Hessian Estimates for the Sigma-2 Equation in Dimension 3

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Abstract

We derive a priori interior Hessian estimates for the special Lagrangian equation $\sigma_2 = 1$ in dimension 3. © 2008 Wiley Periodicals, Inc.

1 Introduction

In this article, we derive an *interior a priori* Hessian estimate for the σ_2 equation

$$\sigma_2(D^2u) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1 \tag{1.1}$$

in dimension 3, where λ_i are the eigenvalues of the Hessian D^2u. We attack (1.1) via its special Lagrangian equation form

$$\sum_{i=1}^{n} \arctan\lambda_i = \Theta \tag{1.2}$$

with $n = 3$ and $\Theta = \frac{\pi}{2}$. Equation (1.2) stems from the special Lagrangian geometry [4]. The Lagrangian graph $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ is called special when the phase or the argument of the complex number $(1 + \sqrt{-1}\,\lambda_1)\cdots(1 + \sqrt{-1}\,\lambda_n)$ is constant Θ, and it is special if and only if $(x, Du(x))$ is a (volume minimizing) minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$ [4, theorem 2.3, prop. 2.17].

We state our result in the following:

THEOREM 1.1 *Let u be a smooth solution to* (1.1) *on $B_R(0) \subset \mathbb{R}^3$. Then we have*

$$|D^2u(0)| \leq C(3)\exp\left[C(3)\max_{B_R(0)}\frac{|Du|^3}{R^3}\right].$$

By Trudinger's gradient estimates for σ_k-equations [9], we can bound D^2u in terms of the solution u in $B_{2R}(0)$ as

$$|D^2u(0)| \leq C(3)\exp\left[C(3)\max_{B_{2R}(0)}\frac{|u|^3}{R^6}\right].$$

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One immediate consequence of the above estimates is a Liouville-type result for global solutions with quadratic growth to (1.1); namely, any such solution must be quadratic (cf. [15, 16]). Another consequence is the regularity (analyticity) of the C^0 viscosity solutions to (1.1) or (1.2) with $n = 3$ and $\Theta = \pm\frac{\pi}{2}$.

In the 1950s, Heinz [5] derived a Hessian bound for $\sigma_2(D^2u) = \lambda_1\lambda_2 = \det(D^2u) = 1$, the two-dimensional Monge-Ampère equation, which is equivalent to (1.2) with $n = 2$ and $\Theta = \pm\frac{\pi}{2}$. In the 1970s Pogorelov [8] constructed his famous counterexamples, namely, irregular solutions to three-dimensional Monge-Ampère equations $\sigma_3(D^2u) = \lambda_1\lambda_2\lambda_3 = \det(D^2u) = 1$; see generalizations of the counterexamples for σ_k-equations with $k \geq 3$ in [10]. Hessian estimates for solutions with certain strict convexity constraints to Monge-Ampère equations and σ_k-equations ($k \geq 2$) were derived by Pogorelov [8] and Chou and Wang [3], respectively, using the Pogorelov technique. Urbas [11, 12] and Bao and Chen [2] obtained (pointwise) Hessian estimates in terms of certain integrals of the Hessian, respectively, for σ_k-equations and the special Lagrangian equation (1.2) with $n = 3$ and $\Theta = \pi$,

The heuristic idea of the proof of Theorem 1.1 is as follows: The function $b = \ln\sqrt{1+\lambda_{\max}^2}$ is subharmonic so that b at any point is bounded by its integral over a ball around the point on the minimal surface by Michael and Simon's mean value inequality [6]. This special choice of b is not only subharmonic but, even stronger, satisfies a Jacobi inequality. This Jacobi inequality leads to a bound on the integral of b by the volume of the ball on the minimal surface. Taking advantage of the divergence form of the volume element of the minimal Lagrangian graph, we bound the volume in terms of the height of the special Lagrangian graph, which is the gradient of the solution to equation (1.2).

Now the challenging regularity problem for sigma-2 equations in dimension 4 and higher still remains open to us.

Notation. $D_i = \partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$, $u_i = \partial_i u$, $u_{ji} = \partial_{ij} u$, etc., but $\lambda_1, \ldots, \lambda_n$ and

$$b_1 = \ln\sqrt{1+\lambda_1^2}, \qquad b_2 = \frac{\ln\sqrt{1+\lambda_1^2} + \ln\sqrt{1+\lambda_2^2}}{2},$$

do not represent the partial derivatives. Further, h_{ijk} will denote (the second fundamental form)

$$h_{ijk} = \frac{1}{\sqrt{1+\lambda_i^2}}\frac{1}{\sqrt{1+\lambda_j^2}}\frac{1}{\sqrt{1+\lambda_k^2}}\,u_{ijk},$$

when D^2u is diagonalized. Finally, $C(n)$ will denote various constants depending only on dimension n.

2 Preliminary Inequalities

Taking the gradient of both sides of the special Lagrangian equation (1.2), we have

$$\sum_{i,j=1}^{n} g^{ij}\partial_{ij}(x, Du(x)) = 0, \tag{2.1}$$

where (g^{ij}) is the inverse of the induced metric $g = (g_{ij}) = I + D^2uD^2u$ on the surface $(x, Du(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$. Simple geometric manipulation of (2.1) yields the usual form of the minimal surface equation

$$\triangle_g(x, Du(x)) = 0,$$

where the Laplace-Beltrami operator of the metric g is given by

$$\triangle_g = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \partial_i\big(\sqrt{\det g}\, g^{ij}\partial_j\big).$$

Because we are using harmonic coordinates $\triangle_g x = 0$, we see that $\triangle_g$ also equals the linearized operator of the special Lagrangian equation (1.2) at u,

$$\triangle_g = \sum_{i,j=1}^{n} g^{ij}\partial_{ij}.$$

The gradient and inner product with respect to the metric g are

$$\nabla_g v = \left(\sum_{k=1}^{n} g^{1k}v_k, \cdots, \sum_{k=1}^{n} g^{nk}v_k\right),$$

$$\langle\nabla_g v, \nabla_g w\rangle_g = \sum_{i,j=1}^{n} g^{ij}v_iw_j, \quad \text{in particular, } |\nabla_g v|^2 = \langle\nabla_g v, \nabla_g v\rangle_g.$$

We begin with some geometric calculations.

LEMMA 2.1 *Let u be a smooth solution to* (1.2). *Suppose that the Hessian D^2u is diagonalized and the eigenvalue λ_1 is distinct from all other eigenvalues of D^2u at point p. Set $b_1 = \ln\sqrt{1+\lambda_1^2}$ near p. Then we have at p*

$$|\nabla_g b_1|^2 = \sum_{k=1}^{n} \lambda_1^2 h_{11k}^2 \tag{2.2}$$

and

$$\triangle_g b_1 =$$

$$(1+\lambda_1^2)h_{111}^2 + \sum_{k>1}\left(\frac{2\lambda_1}{\lambda_1-\lambda_k} + \frac{2\lambda_1^2\lambda_k}{\lambda_1-\lambda_k}\right)h_{kk1}^2 \tag{2.3}$$

$$+\sum_{k>1}\left[1+\frac{2\lambda_1}{\lambda_1-\lambda_k}+\frac{\lambda_1^2(\lambda_1+\lambda_k)}{\lambda_1-\lambda_k}\right]h_{11k}^2 \tag{2.4}$$

$$+\sum_{k>j>1} 2\lambda_1\left[\frac{1+\lambda_k^2}{\lambda_1-\lambda_k}+\frac{1+\lambda_j^2}{\lambda_1-\lambda_j}+(\lambda_j+\lambda_k)\right]h_{kj1}^2. \tag{2.5}$$

PROOF: We first compute the derivatives of the smooth function b_1 near p. We may implicitly differentiate the characteristic equation

$$\det(D^2u-\lambda_1 I)=0$$

near any point where λ_1 is distinct from the other eigenvalues. Then we get at p

$$\partial_e\lambda_1 = \partial_e u_{11},$$

$$\partial_{ee}\lambda_1 = \partial_{ee}u_{11} + \sum_{k>1} 2\,\frac{(\partial_e u_{1k})^2}{\lambda_1-\lambda_k},$$

with arbitrary unit vector $e\in\mathbb{R}^n$.

Thus we have (2.2) at p

$$|\nabla_g b_1|^2 = \sum_{k=1}^n g^{kk}\left(\frac{\lambda_1}{1+\lambda_1^2}\partial_k u_{11}\right)^2 = \sum_{k=1}^n \lambda_1^2 h_{11k}^2,$$

where we used the notation $h_{ijk}=\sqrt{g^{ii}}\sqrt{g^{jj}}\sqrt{g^{kk}}\,u_{ijk}$.

From

$$\partial_{ee}b_1 = \partial_{ee}\ln\sqrt{1+\lambda_1^2} = \frac{\lambda_1}{1+\lambda_1^2}\,\partial_{ee}\lambda_1 + \frac{1-\lambda_1^2}{(1+\lambda_1^2)^2}(\partial_e\lambda_1)^2,$$

we conclude that at p

$$\partial_{ee}b_1 = \frac{\lambda_1}{1+\lambda_1^2}\left[\partial_{ee}u_{11}+\sum_{k>1}2\,\frac{(\partial_e u_{1k})^2}{\lambda_1-\lambda_k}\right]+\frac{1-\lambda_1^2}{(1+\lambda_1^2)^2}(\partial_e u_{11})^2$$

and

$$\triangle_g b_1 = \sum_{\gamma=1}^n g^{\gamma\gamma}\partial_{\gamma\gamma}b_1$$

$$= \sum_{\gamma=1}^n g^{\gamma\gamma}\frac{\lambda_1}{1+\lambda_1^2}\left[\partial_{\gamma\gamma}u_{11}+\sum_{k>1}2\frac{(u_{1k\gamma})^2}{\lambda_1-\lambda_k}\right]+\sum_{\gamma=1}^n\frac{1-\lambda_1^2}{(1+\lambda_1^2)^2}\,g^{\gamma\gamma}u_{11\gamma}^2. \tag{2.6}$$

Next we substitute the fourth-order derivative terms $\partial_{\gamma\gamma}u_{11}$ in the above by lower-order derivative terms. Differentiating the minimal surface equation (2.1) $\sum_{\alpha,\beta=1}^{n} g^{\alpha\beta}u_{j\alpha\beta}=0$, we obtain

$$
\begin{aligned}
\triangle_g u_{ij} &= \sum_{\alpha,\beta=1}^{n} g^{\alpha\beta}u_{ji\alpha\beta} = \sum_{\alpha,\beta=1}^{n} -\partial_i g^{\alpha\beta}u_{j\alpha\beta} \\
&= \sum_{\alpha,\beta,\gamma,\delta=1}^{n} g^{\alpha\gamma}\partial_i g_{\gamma\delta}g^{\delta\beta}u_{j\alpha\beta} \\
&= \sum_{\alpha,\beta=1}^{n} g^{\alpha\alpha}g^{\beta\beta}(\lambda_\alpha+\lambda_\beta)u_{\alpha\beta i}u_{\alpha\beta j}, \qquad (2.7)
\end{aligned}
$$

where we used

$$
\partial_i g_{\gamma\delta} = \partial_i\left(\delta_{\gamma\delta}+\sum_{\varepsilon=1}^{n}u_{\gamma\varepsilon}u_{\varepsilon\delta}\right) = u_{\gamma\delta i}(\lambda_\gamma+\lambda_\delta)
$$

with diagonalized D^2u. Plugging (2.7) with $i=j=1$ into (2.6), we have at p

$$
\begin{aligned}
\triangle_g b_1 &= \frac{\lambda_1}{1+\lambda_1^2}\left[\sum_{\alpha,\beta=1}^{n} g^{\alpha\alpha}g^{\beta\beta}(\lambda_\alpha+\lambda_\beta)u_{\alpha\beta 1}^2 + \sum_{\gamma=1}^{n}\sum_{k>1}2\frac{u_{1k\gamma}^2}{\lambda_1-\lambda_k}\,g^{\gamma\gamma}\right] \\
&\quad + \sum_{\gamma=1}^{n}\frac{1-\lambda_1^2}{(1+\lambda_1^2)^2}g^{\gamma\gamma}u_{11\gamma}^2 \\
&= \lambda_1\sum_{\alpha,\beta=1}^{n}(\lambda_\alpha+\lambda_\beta)h_{\alpha\beta 1}^2 + \sum_{k>1}\sum_{\gamma=1}^{n}\frac{2\lambda_1(1+\lambda_k^2)}{\lambda_1-\lambda_k}\,h_{\gamma k1}^2 \\
&\quad + \sum_{\gamma=1}^{n}(1-\lambda_1^2)h_{11\gamma}^2,
\end{aligned}
$$

where we used the notation $h_{ijk}=\sqrt{g^{ii}}\sqrt{g^{jj}}\sqrt{g^{kk}}u_{ijk}$. Regrouping those terms $h_{\heartsuit\heartsuit 1}$, $h_{11\heartsuit}$, and $h_{\heartsuit\clubsuit 1}$ in the last expression, we have

$$
\begin{aligned}
\triangle_g b_1 &= (1-\lambda_1^2)h_{111}^2 + \sum_{\alpha=1}^{n}2\lambda_1\lambda_\alpha h_{\alpha\alpha 1}^2 + \sum_{k>1}\frac{2\lambda_1(1+\lambda_k^2)}{\lambda_1-\lambda_k}\,h_{kk1}^2 \\
&\quad + \sum_{k>1}(1-\lambda_1^2)h_{11k}^2 + \sum_{k>1}2\lambda_1(\lambda_k+\lambda_1)h_{k11}^2 + \sum_{k>1}\frac{2\lambda_1(1+\lambda_k^2)}{\lambda_1-\lambda_k}\,h_{1k1}^2 \\
&\quad + \sum_{k>j>1}2\lambda_1(\lambda_j+\lambda_k)h_{jk1}^2 + \sum_{\substack{j,k>1,\\ j\neq k}}\frac{2\lambda_1(1+\lambda_k^2)}{\lambda_1-\lambda_k}\,h_{jk1}^2.
\end{aligned}
$$

After simplifying the above expression, we have the second formula in Lemma 2.1. □

LEMMA 2.2 *Let u be a smooth solution to* (1.2) *with $n = 3$ and $\Theta \geq \frac{\pi}{2}$. Suppose that the ordered eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of the Hessian D^2u satisfy $\lambda_1 > \lambda_2$ at point p. Set*

$$b_1 = \ln \sqrt{1 + \lambda_{\max}^2} = \ln \sqrt{1 + \lambda_1^2}.$$

Then we have at p

$$\triangle_g b_1 \geq \frac{1}{3} |\nabla_g b_1|^2. \tag{2.8}$$

PROOF : We assume that the Hessian D^2u is diagonalized at point p.

Step 1. Recall $\theta_i = \arctan \lambda_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\theta_1 + \theta_2 + \theta_3 = \Theta \geq \frac{\pi}{2}$. It is easy to see that $\theta_1 \geq \theta_2 > 0$ and $\theta_i + \theta_j \geq 0$ for any pair. Consequently, $\lambda_1 \geq \lambda_2 > 0$ and $\lambda_i + \lambda_j \geq 0$ for any pair of distinct eigenvalues. It follows that (2.5) in the formula for $\triangle_g b_1$ is positive; then from (2.3) and (2.4) we have the inequality

$$\triangle_g b_1 \geq \lambda_1^2 \left(h_{111}^2 + \sum_{k>1} \frac{2\lambda_k}{\lambda_1 - \lambda_k} h_{kk1}^2 \right) + \lambda_1^2 \sum_{k>1} \left(1 + \frac{2\lambda_k}{\lambda_1 - \lambda_k} \right) h_{11k}^2. \tag{2.9}$$

Combining (2.9) and (2.2) gives

$$\begin{aligned} \triangle_g\, b_1 - \frac{1}{3} |\nabla_g b_1|^2 \geq \\ \lambda_1^2 \left(\frac{2}{3} h_{111}^2 + \sum_{k>1} \frac{2\lambda_k}{\lambda_1 - \lambda_k} h_{kk1}^2 \right) + \lambda_1^2 \sum_{k>1} \frac{2(\lambda_1 + 2\lambda_k)}{3(\lambda_1 - \lambda_k)} h_{11k}^2. \end{aligned} \tag{2.10}$$

Step 2. We show that the second term on the right-hand side of (2.10) is nonnegative. Note that $\lambda_1 + 2\lambda_k \geq \lambda_1 + 2\lambda_3$. We only need to show that $\lambda_1 + 2\lambda_3 \geq 0$ in the case that $\lambda_3 < 0$ or equivalently $\theta_3 < 0$. From $\theta_1 + \theta_2 + \theta_3 = \Theta \geq \frac{\pi}{2}$, we have

$$\frac{\pi}{2} > \theta_3 + \frac{\pi}{2} = \left(\frac{\pi}{2} - \theta_1 \right) + \left(\frac{\pi}{2} - \theta_2 \right) + \Theta - \frac{\pi}{2} \geq 2 \left(\frac{\pi}{2} - \theta_1 \right).$$

It follows that

$$-\frac{1}{\lambda_3} = \tan \left(\theta_3 + \frac{\pi}{2} \right) > 2 \tan \left(\frac{\pi}{2} - \theta_1 \right) = \frac{2}{\lambda_1};$$

then

$$\lambda_1 + 2\lambda_3 > 0. \tag{2.11}$$

Step 3. We show that the first term in (2.10) is nonnegative by proving

$$\frac{2}{3}h_{111}^2 + \frac{2\lambda_2}{\lambda_1-\lambda_2}h_{221}^2 + \frac{2\lambda_3}{\lambda_1-\lambda_3}h_{331}^2 \geq 0. \tag{2.12}$$

We only need to show it for $\lambda_3 < 0$. Directly from the minimal surface equation (2.1)

$$h_{111} + h_{221} + h_{331} = 0;$$

we bound

$$h_{331}^2 = (h_{111}+h_{221})^2 \leq \left(\frac{2}{3}h_{111}^2 + \frac{2\lambda_2}{\lambda_1-\lambda_2}h_{221}^2\right)\left(\frac{3}{2} + \frac{\lambda_1-\lambda_2}{2\lambda_2}\right).$$

It follows that

$$\frac{2}{3}h_{111}^2 + \frac{2\lambda_2}{\lambda_1-\lambda_2}h_{221}^2 + \frac{2\lambda_3}{\lambda_1-\lambda_3}h_{331}^2 \geq \\ \left(\frac{2}{3}h_{111}^2 + \frac{2\lambda_2}{\lambda_1-\lambda_2}h_{221}^2\right)\left[1 + \frac{2\lambda_3}{\lambda_1-\lambda_3}\left(\frac{3}{2} + \frac{\lambda_1-\lambda_2}{2\lambda_2}\right)\right].$$

The last term becomes

$$1 + \frac{2\lambda_3}{\lambda_1-\lambda_3}\left(\frac{3}{2} + \frac{\lambda_1-\lambda_2}{2\lambda_2}\right) = \frac{\sigma_2}{(\lambda_1-\lambda_3)\lambda_2} > 0.$$

The above inequality is from the observation

$$\operatorname{Re}\prod_{i=1}^{3}\left(1+\sqrt{-1}\lambda_i\right) = 1-\sigma_2 \leq 0$$

for $\frac{3\pi}{2} > \theta_1 + \theta_2 + \theta_3 = \Theta \geq \frac{\pi}{2}$. Therefore (2.12) holds.

We have proved the pointwise Jacobi inequality (2.8) in Lemma 2.2. □

LEMMA 2.3 *Let u be a smooth solution to* (1.2) *with $n = 3$ and $\Theta \geq \frac{\pi}{2}$. Suppose that the ordered eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of the Hessian D^2u satisfy $\lambda_2 > \lambda_3$ at point p. Set*

$$b_2 = \frac{1}{2}\left(\ln\sqrt{1+\lambda_1^2} + \ln\sqrt{1+\lambda_2^2}\right).$$

Then b_2 satisfies at p

$$\triangle_g b_2 \geq 0. \tag{2.13}$$

Further, suppose that $\lambda_1 \equiv \lambda_2$ in a neighborhood of p. Then b_2 satisfies at p

$$\triangle_g b_2 \geq \frac{1}{3}|\nabla_g b_2|^2. \tag{2.14}$$

PROOF: We assume that Hessian D^2u is diagonalized at point p. We may use Lemma 2.1 to obtain expressions for both $\triangle_g \ln\sqrt{1+\lambda_1^2}$ and $\triangle_g \ln\sqrt{1+\lambda_2^2}$, whenever the eigenvalues of D^2u are distinct. From (2.3), (2.4), and (2.5), we have

$$\begin{aligned}
&\triangle_g \ln\sqrt{1+\lambda_1^2}+\triangle_g \ln\sqrt{1+\lambda_2^2}\\
&\quad=(1+\lambda_1^2)h_{111}^2+\sum_{k>1}\frac{2\lambda_1(1+\lambda_1\lambda_k)}{\lambda_1-\lambda_k}\,h_{kk1}^2\\
&\qquad+\sum_{k>1}\left[1+\lambda_1^2+2\lambda_1\left(\frac{1+\lambda_1\lambda_k}{\lambda_1-\lambda_k}\right)\right]h_{11k}^2\\
&\qquad+2\lambda_1\left[\frac{1+\lambda_3^2}{\lambda_1-\lambda_3}+\frac{1+\lambda_2^2}{\lambda_1-\lambda_2}+(\lambda_3+\lambda_2)\right]h_{321}^2\\
&\qquad+(1+\lambda_2^2)h_{222}^2+\sum_{k\neq2}\frac{2\lambda_2(1+\lambda_2\lambda_k)}{\lambda_2-\lambda_k}\,h_{kk2}^2\\
&\qquad+\sum_{k\neq2}\left[1+\lambda_2^2+2\lambda_2\left(\frac{1+\lambda_2\lambda_k}{\lambda_2-\lambda_k}\right)\right]h_{22k}^2\\
&\qquad+2\lambda_2\left[\frac{1+\lambda_3^2}{\lambda_2-\lambda_3}+\frac{1+\lambda_1^2}{\lambda_2-\lambda_1}+(\lambda_3+\lambda_1)\right]h_{321}^2.
\end{aligned}\tag{2.15}$$

The function b_2 is symmetric in λ_1 and λ_2; thus b_2 is smooth even when $\lambda_1=\lambda_2$ provided that $\lambda_2>\lambda_3$. We simplify (2.15) to the following, which holds by continuity wherever $\lambda_1\geq\lambda_2>\lambda_3$:

$$2\,\triangle_g\, b_2=$$

$$(1+\lambda_1^2)h_{111}^2+(3+\lambda_2^2+2\lambda_1\lambda_2)h_{221}^2+\left(\frac{2\lambda_1}{\lambda_1-\lambda_3}+\frac{2\lambda_1^2\lambda_3}{\lambda_1-\lambda_3}\right)h_{331}^2\tag{2.16}$$

$$+(3+\lambda_1^2+2\lambda_1\lambda_2)h_{112}^2+(1+\lambda_2^2)h_{222}^2+\left(\frac{2\lambda_2}{\lambda_2-\lambda_3}+\frac{2\lambda_2^2\lambda_3}{\lambda_2-\lambda_3}\right)h_{332}^2\tag{2.17}$$

$$+\left[\frac{3\lambda_1-\lambda_3+\lambda_1^2(\lambda_1+\lambda_3)}{\lambda_1-\lambda_3}\right]h_{113}^2+\left[\frac{3\lambda_2-\lambda_3+\lambda_2^2(\lambda_2+\lambda_3)}{\lambda_2-\lambda_3}\right]h_{223}^2\tag{2.18}$$

$$+2\left[1+\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1+\frac{\lambda_1(1+\lambda_3^2)}{\lambda_1-\lambda_3}+\frac{\lambda_2(1+\lambda_3^2)}{\lambda_2-\lambda_3}\right]h_{123}^2.\tag{2.19}$$

Using the assumption $\lambda_1\geq\lambda_2\geq\lambda_3$ coupled with the relations $\lambda_1\geq\lambda_2>0$, $\lambda_i+\lambda_j>0$, and $\sigma_2\geq1$ derived in the proof of Lemma 2.2, we see that (2.19) and (2.18) are nonnegative. We only need to justify the nonnegativity of (2.16) and

(2.17) for $\lambda_3 < 0$. From the minimal surface equation (2.1), we know

$$\begin{aligned} h_{332}^2 &= (h_{112} + h_{222})^2 \\ &\leq \left[\left(\lambda_1^2 + 2\lambda_1\lambda_2\right)h_{112}^2 + \lambda_2^2 h_{222}^2\right]\left(\frac{1}{\lambda_1^2 + 2\lambda_1\lambda_2} + \frac{1}{\lambda_2^2}\right). \end{aligned}$$

It follows that

$$\begin{aligned} (2.17) &\geq (\lambda_1^2 + 2\lambda_1\lambda_2)h_{112}^2 + \lambda_2^2 h_{222}^2 + \frac{2\lambda_2^2\lambda_3}{\lambda_2 - \lambda_3} h_{332}^2 \\ &\geq \left[(\lambda_1^2 + 2\lambda_1\lambda_2)h_{112}^2 + \lambda_2^2 h_{222}^2\right]\left[1 + \frac{2\lambda_2^2\lambda_3}{\lambda_2 - \lambda_3}\left(\frac{1}{\lambda_1^2 + 2\lambda_1\lambda_2} + \frac{1}{\lambda_2^2}\right)\right]. \end{aligned}$$

The last term becomes

$$\frac{2\lambda_2^2\lambda_3}{\lambda_2 - \lambda_3}\left(\frac{\lambda_2 - \lambda_3}{2\lambda_2^2\lambda_3} + \frac{1}{\lambda_1^2 + 2\lambda_1\lambda_2} + \frac{1}{\lambda_2^2}\right) = \\ \frac{\lambda_2}{\lambda_2 - \lambda_3}\left[\frac{\sigma_2}{\lambda_1\lambda_2} - \frac{\lambda_3}{(\lambda_1 + 2\lambda_2)}\right] \geq 0.$$

Thus (2.17) is nonnegative. Similarly, (2.16) is nonnegative. We have proved (2.13).

Next we prove (2.14), still assuming D^2u is diagonalized at point p. Plugging $\lambda_1 = \lambda_2$ into (2.16), (2.17), and (2.18), we get

$$\begin{aligned} 2 \triangle_g b_2 \geq{}& \lambda_1^2\left(h_{111}^2 + 3h_{221}^2 + \frac{2\lambda_3}{\lambda_1 - \lambda_3} h_{331}^2\right) \\ &+ \lambda_1^2\left(3h_{112}^2 + h_{222}^2 + \frac{2\lambda_3}{\lambda_1 - \lambda_3} h_{332}^2\right) \\ &+ \lambda_1^2\left(\frac{\lambda_1 + \lambda_3}{\lambda_1 - \lambda_3}\right)(h_{113}^2 + h_{223}^2). \end{aligned}$$

Differentiating the eigenvector equations in the neighborhood where $\lambda_1 \equiv \lambda_2$

$$(D^2u)U = \frac{\lambda_1 + \lambda_2}{2} U, \quad (D^2u)V = \frac{\lambda_1 + \lambda_2}{2} V, \quad \text{and} \quad (D^2u)W = \lambda_3 W,$$

we see that $u_{11e} = u_{22e}$ for any $e \in \mathbb{R}^3$ at point p. Using the minimal surface equation (2.1), we then have

$$h_{11k} = h_{22k} = -\frac{1}{2} h_{33k}$$

at point p. Thus

$$\triangle_g b_2 \geq \lambda_1^2\left[2\left(\frac{\lambda_1 + \lambda_3}{\lambda_1 - \lambda_3}\right)h_{111}^2 + 2\left(\frac{\lambda_1 + \lambda_3}{\lambda_1 - \lambda_3}\right)h_{112}^2 + \left(\frac{\lambda_1 + \lambda_3}{\lambda_1 - \lambda_3}\right)h_{113}^2\right].$$

The gradient $|\nabla_g b_2|^2$ has the expression at p

$$\begin{aligned}|\nabla_g b_2|^2 &= \sum_{k=1}^{3} g^{kk}\left(\frac{1}{2}\frac{\lambda_1}{1+\lambda_1^2}\,\partial_k u_{11} + \frac{1}{2}\frac{\lambda_2}{1+\lambda_2^2}\,\partial_k u_{22}\right)^2 \\ &= \sum_{k=1}^{3} \lambda_1^2 h_{11k}^2.\end{aligned}$$

Thus at p

$$\begin{aligned}\triangle_g b_2 - \frac{1}{3}|\nabla_g b_2|^2 \ge \lambda_1^2 \Bigg\{ &\left[2\left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3}\right) - \frac{1}{3}\right] h_{111}^2 + \left[2\left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3}\right) - \frac{1}{3}\right] h_{112}^2 \\ &+ \left(\frac{\lambda_1+\lambda_3}{\lambda_1-\lambda_3} - \frac{1}{3}\right) h_{113}^2 \Bigg\} \\ \ge 0,&\end{aligned}$$

where we again used $\lambda_1 + 2\lambda_3 > 0$ from (2.11). We have proved (2.14) of Lemma 2.3. □

PROPOSITION 2.4 *Let u be a smooth solution to the special Lagrangian equation* (1.2) *with $n = 3$ and $\Theta = \frac{\pi}{2}$ on $B_4(0) \subset \mathbb{R}^3$. Set*

$$b = \max\left\{\ln\sqrt{1+\lambda_{\max}^2},\, K\right\}$$

with $K = 1 + \ln\sqrt{1+\tan^2(\frac{\pi}{6})}$. Then b satisfies the integral Jacobi inequality

$$\int_{B_4} -\langle \nabla_g \varphi, \nabla_g b\rangle_g \, dv_g \ge \frac{1}{3}\int_{B_4} \varphi |\nabla_g b|^2 \, dv_g \tag{2.20}$$

for all nonnegative $\varphi \in C_0^\infty(B_4)$.

PROOF: If $b_1 = \ln\sqrt{1+\lambda_{\max}^2}$ is smooth everywhere, then the pointwise Jacobi inequality (2.8) in Lemma 2.2 already implies the integral Jacobi (2.20). It is known that $\lambda_{\max}$ is always a Lipschitz function of the entries of the Hessian $D^2 u$. Now u is smooth in x, so $b_1 = \ln\sqrt{1+\lambda_{\max}^2}$ is Lipschitz in terms of x. If b_1 (or equivalently $\lambda_{\max}$) is not smooth, then the two largest eigenvalues $\lambda_1(x)$ and $\lambda_2(x)$ coincide, and $b_1(x) = b_2(x)$, where $b_2(x)$ is the average

$$b_2 = \frac{\ln\sqrt{1+\lambda_1^2} + \ln\sqrt{1+\lambda_2^2}}{2}.$$

We prove the integral Jacobi inequality (2.20) for a possibly singular $b_1(x)$ in two cases. Set

$$S = \{x \mid \lambda_1(x) = \lambda_2(x)\}.$$

Case 1. S has measure zero. For small $\tau > 0$, let

$$\begin{aligned}\Omega &= B_4 \setminus \{x \mid b_1(x) \le K\} = B_4 \setminus \{x \mid b(x) = K\},\\ \Omega_1(\tau) &= \{x \mid b(x) = b_1(x) > b_2(x) + \tau\} \cap \Omega,\\ \Omega_2(\tau) &= \{x \mid b_2(x) \le b(x) = b_1(x) < b_2(x) + \tau\} \cap \Omega.\end{aligned}$$

Now $b(x) = b_1(x)$ is smooth in $\overline{\Omega_1(\tau)}$. We claim that $b_2(x)$ is smooth in $\overline{\Omega_2(\tau)}$. We know $b_2(x)$ is smooth wherever $\lambda_2(x) > \lambda_3(x)$. If (the Lipschitz) $b_2(x)$ is not smooth at $x_* \in \overline{\Omega_2(\tau)}$, then

$$\begin{aligned}\ln\sqrt{1+\lambda_3^2} = \ln\sqrt{1+\lambda_2^2} &\ge \ln\sqrt{1+\lambda_1^2} - 2\tau\\ &\ge \ln\sqrt{1+\tan^2(\tfrac{\pi}{6})} + 1 - 2\tau,\end{aligned}$$

by the choice of K. For small enough τ, we have $\lambda_2 = \lambda_3 > \tan(\frac{\pi}{6})$ and a contradiction

$$(\theta_1 + \theta_2 + \theta_3)(x_*) > \frac{\pi}{2}.$$

Note that

$$\begin{aligned}&\int_{B_4} -\langle \nabla_g \varphi, \nabla_g b\rangle_g \, dv_g\\ &\quad= \int_{\Omega} -\langle \nabla_g \varphi, \nabla_g b\rangle_g \, dv_g\\ &\quad= \lim_{\tau\to 0^+}\Bigg[\int_{\Omega_1(\tau)} -\langle \nabla_g \varphi, \nabla_g b\rangle_g \, dv_g + \int_{\Omega_2(\tau)} -\langle \nabla_g \varphi, \nabla_g (b_2+\tau)\rangle_g \, dv_g\Bigg].\end{aligned}$$

By the smoothness of b in $\Omega_1(\tau)$ and b_2 in $\Omega_2(\tau)$, and also inequalities (2.8) and (2.13), we have

$$\begin{aligned}&\int_{\Omega_1(\tau)} -\langle \nabla_g \varphi, \nabla_g b\rangle_g \, dv_g + \int_{\Omega_2(\tau)} -\langle \nabla_g \varphi, \nabla_g (b_2+\tau)\rangle_g \, dv_g\\ &\quad= \int_{\partial\Omega_1(\tau)} -\varphi \partial_{\gamma_g^1} b \, dA_g + \int_{\Omega_1(\tau)} \varphi \bigtriangleup_g b_1 \, dv_g\\ &\qquad+ \int_{\partial\Omega_2(\tau)} -\varphi \partial_{\gamma_g^2} (b_2+\tau) dA_g + \int_{\Omega_2(\tau)} \varphi \bigtriangleup_g (b_2+\tau) dv_g\end{aligned}$$

$$\geq \int_{\partial\Omega_1(\tau)} -\varphi\partial_{\gamma_g^1} b\, dA_g + \int_{\partial\Omega_2(\tau)} -\varphi\partial_{\gamma_g^2}(b_2+\tau) dA_g + \frac{1}{3}\int_{\Omega_1(\tau)} \varphi|\nabla_g b_1|^2\, dv_g,$$

where γ_g^1 and γ_g^2 are the outward conormals of $\partial\Omega_1(\tau)$ and $\partial\Omega_2(\tau)$ with respect to the metric g.

Observe that if b_1 is not smooth on any part of $\partial\Omega\backslash\partial B_4$, which is the K-level set of b_1, then on this portion $\partial\Omega\backslash\partial B_4$ is also the K-level set of b_2, which is smooth near this portion. Applying Sard's theorem, we can perturb K so that $\partial\Omega$ is piecewise C^1. Applying Sard's theorem again, we find a subsequence of positive τ going to 0, so that the boundaries $\partial\Omega_1(\tau)$ and $\partial\Omega_2(\tau)$ are piecewise C^1.

Then we show the above boundary integrals are nonnegative. The boundary integral portion along $\partial\Omega$ is easily seen to be nonnegative, because either $\varphi = 0$ or $-\partial_{\gamma_g^1} b \geq 0, -\partial_{\gamma_g^2}(b_2+\tau) \geq 0$ there. The boundary integral portion in the interior of Ω is also nonnegative, because there we have

$$b = b_2 + \tau \quad (\text{and } b \geq b_2+\tau \text{ in } \Omega_1(\tau)),$$
$$-\partial_{\gamma_g^1} b - \partial_{\gamma_g^2}(b_2+\tau) = \partial_{\gamma_g^2} b - \partial_{\gamma_g^2}(b_2+\tau) \geq 0.$$

Taking the limit along the (Sard) sequence of τ going to 0, we obtain $\Omega_1(\tau) \to \Omega$ up to a set of measure zero, and

$$\int_{B_4} -\langle\nabla_g\varphi, \nabla_g b\rangle_g\, dv_g = \int_{\Omega} -\langle\nabla_g\varphi, \nabla_g b\rangle_g\, dv_g \geq \frac{1}{3}\int_{\Omega} |\nabla_g b|^2\, dv_g = \frac{1}{3}\int_{B_4} |\nabla_g b|^2\, dv_g.$$

Case 2. S has positive measure. The discriminant

$$\mathcal{D} = (\lambda_1-\lambda_2)^2(\lambda_2-\lambda_3)^2(\lambda_3-\lambda_1)^2$$

is an analytic function in B_4, because the smooth u is actually analytic (cf. [7, p. 203]). So $\mathcal{D}$ must vanish identically. Then we have either $\lambda_1(x) = \lambda_2(x)$ or $\lambda_2(x) = \lambda_3(x)$ at any point $x \in B_4$. In turn, we know that $\lambda_1(x) = \lambda_2(x) = \lambda_3(x) = \tan(\frac{\pi}{6})$ and $b = K > b_1(x)$ at every "boundary" point of S inside B_4, $x \in \partial S \cap \mathring{B}_4$. If the "boundary" set ∂S has positive measure, then $\lambda_1(x) = \lambda_2(x) = \lambda_3(x) = \tan(\frac{\pi}{6})$ everywhere by the analyticity of u, and (2.20) is trivially true. In the case that ∂S has zero measure, $b = b_1 > K$ is smooth up to the boundary of every component of $\{x \mid b(x) > K\}$. By the pointwise Jacobi inequalities (2.14) and (2.8), the integral inequality (2.20) is also valid in case 2. □

3 Proof of Theorem 1.1

We assume that $R = 4$ and u is a solution on $B_4 \subset \mathbb{R}^3$ for simplicity of notation. By scaling $v(x) = u(\frac{R}{4}x)/(\frac{R}{4})^2$, we still get the estimate in Theorem 1.1. Without loss of generality, we assume that the continuous Hessian D^2u sits on the convex branch of $\{(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1\}$ containing $(1, 1, 1)/\sqrt{3}$; then u satisfies (1.2) with $n = 3$ and $\Theta = \frac{\pi}{2}$. By symmetry this also covers the concave branch corresponding to $\Theta = -\frac{\pi}{2}$.

Step 1. By the integral Jacobi inequality (2.20) in Proposition 2.4, b is subharmonic in the integral sense; then b^3 is also subharmonic in the integral sense on the minimal surface $\mathfrak{M} = (x, Du)$:

$$\begin{aligned}\int -\langle \nabla_g \varphi, \nabla_g b^3 \rangle_g \, dv_g &= \int -\langle \nabla_g (3b^2\varphi) - 6b\varphi\nabla_g b, \nabla_g b\rangle_g \, dv_g \\ &\geq \int (\varphi b^2 |\nabla_g b|^2 + 6b\varphi|\nabla_g b|^2) dv_g \geq 0\end{aligned}$$

for all nonnegative $\varphi \in C_0^\infty$, where we approximate $b^2\varphi$ by smooth functions if necessary.

Applying Michael and Simon's mean value inequality [6, theorem 3.4] to the Lipschitz subharmonic function b^3, we obtain

$$b(0) \leq C(3)\Bigg(\int_{\mathfrak{B}_1 \cap \mathfrak{M}} b^3 \, dv_g\Bigg)^{1/3} \leq C(3)\Bigg(\int_{B_1} b^3 \, dv_g\Bigg)^{1/3},$$

where $\mathfrak{B}_r$ is the ball with radius r and center $(0, Du(0))$ in $\mathbb{R}^3 \times \mathbb{R}^3$, and B_r is the ball with radius r and center 0 in $\mathbb{R}^3$. Choose a cutoff function $\varphi \in C_0^\infty(B_2)$ such that $\varphi \geq 0$, $\varphi = 1$ on B_1, and $|D\varphi| \leq 1.1$; we then have

$$\Bigg(\int_{B_1} b^3 \, dv_g\Bigg)^{1/3} \leq \Bigg(\int_{B_2} \varphi^6 b^3 \, dv_g\Bigg)^{1/3} = \Bigg(\int_{B_2} (\varphi b^{1/2})^6 \, dv_g\Bigg)^{1/3}.$$

Applying the Sobolev inequality on the minimal surface $\mathfrak{M}$ [6, theorem 2.1] or [1, theorem 7.3] to $\varphi b^{1/2}$, which we may assume to be C^1 by approximation, we obtain

$$\Bigg(\int_{B_2} (\varphi b^{1/2})^6 \, dv_g\Bigg)^{1/3} \leq C(3) \int_{B_2} |\nabla_g (\varphi b^{1/2})|^2 \, dv_g.$$

Splitting the second integrand as follows:

$$\begin{aligned}|\nabla_g (\varphi b^{1/2})|^2 = \left|\frac{1}{2b^{1/2}}\, \varphi \nabla_g b + b^{1/2}\nabla_g \varphi\right|^2 &\leq \frac{1}{2b}\, \varphi^2 |\nabla_g b|^2 + 2b|\nabla_g \varphi|^2 \\ &\leq \frac{1}{2}\, \varphi^2 |\nabla_g b|^2 + 2b|\nabla_g \varphi|^2,\end{aligned}$$

where we used $b \geq 1$, we get

$$\begin{aligned}
b(0) &\leq C(3) \int_{B_2} |\nabla_g (\varphi b^{1/2})|^2 \, dv_g \\
&\leq C(3) \Big(\int_{B_2} \varphi^2 |\nabla_g b|^2 \, dv_g + \int_{B_2} b |\nabla_g \varphi|^2 \, dv_g \Big) \\
&\leq \underbrace{C(3) \|Du\|_{L^\infty(B_2)}}_{\text{Step 2}} + C(3) \underbrace{[\|Du\|^2_{L^\infty(B_3)} + \|Du\|^3_{L^\infty(B_4)}]}_{\text{Step 3}}.
\end{aligned}$$

Step 2. By (2.20) in Proposition 2.4, b satisfies the Jacobi inequality in the integral sense:

$$3 \triangle_g b \geq |\nabla_g b|^2.$$

Multiplying both sides by the above nonnegative cutoff function $\varphi \in C_0^\infty(B_2)$ and then integrating, we obtain

$$\begin{aligned}
\int_{B_2} \varphi^2 |\nabla_g b|^2 \, dv_g &\leq 3 \int_{B_2} \varphi^2 \triangle_g b \, dv_g \\
&= -3 \int_{B_2} \langle 2\varphi \nabla_g \varphi, \nabla_g b \rangle dv_g \\
&\leq \frac{1}{2} \int_{B_2} \varphi^2 |\nabla_g b|^2 \, dv_g + 18 \int_{B_2} |\nabla_g \varphi|^2 \, dv_g.
\end{aligned}$$

It follows that

$$\int_{B_2} \varphi^2 |\nabla_g b|^2 \, dv_g \leq 36 \int_{B_2} |\nabla_g \varphi|^2 \, dv_g.$$

Observe the ("conformality") identity:

$$\left(\frac{1}{1+\lambda_1^2}, \frac{1}{1+\lambda_2^2}, \frac{1}{1+\lambda_3^2} \right) V = (\sigma_1 - \lambda_1, \ \sigma_1 - \lambda_2, \ \sigma_1 - \lambda_3)$$

where we used the identity $V = \prod_{i=1}^3 \sqrt{(1+\lambda_i^2)} = \sigma_1 - \sigma_3$ with $\sigma_2 = 1$. We then have

$$\begin{aligned}
(3.1) \qquad |\nabla_g \varphi|^2 \, dv_g = \sum_{i=1}^3 \frac{(D_i \varphi)^2}{1+\lambda_i^2} V \, dx &= \sum_{i=1}^3 (D_i \varphi)^2 (\sigma_1 - \lambda_i) dx \\
&\leq 2.42 \triangle u \, dx.
\end{aligned}$$

Thus

$$\int_{B_2} \varphi^2 |\nabla_g b|^2 \, dv_g \le C(3) \int_{B_2} \triangle u \, dx \le C(3) \|Du\|_{L^\infty(B_2)}.$$

Step 3. By (3.1), we get

$$\int_{B_2} b |\nabla_g \varphi|^2 \, dv_g \le C(3) \int_{B_2} b \triangle u \, dx.$$

Choose another cutoff function $\psi \in C_0^\infty(B_3)$ such that $\psi \ge 0$, $\psi = 1$ on B_2, and $|D\psi| \le 1.1$. We have

$$\begin{aligned}\int_{B_2} b \triangle u \, dx \le \int_{B_3} \psi b \triangle u \, dx &= \int_{B_3} -\langle bD\psi + \psi Db, Du\rangle dx \\ &\le \|Du\|_{L^\infty(B_3)} \int_{B_3} (b|D\psi| + \psi |Db|) dx \\ &\le C(3) \|Du\|_{L^\infty(B_3)} \int_{B_3} (b + |Db|) dx.\end{aligned}$$

Now

$$\begin{aligned} b &= \max \left\{ \ln \sqrt{1 + \lambda_{\max}^2}, K \right\} \\ &\le \lambda_{\max} + K < \lambda_1 + \lambda_2 + \lambda_3 + K = \triangle u + K, \end{aligned}$$

where $\lambda_2 + \lambda_3 > 0$ follows from $\arctan \lambda_2 + \arctan \lambda_3 = \frac{\pi}{2} - \arctan \lambda_1 > 0$. Hence

$$\int_{B_3} b \, dx \le C(3)[1 + \|Du\|_{L^\infty(B_3)}].$$

We have left to estimate $\int_{B_3} |Db| dx$:

$$\begin{aligned}\int_{B_3} |Db| dx &\le \int_{B_3} \sqrt{\sum_{i=1}^{3} \frac{(D_i b)^2}{(1 + \lambda_i^2)} (1 + \lambda_1^2)(1 + \lambda_2^2)(1 + \lambda_3^2)} \, dx \\ &= \int_{B_3} |\nabla_g b| V \, dx \\ &\le \left(\int_{B_3} |\nabla_g b|^2 V \, dx \right)^{1/2} \left(\int_{B_3} V \, dx \right)^{1/2}.\end{aligned}$$

Repeating the "Jacobi" argument from Step 2, we see

$$\int_{B_3} |\nabla_g b|^2 V\, dx \le C(3)\|Du\|_{L^\infty(B_4)}.$$

Then by the Sobolev inequality on the minimal surface $\mathfrak{M}$, we have

$$\int_{B_3} V\, dx = \int_{B_3} dv_g \le \int_{B_4} \phi^6\, dv_g \le C(3)\bigg(\int_{B_4} |\nabla_g \phi|^2\, dv_g\bigg)^3,$$

where the nonnegative cutoff function $\phi \in C_0^\infty(B_4)$ satisfies $\phi = 1$ on B_3 and $|D\phi| \le 1.1$.

Applying the conformality equality (3.1) again, we obtain

$$\int_{B_4} |\nabla_g \phi|^2\, dv_g \le C(3) \int_{B_4} \triangle u\, dx \le C(3)\|Du\|_{L^\infty(B_4)}.$$

Thus we get

$$\int_{B_3} V\, dx \le C(3)\|Du\|^3_{L^\infty(B_4)}$$

and

$$\int_{B_3} |Db| dx \le C(3)\|Du\|^2_{L^\infty(B_4)}.$$

In turn, we obtain

$$\int_{B_2} b|\nabla_g \varphi|^2\, dv_g \le C(3)[K\|Du\|_{L^\infty(B_3)} + \|Du\|^2_{L^\infty(B_3)} + \|Du\|^3_{L^\infty(B_4)}].$$

Finally, collecting all the estimates in the above three steps, we arrive at

$$\begin{aligned}\lambda_{\max}(0) &\le \exp\big[C(3)\big(\|Du\|_{L^\infty(B_4)} + \|Du\|^2_{L^\infty(B_4)} + \|Du\|^3_{L^\infty(B_4)}\big)\big]\\ &\le C(3)\exp\big[C(3)\|Du\|^3_{L^\infty(B_4)}\big].\end{aligned}$$

This completes the proof of Theorem 1.1.

Remark. A sharper Hessian estimate and a gradient estimate for the special Lagrangian equation (1.2) with $n = 2$ were derived by an elementary method in [13]. More involved arguments are needed to obtain the Hessian and gradient estimates for (1.2) with $n = 3$ and $|\Theta| > \frac{\pi}{2}$ in [14].

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