A Priori Estimate for Convex Solutions to Special Lagrangian Equations and Its Application

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Abstract

We derive a priori interior Hessian estimates for special Lagrangian equations when the potential is convex. When the phase is very large, we show that continuous viscosity solutions are smooth in the interior of the domain.

1 Introduction

In this paper, we establish a priori interior Hessian estimates for convex solutions to the special Lagrangian equation

$$F(D^2u) = \sum_{i=1}^{n} \arctan \lambda_i = \Theta \tag{1.1}$$

where the λ_i are the eigenvalues of the Hessian D^2u and Θ is a constant.

The fully nonlinear equation (1.1) arises from the special Lagrangian geometry [7]. The gradient graph $(x, Du(x))$ of the potential u is a Lagrangian submanifold in $\mathbb{R}^n \times \mathbb{R}^n$. The Lagrangian graph is called special when the phase, which at each point is the argument of the complex number $(1 + \sqrt{-1}\,\lambda_1) \cdots (1 + \sqrt{-1}\,\lambda_n)$, is a constant Θ; that is, u satisfies equation (1.1). A special Lagrangian graph is a volume-minimizing minimal submanifold in $\mathbb{R}^{2n}$.

We first state the following interior Hessian estimates:

THEOREM 1.1 *Let u be a smooth convex solution to* (1.1) *on a ball* $B_R(0) \subset \mathbb{R}^n$. *Then we have*

$$|D^2u(0)| \le C(n) \exp\left\{ C(n) \left[\operatorname*{osc}_{B_R(0)} \frac{u}{R^2} \right]^{3n-2} \right\},$$

where $C(n)$ is a uniform dimensional constant.

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Observe that all eigenvalues λ_i are positive if the phase Θ is very large:

$$\sum_{i=1}^{n} \arctan \lambda_i = \Theta \geq (n-1)\frac{\pi}{2}.$$

Then a direct consequence of Theorem 1.1 is the following:

COROLLARY 1.2 *Let u be a smooth solution to* (1.1) *with* $|\Theta| \geq (n-1)\frac{\pi}{2}$ *on* $B_R(0) \subset \mathbb{R}^n$. *Then we have*

$$|D^2 u(0)| \leq C(n) \exp\left\{ C(n) \left[\operatorname*{osc}_{B_R(0)} \frac{u}{R^2} \right]^{3n-2} \right\}.$$

In the 1950s Heinz [9] derived a Hessian bound for the two-dimensional Monge-Ampère-type equation, including (1.1) with $n = 2$. In the 1970s Pogorelov [13] constructed irregular solutions to $\sigma_3(D^2 u) = \det(D^2 u) = 1$ in dimension 3, which were generalized to a wider class of σ_k-equations with $k \geq 3$ by Urbas [16]. Hessian estimates for solutions with certain strict convexity constraints to Monge-Ampère equations and σ_k equations with $k \geq 2$ were obtained by Pogorelov [13] and Chou and Wang [5], respectively. Pointwise Hessian estimates in terms of certain integrals of the Hessian for σ_k-equations and for special Lagrangian equation (1.1) with $n = 3$, $\Theta = \pi$, were produced by Urbas [17, 18] and by Bao and Chen [1], respectively. More recently, for (1.1) Hessian estimates have been obtained: when the solutions are convex with small gradients in [19]; when $n = 2$ in [20] (giving a sharper bound than in [9]); when $n = 3$ and $|\Theta| \geq \frac{\pi}{2}$, including the equation $\sigma_2\left(D^2 u\right) = 1$ in dimension 3, in [21, 22].

What Theorem 1.1 says is that the geometry of the special Lagrangian graph with convex potential is simple, namely, the induced metric is quasi-isometric to the flat one up to a factor of the height (or the oscillation of the potential). The idea of our arguments is thus to retrieve this simple geometry. By a Lewy-type rotation, the geometry of the Lagrangian graph is already simple in the rotated coordinate system. Consequently, the subharmonic volume element, in the original coordinates, of the special Lagrangian graph with convex potential is actually a subsolution to a uniformly elliptic equation. Therefore the volume element is bounded pointwise by its integral on the minimal graph by mean value inequalities. (Here one can avoid the "harder" mean value inequality for the nonuniformly elliptic Laplace-Beltrami operator on minimal surfaces.) Using a relative isoperimetric inequality on the rotated coordinate plane, we derive a Sobolev inequality for functions without compact support on the Lagrangian graph. Then we bound the integral of the volume element by that of its gradient. Further, the Laplacian of the volume element bounds its gradient. This Jacobian inequality enables us to bound the integral of the gradient of the volume element by a volume term. The Lewy-type rotation also leads to a bound of the volume by the height of the Lagrangian graph, or the gradient of the potential. Lastly, the gradient of any convex function is dominated by its oscillation.

As an application of our a priori estimates in Corollary 1.2, we have the following regularity result:

THEOREM 1.3 *Any C^0 viscosity solution to* (1.1) *with* $|\Theta| \geq (n-1)\frac{\pi}{2}$ *is analytic inside the domain of the solution.*

A well-known result (cf. [11, cor. 3]) states that all $W^{2,n}$ strong solutions to any (possibly degenerate) elliptic equation in dimension n are also C^0 viscosity solutions. From this and Theorem 1.3 we see that all $W^{2,n}$ strong solutions to (1.1) with $|\Theta| \geq (n-1)\frac{\pi}{2}$ are regular. Previously in [1], the regularity was shown for any $W^{2,3+}$ convex strong solution to $\triangle u = \det D^2 u$, which is (1.1) with $\Theta = \pi$ and $n = 3$.

There are several ways to establish the existence and uniqueness of a C^0 viscosity solution to the Dirichlet problem of the special Lagrangian equation (1.1); see the remarks at the end of the paper.

Throughout the paper, $C(n)$ denotes various positive constants depending only on dimension n.

2 A Jacobi Inequality for the Volume Element

Taking the gradient of both sides of the special Lagrangian equation (1.1), we have for each $k = 1, \ldots, n$

$$\sum_{i,j=1}^{n} g^{ij} \partial_{ij} u_k = 0, \tag{2.1}$$

where (g^{ij}) is the inverse of the induced metric $g = (g_{ij}) = I + D^2 u D^2 u$ on the submanifold $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$. Straight computation using (1.1) shows that the Laplace-Beltrami operator of the metric g

$$\triangle_g = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \partial_i (\sqrt{\det g}\, g^{ij} \partial_j)$$

simplifies to

$$\triangle_g = \sum_{i,j=1}^{n} g^{ij} \partial_{ij}. \tag{2.2}$$

The gradient and inner product with respect to the metric g are

$$\nabla_g v = \left(\sum_{k=1}^{n} g^{1k} v_k, \ldots, \sum_{k=1}^{n} g^{nk} v_k \right),$$

$$\langle \nabla_g v, \nabla_g w \rangle_g = \sum_{i,j=1}^{n} g^{ij} v_i w_j,$$

where $v_k = \partial_k v$ and $w_j = \partial_j w$ for functions v and w. In particular, $|\nabla_g v|^2 = \langle \nabla_g v, \nabla_g v\rangle_g$.

We now derive a Jacobi inequality for the volume element

$$V = \sqrt{\det g} = \prod_{i=1}^{n} (1+\lambda_i^2)^{\frac{1}{2}}.$$

PROPOSITION 2.1 *Suppose that u is a smooth convex solution to* (1.1) *in* $B_R \subset \mathbb{R}^n$. *Then*

$$\triangle_g \ln V \ \geq \frac{1}{n}|\nabla_g \ln V|^2.$$

Consequently,

$$\int_{B_r} |\nabla_g \ln V|^2 \, dv_g \leq \frac{C(n)}{R-r} \int_{B_R} dv_g.$$

PROOF: By differentiating the minimal surface equation (2.1) again and performing some long and straightforward computations, one gets the standard formula for $\triangle_g \ln V$; see, for example, [23, lemma 2.1]. (The general formula for minimal submanifolds of any dimension or codimension originates in [14, p. 90].) At any fixed point, we assume that $D^2 u$ is diagonalized; then

$$\triangle_g \ln V = \sum_{i,j,k=1}^{n} (1+\lambda_i\lambda_j) h_{ijk}^2,$$

where $h_{ijk} = \sqrt{g^{ii}}\sqrt{g^{jj}}\sqrt{g^{kk}} u_{ijk}$ are the second fundamental form of the graph. Gathering all terms containing $h_{ijj}^2 = h_{jij}^2 = h_{jji}^2$ for a fixed i, we have

$$\begin{aligned}(1+\lambda_i^2)h_{iii}^2 + \sum_{j\neq i}(1+\lambda_j^2)h_{jji}^2 + \sum_{j\neq i}(1+\lambda_i\lambda_j)h_{ijj}^2 + \sum_{j\neq i}(1+\lambda_j\lambda_i)h_{jij}^2 \\ = (1+\lambda_i^2)h_{iii}^2 + \sum_{j\neq i}(3+\lambda_j^2+2\lambda_i\lambda_j)h_{jji}^2.\end{aligned}$$

Thus

$$\begin{aligned}\triangle_g \ln V = \sum_{i=1}^{n}\Bigg[(1+\lambda_i^2)h_{iii}^2 + \sum_{j\neq i}(3+\lambda_j^2+2\lambda_i\lambda_j)h_{jji}^2\Bigg] \\ + 2\sum_{i<j<k}(3+\lambda_i\lambda_j+\lambda_j\lambda_k+\lambda_k\lambda_i)h_{ijk}^2.\end{aligned}$$

To bound the gradient, we compute (still at the same fixed point with $D^2 u$ diagonalized)

$$\partial_i \ln V = \sum_{j=1}^{n} g^{jj}\lambda_j u_{jji}.$$

Then

$$|\nabla_g \ln V|_g^2 = \sum_{i=1}^n g^{ii}\Big(\sum_{j=1}^n g^{jj}\lambda_j u_{jji}\Big)^2$$
$$= \sum_{i=1}^n \Big(\sum_{j=1}^n \lambda_j h_{jji}\Big)^2 \le n \sum_{i,j=1}^n \lambda_j{}^2 h_{jji}^2.$$

From the convexity of u, we have

$$\triangle_g \ln V - \frac{1}{n}|\nabla_g \ln V|^2 \ge \sum_{i=1}^n h_{iii}^2 + \sum_{j\ne i}[3+2\lambda_i\lambda_j]h_{jji}^2 \ge 0.$$

Next, for any smooth cutoff function $\varphi \in C_0^\infty(B_R)$,

$$\int_{B_R} \varphi^2|\nabla_g \ln V|^2\, dv_g \le n\int_{B_R}\varphi^2\,\triangle_g \ln V\, dv_g$$
$$= -n\int_{B_R}\langle 2\varphi\nabla_g\varphi, \nabla_g \ln V\rangle_g\, dv_g$$
$$\le \frac{1}{2}\int_{B_R}\varphi^2|\nabla_g \ln V|^2\, dv_g + 2n^2\int_{B_R}|\nabla_g\varphi|^2\, dv_g.$$

In particular, choosing φ to be 1 on B_r with gradient bounded by $\frac{2}{R-r}$ in B_R, we see

$$\int_{B_r}|\nabla_g \ln V|^2\, dv_g \le \int_{B_R}\varphi^2|\nabla_g \ln V|^2\, dv_g$$
$$\le 4n^2\int_{B_R}|\nabla_g\varphi|^2\, dv_g$$
$$\le \frac{C(n)}{R-r}\int_{B_R} dv_g.$$

This completes the proof of Proposition 2.1. □

3 Hessian Estimates for Smooth Convex Solutions

We assume that $R = 7$ and u is a solution on $B_7 \subset \mathbb{R}^n$ for simplicity of notation. By scaling $u(\frac{R}{7}x)/(\frac{R}{7})^2$, we still get the estimate in Theorem 1.1. We now present the proof of Theorem 1.1, which consists of five steps.

Step 1. We take a new representation $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ for the original (special) Lagrangian graph $\mathfrak{M} = (x, Du(x))$ in a new coordinate system of $\mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{C}^n$, $\bar{z} = e^{-\sqrt{-1}\,\pi/4} z$ with $z = x + \sqrt{-1}\, y$ and $\bar{z} = \bar{x} + \sqrt{-1}\, \bar{y}$:

$$\begin{cases} \bar{x} = \frac{\sqrt{2}}{2} x + \frac{\sqrt{2}}{2} Du(x) \\ \bar{y} = D\bar{u} = -\frac{\sqrt{2}}{2} x + \frac{\sqrt{2}}{2} Du(x). \end{cases} \tag{3.1}$$

The following is contained in [23, p. 122].

PROPOSITION 3.1 *Let u be a smooth convex function $B_R(0) \subset \mathbb{R}^n$. Then the Lagrangian submanifold $\mathfrak{M} = (x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ can be represented as a gradient graph $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ of the new potential $\bar{u}$ in a domain containing a ball of radius*

$$\bar{R} \geq \frac{\sqrt{2}\, R}{2} \tag{3.2}$$

such that in these coordinates the new Hessian satisfies

$$-I \;\leq D^2\bar{u} \leq I. \tag{3.3}$$

Also define

$$\bar{\Omega}_r = \bar{x}(B_r(0)).$$

Then we have from (3.2)

$$\operatorname{dist}(\bar{\Omega}_1, \partial\bar{\Omega}_5) \geq \frac{4}{\sqrt{2}} > 2. \tag{3.4}$$

We see from (3.1) that $|\bar{x}| \leq \rho$ for $\bar{x} \in \bar{\Omega}_6$ with

$$\rho = 6\,\frac{\sqrt{2}}{2} + \|Du\|_{L^\infty(B_6)}\,\frac{\sqrt{2}}{2}. \tag{3.5}$$

From (3.3), it follows that the induced metric on $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ in $\bar{x}$-coordinates is bounded by

$$d\bar{x}^2 \leq g(\bar{x}) \leq 2d\bar{x}^2. \tag{3.6}$$

Step 2. We recall a relative isoperimetric inequality in [22] and include a proof for completeness.

LEMMA 3.2 *Let $\Omega_1 \subset \Omega_2 \subset B_\rho \subset \mathbb{R}^n$. Suppose that* $\operatorname{dist}(\Omega_1, \partial\Omega_2) \geq 2$; *also A and A^c are disjoint measurable sets such that $A \cup A^c = \Omega_2$. Then*

$$\min\{|A \cap \Omega_1|, |A^c \cap \Omega_1|\} \leq C(n)\rho^n |\partial A \cap \partial A^c|^{\frac{n}{n-1}}.$$

PROOF: Define a continuous function on Ω_1

$$\xi(x) = \frac{|A \cap B_1(x)|}{|B_1|}.$$

Case 1. $\xi(x_*) = \frac{1}{2}$ for some $x_* \in \Omega_1$. We know $B_1(x_*) \subset \Omega_2$. From the classical relative isoperimetric inequality for balls (cf. [10, theorem 5.3.2]), we have

$$\frac{|B_1|}{2} \leq C(n)|\partial A \cap \partial A^{\mathrm{c}} \cap B_1(x_*)|^{\frac{n}{n-1}} \leq C(n)|\partial A \cap \partial A^{\mathrm{c}}|^{\frac{n}{n-1}}.$$

It then follows that

$$\min\{|A \cap \Omega_1|, |A^{\mathrm{c}} \cap \Omega_1|\} \leq |\Omega_1| < |B_\rho| = \rho^n |B_1| \leq C(n)\rho^n |\partial A \cap \partial A^{\mathrm{c}}|^{\frac{n}{n-1}}.$$

Case 2.1. $\xi(x) > \frac{1}{2}$ for all $x \in \Omega_1$. Cover Ω_1 by at most $N \leq C(n)\rho^n$ unit balls $B_1(x_i)$ for some uniform constant $C(n)$. Note that all these balls are inside Ω_2. By the classical relative isoperimetric inequality for balls again,

$$|A^{\mathrm{c}} \cap B_1(x_i)| = \min\{|A \cap B_1(x_i)|, |A^{\mathrm{c}} \cap B_1(x_i)|\} \leq C(n)|\partial A \cap \partial A^{\mathrm{c}}|^{\frac{n}{n-1}}.$$

Summing this inequality over all covers, we get

$$|A^c \cap \Omega_1| \leq \sum_{i=1}^{N} |A^c \cap B_1(x_i)| \leq C(n)\rho^n C(n)|\partial A \cap \partial A^{\mathrm{c}}|^{\frac{n}{n-1}};$$

then the conclusion of the lemma follows.

Case 2.2. $\xi(x) < \frac{1}{2}$ for all $x \in \Omega_1$. Repeat the argument in Case 2.1 with A^{c} replaced by A; we still have the conclusion of the lemma.

The proof of Lemma 3.2 is complete. □

Using the relative isoperimetric inequality in Lemma 3.2, we proceed with the following Sobolev inequality on the special Lagrangian graph for functions without compact support. This is inspired by [2].

PROPOSITION 3.3 *Let u be a smooth convex function on $B_5(0) \subset \mathbb{R}^n$. Let f be a smooth positive function on the Lagrangian surface $\mathfrak{M} = (x, Du(x))$. Then*

$$\left[\int_{B_1} |(f-\iota)^+|^{\frac{n}{n-1}}\, dv_g\right]^{\frac{n-1}{n}} \leq C(n)\rho^{2(n-1)} \int_{B_5} |\nabla_g (f-\iota)^+| dv_g,$$

where ρ was defined in (3.5) *and $\iota = \frac{2}{|B_1|}\int_{B_5(0)} f\, dx$.*

PROOF: Let $M = \|f\|_{L^\infty(B_1)}$. We may assume $\iota < M$ since otherwise the desired result holds trivially. By Sard's theorem, the level set $\{x \mid f(x) = t\}$ is C^1 for almost all t. We first show that for all such $t \in [\iota, M]$,

$$\left|\{x \mid f(x) > t\} \cap B_1\right|_g \leq C(n)\rho^{2n}\left|\{x \mid f(x) = t\} \cap B_5\right|_g^{n/(n-1)}. \tag{3.7}$$

Here $|\cdot|_g$ denotes the area or volume with respect to the induced metric; $|\cdot|$ denotes the same with respect to the Euclidean metric.

From $t > \frac{2}{|B_1|}\int_{B_5(0)} f\,dx$ and $f > 0$, it follows that $|\{x \mid f(x) > t\} \cap B_1| < |B_1|/2$ and consequently

$$\left|\{x \mid f(x) \le t\} \cap B_1\right| > \frac{|B_1|}{2}. \tag{3.8}$$

Now we use instead the coordinates for $\mathfrak{M} = (\bar{x}, D\bar{u}(\bar{x}))$ given by the Lewy rotation (3.1). Let

$$A_t = \{\bar{x} \mid f(\bar{x}) > t\} \cap \bar{\Omega}_5,$$

where we are treating f as a function on the special Lagrangian surface $\mathfrak{M}$. Applying Lemma 3.2 with (3.4) and (3.5), we see that

$$\min\{|A_t \cap \bar{\Omega}_1|, |A_t^{\mathrm{c}} \cap \bar{\Omega}_1|\} \le C(n)\rho^n |\partial A_t \cap \partial A_t^{\mathrm{c}}|^{\frac{n}{n-1}}.$$

If $|A_t \cap \bar{\Omega}_1| \le |A_t^{\mathrm{c}} \cap \bar{\Omega}_1|$, then we have from (3.6)

$$\begin{aligned} |A_t \cap \bar{\Omega}_1|_{g(\bar{x})} &\le 2^{\frac{n}{2}} |A_t \cap \bar{\Omega}_1| \\ &\le C(n)\rho^n |\partial A_t \cap \partial A_t^{\mathrm{c}}|^{\frac{n}{n-1}} \\ &\le C(n)\rho^n |\partial A_t \cap \partial A_t^{\mathrm{c}}|_{g(\bar{x})}^{n/(n-1)}. \end{aligned}$$

If $|A_t \cap \bar{\Omega}_1| > |A_t^{\mathrm{c}} \cap \bar{\Omega}_1|$, still we have

$$|A_t \cap \bar{\Omega}_1| \le C(n)\rho^n |A_t^{\mathrm{c}} \cap \bar{\Omega}_1|,$$

since by (3.8), $|A_t^{\mathrm{c}} \cap \bar{\Omega}_1| > |B_1|/2^{n+1}$. Thus

$$\begin{aligned} |A_t \cap \bar{\Omega}_1|_{g(\bar{x})} &\le C(n)\rho^n |A_t^{\mathrm{c}} \cap \bar{\Omega}_1| \\ &\le C(n)\rho^{2n} |\partial A_t \cap \partial A_t^{\mathrm{c}}|_{g(\bar{x})}^{n/(n-1)}. \end{aligned}$$

In either case we have the desired isoperimetric inequality (now given in the new coordinates for $\mathfrak{M}$), which holds for $\iota < t < M$,

$$|A_t \cap \bar{\Omega}_1|_{g(\bar{x})} \le C(n)\rho^{2n} |\partial A_t \cap \partial A_t^{\mathrm{c}}|_{g(\bar{x})}^{n/(n-1)},$$

or equivalently (3.7) in the original coordinates.

We then proceed to prove the Sobolev inequality via the Federer-Fleming argument (cf. [2, theorem 3]). First we recall the Hardy-Littlewood-Polya inequality for any nonnegative, nonincreasing integrand $\eta(t)$:

$$\left[\int_0^T \eta(t)^q\, dt^q\right]^{\frac{1}{q}} \le \int_0^T \eta(t)dt \tag{3.9}$$

with $q \ge 1$. This inequality follows from the fact that the left equals the right when $T = 0$, and the T-derivative of the left is less than or equal to that of the right.

Therefore, we have

$$
\begin{aligned}
&\left[\int_{B_1} |(f-\iota)^+|^{\frac{n}{n-1}}\, dv_g\right]^{\frac{n-1}{n}} \\
&\quad = \left[\int_0^{M-\iota} \big|\{x \mid f(x)-\iota > t\} \cap B_1\big|_g \, dt^{\frac{n}{n-1}}\right]^{\frac{n-1}{n}} \\
&\quad \le \int_0^{M-\iota} \big|\{x \mid f(x)-\iota > t\} \cap B_1\}\big|_g^{(n-1)/n}\, dt \\
&\quad \le C(n)\rho^{2(n-1)} \int_\iota^M \big|\{x \mid f(x) = t\} \cap B_5\big|_g \, dt \\
&\quad \le C(n)\rho^{2(n-1)} \int_{B_5} |\nabla_g (f-\iota)^+| dv_g,
\end{aligned}
$$

by (3.9), (3.7), and the co-area formula for the above three inequalities, respectively. The proposition is thus proved. □

Step 3. We consider the function $f = \ln V$ on the special Lagrangian manifold $\mathfrak{M}$, where V is the volume element with respect to the original x-coordinate system. Observe that $\ln V(\bar{x})$ in the rotated $\bar{x}$-coordinate system satisfies

$$
\sum_{i,j=1}^{n} g^{ij}(\bar{x}) \frac{\partial^2 \ln V(\bar{x})}{\partial \bar{x}_i\, \partial \bar{x}_j} = \triangle_{g(\bar{x})} \ln V(\bar{x}) \ge 0
$$

because of (2.2) and Proposition 2.1. Note also that the above nondivergence and divergence elliptic operators are both uniformly elliptic by (3.6).

Via the De Giorgi–Moser iteration (cf. [6, theorem 8.17]), we have

$$
\begin{aligned}
(\ln V - \iota)^+(0) &= (\ln V - \iota)^+(\bar{0}) \\
&\le C(n)\left[\int_{B_{1/\sqrt{2}}(\bar{0})} |(\ln V-\iota)^+(\bar{x})|^{\frac{n}{n-1}}\, d\bar{x}\right]^{\frac{n-1}{n}} \\
&\le C(n)\left[\int_{B_{1/\sqrt{2}}(\bar{0})} |(\ln V-\iota)^+(\bar{x})|^{\frac{n}{n-1}}\, dv_{g(\bar{x})}\right]^{\frac{n-1}{n}} \\
&\le C(n)\left[\int_{B_1(0)} |(\ln V-\iota)^+(x)|^{\frac{n}{n-1}}\, dv_{g(x)}\right]^{\frac{n-1}{n}}
\end{aligned}
$$

where

$$
\iota = \frac{2}{|B_1(0)|} \int_{B_5(0)} \ln V\, dx.
$$

Using the nondivergence structure of the Laplace-Beltrami operator (2.2), one may also get the above mean value inequality by a local maximum principle [6, theorem 9.20].

Thus by Proposition 3.3 and Proposition 2.1, we obtain

$$
\begin{aligned}
\ln V(0) &\le C(n)\rho^{2(n-1)} \int_{B_5} |\nabla_g (\ln V - \iota)^+| dv_g + C(n) \int_{B_5} \ln V \, dx \\
&\le C(n)\rho^{2(n-1)} \Big(\int_{B_5} |\nabla_g \ln V|^2 \, dv_g \Big)^{\frac{1}{2}} \Big(\int_{B_5} V \, dx \Big)^{\frac{1}{2}} + C(n) \int_{B_5} V \, dx \\
&\le C(n)\rho^{2(n-1)} \int_{B_6} V \, dx. \qquad (3.10)
\end{aligned}
$$

Step 4. We finish the proof of Theorem 1.1 by bounding $\int_{B_6} V \, dx$. Returning to our rotated coordinates, from (3.6) we see

$$
V \, dx = \bar{V} \, d\bar{x} \le 2^{\frac{n}{2}} \, d\bar{x}.
$$

Since $\bar{\Omega}_6 = \bar{x}(B_6(0))$,

$$
\int_{B_6} V \, dx = \int_{\bar{\Omega}_6} \bar{V} d\bar{x} \le 2^{\frac{n}{2}} \int_{\bar{\Omega}_6} d\bar{x} \le C(n)\rho^n.
$$

Then (3.5) leads to

$$
\int_{B_6} V \, dx \le C(n)[1 + \|Du\|^n_{L^\infty(B_6)}].
$$

Finally, from the above and (3.10) we conclude

$$
\ln V(0) \le C(n)\rho^{2(n-1)}[1 + \|Du\|^n_{L^\infty(B_6)}]
$$

or

$$
|D^2 u(0)| \le C(n) \exp\big\{C(n)\|Du\|^{3n-2}_{L^\infty(B_6)}\big\}.
$$

Step 5. Now the gradient of the convex function u is bounded by its oscillation

$$
\|Du\|_{L^\infty(B_6)} \le \operatorname*{osc}_{B_7} u.
$$

By scaling, we arrive at the conclusion of Theorem 1.1.

4 Interior Regularity for Special Lagrangian Equations with Very Large Phase

In this section, we apply our a priori interior estimates of Corollary 1.2 to show Theorem 1.3, which asserts that any C^0 viscosity solution to special Lagrangian equation (1.1) with very large phase $|\Theta| \ge (n-1)\frac{\pi}{2}$ is regular, that is, analytic inside the domain of the C^0 viscosity solution.

PROOF: By symmetry, we assume $\Theta \geq (n-1)\frac{\pi}{2}$. By scaling, we assume that the C^0 viscosity solution u satisfies (1.1) in the unit ball around any interior point of the domain of u. We approximate the C^0 boundary data $u|_{\partial B_1} = \varphi$ by smooth data φ_ε. Notice that the level set of $F(D^2 u) = \arctan\lambda_1 + \cdots + \arctan\lambda_n$ in λ-space, corresponding to the elliptic equation (1.1),

$$\Sigma = \{(\lambda_1, \ldots, \lambda_n) \mid \arctan\lambda_1 + \cdots + \arctan\lambda_n = \Theta\} \tag{4.1}$$

is convex for $\Theta \geq (n-1)\frac{\pi}{2}$. By [4, theorem 4] (see also a simplification [15]), we obtain a smooth solution u_ε to (1.1) with smooth boundary data φ_ε in B_1.

Applying the (easy) comparison theorem to the C^0 viscosity solution u and the C^2 solution u_ε,

$$\|u - u_\varepsilon\|_{L^\infty(B_1)} \leq \|u - u_\varepsilon\|_{L^\infty(\partial B_1)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{4.2}$$

In fact, one can deduce (4.2) from the definition of viscosity solution as follows: Suppose

$$M \stackrel{\text{def}}{=} \max_{B_1}(u - u_\varepsilon) > \max_{\partial B_1}(u - u_\varepsilon) \stackrel{\text{def}}{=} M_\partial;$$

then the function

$$u - \left[u_\varepsilon - \frac{(M - M_\partial)}{2}|x|^2\right]$$

achieves its maximum at an interior point x_0. By the definition of viscosity solution,

$$F\left(D^2\left(u_\varepsilon - \frac{(M - M_\partial)}{2}|x|^2\right)\right) \geq \Theta$$

at x_0. But F is strictly elliptic,

$$\Theta = F(D^2 u_\varepsilon) > F(D^2 u_\varepsilon(x_0) - (M - M_\partial)I) \geq \Theta.$$

This contradiction shows that $\max_{B_1}(u - u_\varepsilon) \leq \max_{\partial B_1}(u - u_\varepsilon)$. Similarly, one proves $\min_{B_1}(u - u_\varepsilon) \geq \min_{\partial B_1}(u - u_\varepsilon)$. Therefore (4.2) holds.

From our Corollary 1.2 combined with the Evans-Krylov-Safonov theory (cf. [6, theorem 17.15]), we have

$$\|u_\varepsilon\|_{C^{2,\alpha}(\Omega)} \leq C(\Omega, \|u\|_{L^\infty(B_1)})$$

for any subdomain $\Omega \subset \mathring{B}_1$. Thus u is $C^{2,\alpha}$ inside B_1. Now by the classical elliptic theory (cf. [6, theorem 17.16] and [12, p. 203]), u is smooth, even analytic inside B_1. □

In closing, we make some remarks on the existence and uniqueness of the C^0 viscosity solution of the Dirichlet problem for special Lagrangian equation (1.1). As we see in the above, the approximation and the existence of smooth solutions with smooth boundary data to (1.1) with $|\Theta| \geq (n-1)\frac{\pi}{2}$ in any (strongly) convex domain already lead to the existence and uniqueness of the C^0 viscosity solution to the Dirichlet problem with C^0 boundary data.

Notice that (1.1) is strictly elliptic in the sense

$$F_{u_{ij}}(D^2u) > 0.$$

The existence and uniqueness of the C^0 viscosity solution for the Dirichlet problem of (1.1) follow from those for general strictly elliptic equations. This general result is known. It follows from the comparison principle for C^0 viscosity solutions to strictly elliptic equations. In fact, strict ellipticity (rather than uniform ellipticity) is enough for the proof of uniqueness (also comparison principle) for fully nonlinear elliptic equations presented in [3, pp. 43–46]. For details, see, for example, [24].

Recently there has been a new approach toward the existence and uniqueness of a C^0 viscosity solution to the special Lagrangian equation (1.1), as stated in [8, p. 46]. Actually, theorem 6.2 in [8] on the Dirichlet problem for general fully nonlinear elliptic equations with starlike/convex level set already applies to (1.1) with $|\Theta| \geq (n-1)\frac{\pi}{2}$, even $|\Theta| \geq (n-2)\frac{\pi}{2}$. This is because the level set (4.1) Σ is convex for large phase $|\Theta| \geq (n-2)\frac{\pi}{2}$; see lemma 2.1 in [25].

We point out that the above two conditions on the fully nonlinear elliptic equations, strict ellipticity and starlike/convex level set, are independent of each other.

The regularity for C^0 viscosity solutions to the general special Lagrangian equation (1.1), without a convexity assumption on solutions or restriction on phase, is far from being settled.

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