1)\delta}{n}\sin\frac{\delta}{n}}{\sin\delta}\left\langle x^{\alpha}, Du\left(x^{\alpha}\right) + x\cot\delta\right\rangle \\
&\geq \left|x^{\alpha}\right|^2\left(\frac{1}{2\cos\frac{\delta}{n}}\right)^2.
\end{aligned}$$

It follows that $\mathfrak{M}$ is a special Lagrangian graph over $\bar{x}$. The Lagrangian graph is the gradient graph of a potential function $\bar{u}$ (cf. [HL, Lemma 2.2]), that is, $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$. The eigenvalues $\bar{\lambda}_i$ of the Hessian $D^2\bar{u}$ are determined by

$$\bar{\theta}_i = \arctan\bar{\lambda}_i = \theta_i - \frac{\delta}{n} \in \left(-\frac{\pi}{2} + \frac{(n-1)\,\delta}{n}, \frac{\pi}{2} - \frac{\delta}{n}\right). \tag{2.4}$$

Then

$$\sum_{i=1}^{n}\bar{\theta}_i = \frac{(n-2)\,\pi}{2},$$

that is, $\bar{u}$ satisfies the special Lagrangian equation (1.1) of phase $(n-2)\,\pi/2$. The lower bound on $\bar{R}$ follows immediately from (2.3). □

2.3. Relative isoperimetric inequality. We end with the last main result of this section, Proposition 2.3. This relative isoperimetric inequality is needed in the proof of Theorem 1.1 to prove a key ingredient, namely a Sobolev inequality for functions without compact support. Proposition 2.3 is proved from the following classical relative isoperimetric inequality for balls.

LEMMA 2.1. *Let A and A^c are disjoint measurable sets such that $A \cup A^c = B_1(0) \subset \mathbb{R}^n$. Then*

$$\min\{|A|, |A^c|\} \le C(n)\, |\partial A \cap \partial A^c|^{n/n-1}. \tag{2.5}$$

Proof. See for example [LY, Theorem 5.3.2.]. □

PROPOSITION 2.3. *Let $\Omega_1 \subset \Omega_2 \subset B_\rho \subset \mathbb{R}^n$. Suppose that $dist(\Omega_1, \partial\Omega_2) \ge 2$, also A and A^c are disjoint measurable sets such that $A \cup A^c = \Omega_2$. Then*

$$\min\{|A \cap \Omega_1|, |A^c \cap \Omega_1|\} \le C(n)\rho^n\, |\partial A \cap \partial A^c|^{n/n-1}.$$

Proof. Define a continuous function on Ω_1

$$\xi(x) = \frac{|A \cap B_1(x)|}{|B_1|}.$$

Case 1. $\xi(x_*) = 1/2$ for some $x_* \in \Omega_1$. We know $B_1(x_*) \subset \Omega_2$. From Lemma 2.1, we have

$$\frac{|B_1|}{2} \le C(n)\, |\partial A \cap \partial A^c \cap B_1(x_*)|^{\frac{n}{n-1}} \le C(n)|\partial A \cap \partial A^c|^{\frac{n}{n-1}}.$$

It then follows

$$\min\{|A \cap \Omega_1|,\ |A^c \cap \Omega_1|\} \le |\Omega_1| < |B_\rho| = \rho^n|B_1| \le C(n)\rho^n|\partial A \cap \partial A^c|^{\frac{n}{n-1}}.$$

Case 2.1. $\xi(x) > 1/2$ for all $x \in \Omega_1$. Cover Ω_1 by at most $N \le C(n)\,\rho^n$ unit balls $B_1(x_i)$ for some uniform constant $C(n)$. Note that all these balls are inside Ω_2. By the classical relative isoperimetric inequality for balls again,

$$|A^c \cap B_1(x_i)| = \min\{|A \cap B_1(x_i)|, |A^c \cap B_1(x_i)|\} \le C(n)|\partial A \cap \partial A^c|^{\frac{n}{n-1}}.$$

Summing this inequality over all covers, we get

$$|A^c \cap \Omega_1| \le \sum_{i=1}^{N} |A^c \cap B_1(x_i)| \le C(n)\rho^n C(n)\, |\partial A \cap \partial A^c|^{\frac{n}{n-1}},$$

then the conclusion of the proposition follows.

Case 2.2. $\xi(x) < 1/2$ for all $x \in \Omega_1$. Repeat the argument in Case 2.1 with A^c replaced by A, we still have the conclusion of the proposition.

The proof of Proposition 2.3 is complete. □

Remark. Considering dumbbell type regions, we see that the order of dependence on ρ is sharp in Proposition 2.3.

3. Proof of Theorem 1.1. We assume that $R = 8$ and u is a solution on $B_8 \subset \mathbb{R}^3$ for simplicity of notation. By scaling $v(x) = u\left(\frac{R}{8}x\right) / \left(\frac{R}{8}\right)^2$, we still get the estimate in Theorem 1.1. We consider the cases when $\Theta = \pi/2 + \delta$ for $\delta \in (0, \pi)$. The cases $\Theta < -\pi/2$ follow by symmetry.

Step 1. As preparation for the proof of Theorem 1.1, we take the phase $\pi/2$ representation $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ in Proposition 2.2 for the original special Lagrangian graph $\mathfrak{M} = (x, Du(x))$ with $x \in B_8$. The critical phase representation is

$$\begin{cases} \bar{x} = x\cos\frac{\delta}{3} + Du(x)\sin\frac{\delta}{3} \\ D\bar{u} = -x\sin\frac{\delta}{3} + Du(x)\cos\frac{\delta}{3}. \end{cases} \tag{3.1}$$

Define

$$\bar{\Omega}_r = \bar{x}(B_r(0)).$$

Then we have from (2.3)

$$\operatorname{dist}(\bar{\Omega}_1, \partial\bar{\Omega}_5) \geq \frac{4}{2\cos\delta/3} > 2. \tag{3.2}$$

We see from (3.1) that $|\bar{x}| \leq \rho$ for $\bar{x} \in \bar{\Omega}_8$ with

$$\rho = 8\cos\frac{\delta}{3} + \|Du\|_{L^\infty(B_8)}\sin\frac{\delta}{3} \tag{3.3}$$

and that $|D\bar{u}(\bar{x})| \leq \kappa$ (for $\bar{x} \in \bar{\Omega}_8$) with

$$\kappa = 8\sin\frac{\delta}{3} + \|Du\|_{L^\infty(B_8)}\cos\frac{\delta}{3}. \tag{3.4}$$

The eigenvalues of the new potential $\bar{u}$ satisfy (2.4), thus the interior Hessian bound in [WY2, Theorem 1.1] gives

$$\left|D^2\bar{u}(\bar{x})\right| \leq C(3)\exp\left[C(3)\kappa^3\right]$$

for $\bar{x} \in \bar{\Omega}_5$. It follows that the induced metric on $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ in $\bar{x}$ coordinates is bounded on $\bar{\Omega}_5$ by

$$d\bar{x}^2 \leq \bar{g}\,(\bar{x}) \leq \mu(\kappa, \delta) d\bar{x}^2, \tag{3.5}$$

where

$$\mu(\kappa, \delta) = \min\left\{1 + C(3)\exp\left[C(3)\kappa^3\right], \left[1 + \left(\cot\frac{\delta}{3}\right)^2\right]\right\}. \tag{3.6}$$

Step 2. Relying on the above set-up and the relative isoperimetric inequality in Proposition 2.3, we proceed with the following Sobolev inequality for functions without compact support.

PROPOSITION 3.1. *Let u be a smooth solution to (1.1) with $\Theta = \pi/2 + \delta$ on $B_5(0) \subset \mathbb{R}^3$. Let f be a smooth positive function on the special Lagrangian surface $\mathfrak{M} = (x, Du\,(x))$. Then*

$$\left[\int_{B_1} |(f-\iota)^+|^{3/2}\, dv_g\right]^{2/3} \leq C(3)\rho^4\mu(\kappa,\delta)\int_{B_5} |\nabla_g (f-\iota)^+|\, dv_g,$$

where ρ, κ, and μ were defined in (3.3), (3.4), and (3.6); also $\iota = \int_{B_5(0)} f dx$.

Proof. Step 2.1. Let $M = ||f||_{L^\infty(B_1)}$. We may assume $\iota < M$. By Sard's theorem, the level set $\{x|\, f(x) = t\}$ is C^1 for almost all t. We first show that for all such $t \in [\iota, M]$,

$$|\{x|\, f(x) > t\} \cap B_1|_g \leq C(3)\rho^6 \left[\mu(\kappa,\delta)|\{x|\, f(x) = t\} \cap B_5|_g\right]^{3/2}. \tag{3.7}$$

Here $|\ |_g$ denotes the area or volume with respect to the induced metric; $|\ |$ denotes the same with respect to the Euclidean metric, as in Lemma 2.1 and Proposition 2.3.

From $t > \int_{B_5} f dx$, it follows that $|\{x|\, f(x) > t\} \cap B_1| < 1$ and consequently

$$|\{x|\, f(x) \leq t\} \cap B_1| > |B_1| - 1 > 1. \tag{3.8}$$

Now we use instead the coordinates for $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ given by the Lewy type rotation (3.1). Let

$$A_t = \{\bar{x}|\, f(\bar{x}) > t\} \cap \bar{\Omega}_5,$$

where we are treating f as a function on the special Lagrangian surface $\mathfrak{M}$. Applying Proposition 2.3 with (3.2) and (3.3), we see that

$$\min\left\{\left|A_t \cap \bar{\Omega}_1\right|, \left|A_t^c \cap \bar{\Omega}_1\right|\right\} \leq C(3)\rho^3 \left|\partial A_t \cap \partial A_t^c\right|^{3/2}.$$

Now either $\left|A_t \cap \bar{\Omega}_1\right| \leq \left|A_t^c \cap \bar{\Omega}_1\right|$, or vice versa.

If $\left|A_t \cap \bar{\Omega}_1\right| \leq \left|A_t^c \cap \bar{\Omega}_1\right|$, then we have from (3.5)

$$\begin{aligned}\left|A_t \cap \bar{\Omega}_1\right|_{\bar{g}} &\leq [\mu(\kappa,\delta)]^{3/2}\left|A_t \cap \bar{\Omega}_1\right| \\ &\leq C(3)\rho^3\,[\mu(\kappa,\delta)]^{3/2}\left|\partial A_t \cap \partial A_t^c\right|^{3/2} \\ &\leq C(3)\rho^3\,[\mu(\kappa,\delta)]^{3/2}\left|\partial A_t \cap \partial A_t^c\right|_{\bar{g}}^{3/2}.\end{aligned}$$

Otherwise, if $\left|A_t \cap \bar{\Omega}_1\right| > \left|A_t^c \cap \bar{\Omega}_1\right|$, still we have that

$$\left|A_t \cap \bar{\Omega}_1\right| \leq C(3)\rho^3\left|A_t^c \cap \bar{\Omega}_1\right|$$

as $\left|A_t^c \cap \bar{\Omega}_1\right| \geq 1/2^3$ from (3.8) and (2.3), and $\rho \geq 8\cos\frac{\pi}{3}$ from (3.3). Thus

$$\begin{aligned}\left|A_t \cap \bar{\Omega}_1\right|_{\bar{g}} &\leq [\mu(\kappa,\delta)]^{3/2}\,C(3)\rho^3\left|A_t^c \cap \bar{\Omega}_1\right| \\ &\leq C(3)\rho^6\,[\mu(\kappa,\delta)]^{3/2}\left|\partial A_t \cap \partial A_t^c\right|_{\bar{g}}^{3/2}.\end{aligned}$$

In either case we have the desired isoperimetric inequality (now given in the new coordinates for $\mathfrak{M}$) which holds for $\iota < t < M$

$$\left|A_t \cap \bar{\Omega}_1\right|_{\bar{g}} \leq C(3)\rho^6\left[\mu(\kappa,\delta)\left|\partial A_t \cap \partial A_t^c\right|_{\bar{g}}\right]^{3/2},$$

or equivalently (3.7) in the original coordinates.

Step 2.2. With this isoperimetric inequality in hand, the following proof is standard (cf. [LY, Theorem 5.3.1]).

$$\begin{aligned}\left[\int_{B_1}\left|(f-\iota)^+\right|^{3/2}dv_g\right]^{2/3} &= \left(\int_0^{M-\iota}\left|\{x|\,f(x)-\iota>t\}\cap B_1\right|_g dt^{3/2}\right)^{2/3} \\ &\leq \int_0^{M-\iota}\left|\{x|\,f(x)-\iota>t\}\cap B_1\}\right|_g^{2/3}dt \\ &\leq C(3)\rho^4\mu(\kappa,\delta)\int_\iota^M\left|\{x|\,f(x)=t\}\cap B_5\right|_g dt \\ &\leq C(3)\rho^4\mu(\kappa,\delta)\int_{B_5}\left|\nabla_g(f-\iota)^+\right|dv_g,\end{aligned}$$

where the last inequality followed from the the coarea formula; the second inequality from (3.7); and the first inequality from the Hardy-Littlewood-Polya inequality for any nonnegative, nonincreasing integrand $\eta(t)$:

$$\left[\int_0^T \eta(t)^q\,dt^q\right]^{1/q} \leq \int_0^T \eta(t)\,dt.$$

This H-L-P inequality (with $q > 1$) is proved by noting that $s\eta(s) \leq \int_0^s \eta(t)\,dt$ and integrating the inequality

$$q\,[s\eta(s)]^{q-1}\,\eta(s) \leq q\left[\int_0^s \eta(t)\,dt\right]^{q-1}\eta(s) = \frac{d}{ds}\left[\int_0^s \eta(t)\,dt\right]^q.$$

The proposition is thus proved. □

Step 3. We continue the proof of Theorem 1.1. Based on Proposition 2.1, a simple calculation shows that the Lipschitz function $\left[(b-\iota)^+\right]^{3/2}$ is also weakly subharmonic, where $\iota = \int_{B_5(0)} bdx$. We apply Michael and Simon's mean value inequality [MS, Theorem 3.4] to obtain

$$\begin{aligned}(b-\iota)^+(0) &\leq C(3)\left[\int_{B_1} |(b-\iota)^+|^{3/2}\,dv_g\right]^{2/3}\\ &\leq C(3)\rho^4\mu(\kappa,\delta)\int_{B_5} |\nabla_g(b-\iota)^+|\,dv_g,\end{aligned}$$

where the second inequality follows from Proposition 3.1, approximating $(b-\iota)^+$ by smooth functions if necessary. Thus

$$\begin{aligned}b(0) &\leq C(3)\rho^4\mu(\kappa,\delta)\int_{B_5} |\nabla_g b|\,dv_g + \int_{B_5} bdx\\ &\leq C(3)\rho^4\mu(\kappa,\delta)\left(\int_{B_5} |\nabla_g b|^2\,dv_g\right)^{1/2}\left(\int_{B_5} Vdx\right)^{1/2} + \int_{B_5} Vdx\\ &\leq C(3)\rho^4\mu(\kappa,\delta)\int_{B_6} Vdx,\end{aligned} \tag{3.9}$$

where the last inequality is deduced from the following argument. From Proposition 2.1, b satisfies the Jacobi inequality in the integral sense:

$$3\,\triangle_g\, b \geq |\nabla_g b|^2\,.$$

Multiplying both sides by a cut-off function $\varphi \in C_0^\infty(B_6)$ such that $\varphi \geq 0$, $\varphi = 1$ on B_5, and $|D\varphi| \leq 1.1$, then integrating, we obtain

$$\begin{aligned}\int_{B_6} \varphi^2\,|\nabla_g b|^2\,dv_g &\leq 3\int_{B_6} \varphi^2\,\triangle_g\, bdv_g\\ &= -3\int_{B_6} \langle 2\varphi\nabla_g\varphi, \nabla_g b\rangle\,dv_g\\ &\leq \frac{1}{2}\int_{B_6} \varphi^2\,|\nabla_g b|^2\,dv_g + 18\int_{B_6} |\nabla_g\varphi|^2\,dv_g.\end{aligned}$$

It follows that

$$\begin{aligned}\int_{B_5} |\nabla_g b|^2\, dv_g \;&\le\; \int_{B_2} \varphi^2\, |\nabla_g b|^2\, dv_g \le 36 \int_{B_2} |\nabla_g \varphi|^2\, dv_g\\ &\le\; C(3) \int_{B_2} V dx.\end{aligned}$$

Step 4. We finish the proof of Theorem 1.1 by bounding $\int_{B_6} V dx$. Observe

$$V = \left|\left(1+\sqrt{-1}\lambda_1\right)\cdots\left(1+\sqrt{-1}\lambda_3\right)\right| = \frac{\sigma_2 - 1}{|\cos\Theta|} > 0.$$

We control the integral of σ_2 in the following:

$$\begin{aligned}\int_{B_r} \sigma_2 dx \;&=\; \int_{B_r} \frac{1}{2}\left[(\triangle u)^2 - \left|D^2 u\right|^2\right] dx\\ &= \frac{1}{2}\int_{B_r} div\left[\left(\triangle u I - D^2 u\right) Du\right] dx\\ &= \frac{1}{2}\int_{\partial B_r} \left\langle \left(\triangle u I - D^2 u\right) Du, \nu\right\rangle dA,\end{aligned}$$

where ν is the outward normal of B_r. Diagonalizing D^2u, we see easily that

$$\triangle u I - D^2 u = \begin{bmatrix}\lambda_2+\lambda_3 & & \\ & \lambda_3+\lambda_1 & \\ & & \lambda_1+\lambda_2\end{bmatrix} > 0$$

as $\theta_i + \theta_j > 0$ with $\theta_1 + \theta_2 + \theta_3 \ge \pi/2$. Then

$$\int_{B_r} \sigma_2 dx \le \|Du\|_{L^\infty(\partial B_r)} \int_{\partial B_r} \triangle u dA.$$

Integrating the boundary integral from $r = 6$ to $r = 7$, we get

$$\begin{aligned}\int_{B_6} \sigma_2 dx \;&\le\; \|Du\|_{L^\infty(B_7)} \min_{r\in[6,7]} \int_{\partial B_r} \triangle u dA\\ &\le\; \|Du\|_{L^\infty(B_7)} \int_{B_7} \triangle u dx\\ &\le\; C(3)\, \|Du\|^2_{L^\infty(B_7)}.\end{aligned}$$

It follows that for $\Theta \geq \pi/2$

$$\begin{aligned}\int_{B_6} V dx &= \frac{1}{|\cos\Theta|}\int_{B_6}(\sigma_2 - 1)\,dx \leq \frac{1}{|\cos\Theta|}\int_{B_6}\sigma_2 dx \\ &\leq \frac{C(3)}{|\cos\Theta|}\|Du\|^2_{L^\infty(B_7)}\end{aligned}$$

or

$$\int_{B_6} V dx \leq \frac{C(3)}{|\cos\Theta|\,\|Du\|_{L^\infty(B_8)}}\|Du\|^3_{L^\infty(B_8)}. \tag{3.10}$$

In order to get Θ-independent control on the volume, we estimate the volume in another way. By the Sobolev inequality on the minimal surface $\mathfrak{M}$ [MS, Theorem 2.1] or [A, Theorem 7.3], we have

$$\int_{B_6} V dx = \int_{B_6} dv_g \leq \int_{B_7}\phi^6 dv_g \leq C(3)\left[\int_{B_7}|\nabla_g\phi|^2\,dv_g\right]^3,$$

where the nonnegative cut-off function $\phi \in C_0^\infty(B_7)$ satisfies $\phi = 1$ on B_6 and $|D\phi| \leq 1.1$.

Observe the (conformality) identity

$$\begin{aligned}&\left(\frac{1}{1+\lambda_1^2},\ldots,\frac{1}{1+\lambda_3^2}\right)V \\ &\quad= \left(\sin\Theta(\sigma_1-\lambda_1)+\cos\Theta\left(1-\frac{\sigma_3}{\lambda_1}\right),\ldots,\right. \\ &\qquad\left.\sin\Theta(\sigma_1-\lambda_3)+\cos\Theta\left(1-\frac{\sigma_3}{\lambda_3}\right)\right),\end{aligned}$$

which follows from differentiating the complex identity

$$\ln V + \sqrt{-1}\sum_{i=1}^{3}\arctan\lambda_i = \ln\left[1-\sigma_2+\sqrt{-1}(\sigma_1-\sigma_3)\right].$$

We then have

$$\begin{aligned}\int_{B_7}|\nabla_g\phi|^2\,dv_g &= \int_{B_7}\sum_{i=1}^{3}\frac{|\phi_i|^2}{1+\lambda_i^2}V dx \\ &\leq 1.21\int_{B_7}[2\sin\Theta\sigma_1+\cos\Theta(3-\sigma_2)]\,dx \\ &\leq C(3)\left[|\sin\Theta|\,\|Du\|_{L^\infty(B_8)}+|\cos\Theta|\,\|Du\|^2_{L^\infty(B_8)}\right]\end{aligned}$$

for $\Theta \geq \pi/2$, where we used the argument leading to (3.10). Thus we get

$$\int_{B_6} Vdx \leq C(3)\left[|\sin\Theta|\,\|Du\|_{L^\infty(B_8)} + |\cos\Theta|\,\|Du\|_{L^\infty(B_8)}\,\|Du\|_{L^\infty(B_8)}\right]^3 \tag{3.11}$$

Now either $|\cos\Theta|\,\|Du\|_{L^\infty(B_8)} \leq 1$ or $|\cos\Theta|\,\|Du\|_{L^\infty(B_8)} > 1$; combining (3.10) and (3.11), we have in either case

$$\int_{B_6} Vdx \leq C(3)\,\|Du\|^3_{L^\infty(B_8)}.$$

Finally from the above inequality and (3.9), we conclude

$$\begin{aligned} b(0) &\leq C(3)\rho^4\mu(\kappa,\delta)\,\|Du\|^3_{L^\infty(B_8)} \\ &\leq C(3)\rho^4\,\|Du\|^3_{L^\infty(B_8)} \min\left\{1 + C(3)\exp\left[C(3)\kappa^3\right], 1 + \left(\cot\frac{\delta}{3}\right)^2\right\}. \end{aligned}$$

Exponentiating, and recalling (3.3), (3.4) and (3.6), we have the Θ-independent bound

$$\left|D^2u(0)\right| \leq C(3)\exp\left\{C(3)\exp\left[C(3)\,\|Du\|^3_{L^\infty(B_8)}\right]\right\}$$

and the Θ-dependent bound

$$\left|D^2u(0)\right| \leq C(3)\exp\left\{C(3)\left[1+\left(\cot\frac{\delta}{3}\right)^2\right]\left[1+\|Du\|_{L^\infty(B_8)}\sin\frac{\delta}{3}\right]^4\|Du\|^3_{L^\infty(B_8)}\right\}.$$

Simplifying the above expressions, we arrive at the conclusion of Theorem 1.1.

4. Proof of Theorem 1.2. We assume that $R = 1$ by scaling $u(Rx)/R^2$, and $\Theta \geq (n-2)\pi/2$ by symmetry.

Case $\Theta = (n-2)\pi/2$. Set $M = \operatorname{osc}_{B_1} u$. We may assume $M > 0$. By replacing u with $u - \min_{B_1} u + M$, we have $M \leq u \leq 2M$ in B_1. Let

$$w = \eta\,|Du| + Au^2$$

with $\eta = 1 - |x|^2$ and $A = n/M$. We assume that w attains its maximum at an interior point $x^* \in B_1$, otherwise w would take its maximum on the boundary ∂B_1 and the conclusion would be straightforward. Choose a coordinate system so that D^2u is diagonalized at x^*. We assume, say $u_n \geq \frac{|Du|}{\sqrt{n}}$ (> 0) at x^*. For all $i = 1, \ldots, n$, we have at x^*

$$0 = w_i = \eta\,|Du|_i + \eta_i\,|Du| + 2Auu_i,$$

then

$$\frac{u_i u_{ii}}{|Du|} = |Du|_i = -\frac{\eta_i\, |Du| + 2Auu_i}{\eta}. \tag{4.1}$$

In particular, we have $u_{nn} < 0$ by the choice of A. Since the phase $\Theta \geq (n-2)\,\pi/2$, it follows that $\lambda_n = \lambda_{\min}$, $|\lambda_n| \leq \lambda_k$, and

$$g^{nn} = \frac{1}{1+\lambda_n^2} \geq \frac{1}{1+\lambda_k^2} = g^{kk} \tag{4.2}$$

for $k = 1, \ldots, n-1$ at x^*.

Next, we show

$$\triangle_g u \geq 0.$$

When $D^2 u$ is diagonalized,

$$\triangle_g u = \sum_{i=1}^{n} g^{ii} u_{ii} = \sum_{i=1}^{n} \frac{\lambda_i}{1+\lambda_i^2} = \frac{1}{2}\sum_{i=1}^{n} \sin(2\theta_i).$$

Let $S \subset \mathbb{R}^n$ be the hypersurface (with boundary) given by

$$S = \left\{ \theta \mid \theta_1 + \theta_2 + \cdots + \theta_n = \frac{\pi}{2}(n-2), |\theta_i| \leq \frac{\pi}{2} \right\},$$

where $\theta = (\theta_1, \ldots, \theta_n)$. Set $\Gamma(\theta) = \frac{1}{2}\sum_{i=1}^{n} \sin(2\theta_i)$. Suppose that Γ obtains a negative minimum on the interior of S at θ^*. At this point $D\Gamma$ vanishes on $T_{\theta^*}S$, thus we have

$$\cos(2\theta_i) = \cos\left(2\theta_j\right), \text{ then } \theta_i = \pm\theta_j.$$

The only two possible configurations for θ are

$$\begin{aligned} \theta_1 &= \cdots = \theta_n = \frac{(n-2)\,\pi}{2n} \quad \text{or} \\ \theta_1 &= \cdots = \theta_{n-2} = \frac{\pi}{2}, \ \theta_{n-1} = -\theta_n. \end{aligned}$$

In either case, Γ is nonnegative. This contradiction allows us to verify the nonnegativity of Γ along the boundary ∂S. It follows easily that $\Gamma \geq 0$ there by induction on dimension n, as

$$\partial S = \bigcup_{k=1}^{n} \left\{ \theta \mid \theta_1 + \cdots + \widehat{\theta_k} + \cdots + \theta_n = \frac{\pi}{2}(n-3), |\theta_i| \leq \frac{\pi}{2} \right\}$$

and $\Gamma(\theta_1, \theta_2) = 0$ for $\theta_1 + \theta_2 = 0$.

Further, we show

$$\triangle_g |Du| \geq 0.$$

We calculate

$$\begin{aligned}
\triangle_g |Du| &= \sum_{\alpha,\beta=1}^{n} g^{\alpha\beta}\partial_{\alpha\beta} |Du| \\
&= \sum_{\alpha,\beta,i=1}^{n} g^{\alpha\beta}\left(\frac{u_i u_{i\beta\alpha}}{|Du|} + \frac{u_{i\alpha}u_{i\beta}}{|Du|} - \sum_{j=1}^{n}\frac{u_i u_{i\beta}u_j u_{j\alpha}}{|Du|^3}\right) \\
&= \sum_{\alpha,\beta,i=1}^{n} g^{\alpha\beta}\left(\frac{u_{i\alpha}u_{i\beta}}{|Du|} - \sum_{j=1}^{n}\frac{u_i u_{i\beta}u_j u_{j\alpha}}{|Du|^3}\right) \\
&\overset{D^2u \text{ is diagonal}}{=} \sum_{\alpha=1}^{n} g^{\alpha\alpha}\frac{\left(|Du|^2 - u_\alpha^2\right)\lambda_\alpha^2}{|Du|^3} \geq 0,
\end{aligned}$$

where we used the minimality equation (2.1).

Combining the subharmonicity of u and $|Du|$ with (4.2) and (4.1), we have at x^*

$$\begin{aligned}
0 &\geq \triangle_g w = |Du| \triangle_g \eta + 2\sum_{\alpha=1}^{n} g^{\alpha\alpha}\eta_\alpha |Du|_\alpha \\
&\quad + \underbrace{\eta \triangle_g |Du| + 2Au \triangle_g u}_{\geq 0} + 2A\sum_{\alpha=1}^{n} g^{\alpha\alpha}u_\alpha^2 \\
&\geq |Du| \triangle_g \eta + 2\sum_{\alpha=1}^{n} g^{\alpha\alpha}\eta_\alpha \underline{|Du|_\alpha} + 2A\sum_{\alpha=1}^{n} g^{\alpha\alpha}u_\alpha^2 \\
&\geq -2ng^{nn}|Du| - 2\sum_{\alpha=1}^{n} g^{\alpha\alpha}\eta_\alpha\left(\frac{\eta_\alpha |Du| + 2Auu_\alpha}{\eta}\right) + \frac{2A}{n}g^{nn}|Du|^2 \\
&\geq -2ng^{nn}|Du| - 8g^{nn}\frac{|Du|}{\eta} - 8g^{nn}Au\frac{|Du|}{\eta} + \frac{2A}{n}g^{nn}|Du|^2;
\end{aligned}$$

It follows that

$$0 \geq -2n\eta - 8 - 8Au + \frac{2A}{n}\eta |Du|.$$

Then by the assumption $M \leq u \leq 2M$ and $A = n/M$

$$\eta |Du| \left(x^*\right) \leq (n + 4 + 8n) M.$$

So we obtain

$$|Du(0)| \le w(x^*) \le 17nM. \tag{4.3}$$

Case $\Theta > (n-2)\pi/2$. Let $\Theta = \delta + (n-2)\pi/2$. From our special Lagrangian equation (1.1), we know

$$\theta_i + (n-1)\frac{\pi}{2} > (n-2)\frac{\pi}{2} + \delta \quad \text{or} \quad \theta_i > -\frac{\pi}{2} + \delta.$$

We can control the gradient of the convex function $u(x) + \frac{1}{2}\max\{\cot\delta, 0\}|x|^2$ by its oscillation, thus

$$|Du(0)| \le \operatorname*{osc}_{B_1} + \frac{1}{2}\max\{\cot\delta, 0\}. \tag{4.4}$$

In order to get rid of the δ-dependence in the gradient estimate, we need the following.

PROPOSITION 4.1. *Let smooth u satisfy (1.1) with* $\Theta - (n-2)\frac{\pi}{2} = \delta \in (0, \pi/4)$ *on* $B_2(0)$. *Suppose that*

$$\operatorname*{osc}_{B_2} u \le \frac{1}{2\sin\delta}. \tag{4.5}$$

Then

$$|Du(0)| \le C(n)\left(\operatorname*{osc}_{B_2} u + 1\right).$$

Proof. We take the Lewy type rotation in the proof of Proposition 2.2, to obtain the critical phase representation $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ for the original special Lagrangian graph $\mathfrak{M} = (x, Du(x))$ with $x \in B_2$. Recentering the new coordinates, we take

$$\begin{cases} \bar{x} = x\cos\frac{\delta}{n} + Du(x)\sin\frac{\delta}{n} - Du(0)\sin\frac{\delta}{n} \\ D\bar{u}(\bar{x}) = -x\sin\frac{\delta}{n} + Du(x)\cos\frac{\delta}{n} \end{cases}. \tag{4.6}$$

By (2.3) we see that the potential $\bar{u}$ is defined on a ball in $\bar{x}$-space around the origin of radius

$$\bar{R} = \frac{2}{2\cos(\frac{\delta}{n})} > 1.$$

From (4.6) and the estimate (4.3) for the critical potential, we have

$$|Du(0)| = \frac{|D\bar{u}(\bar{0})|}{\cos(\delta/n)} \le C(n)\operatorname*{osc}_{\bar{B}_1} \bar{u}.$$

Next, we estimate the oscillation of $\bar{u}$ in terms of u. We may assume that $\bar{u}(\bar{0}) = 0$. Without loss of generality we assume the maximum of $|\bar{u}|$ on $\bar{B}_1(\bar{0})$ happens along the positive $\bar{x}_1$-axis, and even on the boundary $\partial\bar{B}_1$. Thus we have

$$\operatorname*{osc}_{\bar{B}_1} \bar{u} \leq 2 \left| \int_{\bar{x}_1=0}^{\bar{x}_1=1} \bar{u}_{\bar{x}_1} \, d\bar{x}_1 \right| .$$

In the following, we convert the integral of $\bar{u}_{\bar{x}_1}$ to one in terms of u_{x_1}, then recover the oscillation of $\bar{u}$ from that of u.

We work on the x_1-y_1 plane in the remaining of the proof. Under our above assumption, the $\bar{x}_1$-axis is given by the line

$$y_1 = \tan\left(\frac{\delta}{n}\right) x_1$$

and the curve γ: $(x_1, u_1(x_1))$ with $|x_1| < 2$ forms a graph over the $\bar{x}_1$-axis. Let l_0 be the line perpendicular to the $\bar{x}_1$-axis and intersecting the curve γ at $(0, u_1(0))$ along the y_1-axis. The intersection of l_0 and the $\bar{x}_1$-axis (which is also the origin of the recentered the $\bar{x}_1$-$\bar{y}_1$ plane) has distance to the origin of the x_1-y_1 plane given by

$$|u_1(0)| \sin\left(\frac{\delta}{n}\right) \leq \left(\operatorname*{osc}_{B_1} u + \frac{1}{2}\cot\delta\right) \sin\left(\frac{\delta}{n}\right) \leq 1 \tag{4.7}$$

by the rough bound (4.4) and the condition (4.5). Now let l_1 be the line parallel to l_0 passing through the point $\bar{x}_1 = 1$ along the $\bar{x}_1$-axis.

The integral

$$\int_{\bar{x}_1=0}^{\bar{x}_1=1} \bar{u}_{\bar{x}_1} \, d\bar{x}_1$$

is the signed area between the $\bar{x}_1$-axis and the curve γ, and lying between the lines l_0 and l_1. We convert this to an integral over x_1,

$$\int_{\bar{x}_1=0}^{\bar{x}_1=1} \bar{u}_{\bar{x}_1} \, d\bar{x}_1 = \int_{P(l_0\cap\bar{x}_1\text{-axis})}^{P(l_1\cap\bar{x}_1\text{-axis})} \left[u_1(x_1) - \tan\left(\frac{\delta}{n}\right) x_1\right] dx_1 + K_0 + K_1,$$

where P denotes projection to the x_1-axis, and K_0 as well as K_1 denotes the signed areas to the left or right of the desired region, forming the difference.

It is important to note the following for $j = 1, 2$:

(i) $P(l_j \cap \bar{x}_1\text{-axis})$ is in the x_1-domain of u_1 by (4.7),

$$\begin{aligned} |P(l_0 \cap \bar{x}_1\text{-axis})| &\leq 1 \cdot \cos\left(\frac{\delta}{n}\right) < 1, \\ |P(l_1 \cap \bar{x}_1\text{-axis})| &\leq (1+1) \cdot \cos\left(\frac{\delta}{n}\right) < 2; \end{aligned}$$

(ii) $P(l_j \cap \gamma)$ is also in the x_1-domain of u_1 as the whole Lagrangian surface $\mathfrak{M}$ is a graph over B_2,

$$|P(l_j \cap \gamma)| \leq 2;$$

(iii) the region K_j is bounded by the line l_j, the vertical line $x_1 = P(l_j \cap \bar{x}_1\text{-axis})$, and the curve γ, also each region K_j is on one side of the $\bar{x}_1$-axis.

Thus from (i)

$$\left| \int_{P(l_0 \cap \bar{x}_1\text{-axis})}^{P(l_1 \cap \bar{x}_1\text{-axis})} \left[u_1(x_1) - \tan\left(\frac{\delta}{n}\right) x_1 \right] dx_1 \right| \leq \operatorname*{osc}_{B_2} u + C(n)$$

and from (ii) and (iii)

$$|K_j| \leq \left| \int_{P(l_j \cap \bar{x}_1\text{-axis})}^{P\left[l_j \cap \gamma\right]} \left[u_1(x_1) - \tan\left(\frac{\delta}{n}\right) x_1 \right] dx_1 \right| \leq \operatorname*{osc}_{B_2} u + C(n).$$

It follows that we have the conclusion of Proposition 4.1

$$|Du(0)| \leq C(n) \operatorname*{osc}_{\bar{B}_1} \bar{u} \leq C(n) \left(\operatorname*{osc}_{B_2} u + 1 \right). \qquad \square$$

We finish the proof of Theorem 1.2. For $\delta \geq \pi/4$, the bound (4.4) gives

$$|Du(0)| \leq \operatorname*{osc}_{B_1} u + \frac{1}{2} \leq C(n) \left[\operatorname*{osc}_{B_2} u + 1 \right].$$

For $\delta \leq \pi/4$, if $\operatorname{osc}_{B_2} u \leq 1/(2\sin\delta)$, then Proposition 4.1 gives

$$|Du(0)| \leq C(n) \left[\operatorname*{osc}_{B_2} u + 1 \right].$$

Otherwise, $\operatorname{osc}_{B_2} u > 1/(2\sin\delta)$, and from (4.4)

$$|Du(0)| \leq \operatorname*{osc}_{B_1} u + \operatorname*{osc}_{B_2} u \leq C(n) \left[\operatorname*{osc}_{B_2} u + 1 \right].$$

Applying this estimate on $B_2(x)$ for any $x \in B_1(0)$, we arrive at the conclusion of Theorem 1.2.

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