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Calculus of Variations

Improving Pogorelov's isometric embedding counterexample

Nikolai Nadirashvili · Yu Yuan

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Abstract We construct a $C^{2,1}$ metric of non-negative Gauss curvature with no C^2 local isometric embedding in $\mathbb{R}^3$.

1 Introduction

In this note, we modify Pogorelov's $C^{2,1}$ Riemannian metric in [8] with no local C^2 isometric embedding in $\mathbb{R}^3$ and sign changing Gauss curvature to a similar one with *non-negative* Gauss curvature. Though each effective piece of Pogorelov's metric upon which he drew his contradiction has no negative Gauss curvature, inevitably the Gauss curvature must be *negative* somewhere, as each effective piece of the metric is surrounded by a flat one with Euclidean metric. We add "tails" to those effective pieces and construct matching metric on the tails. Then we obtain the final metric g with $K_g \geq 0$ admitting no local C^2 isometric embedding. To be precise, we state

Theorem 1.1 *There exists a $C^{2,1}$ metric g in $B_1 \subset \mathbb{R}^2$ with Gauss curvature $K_g \geq 0$ such that there is no C^2 isometric embedding of $(B_r(0), g)$ in $\mathbb{R}^3$ for any $r > 0$.*

Remark As Jacobowitz observed [4, p.249], one can improve Pogorelov's $C^{2,1}$ metric to $C^{3,\alpha}$ with no $C^{2,\beta}$ local isometric embedding for $1 > \beta > 2\alpha$. We can also modify this $C^{3,\alpha}$

Dedicate to Professor Wei-Yue Ding on his sixtieth birthday.

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N. Nadirashvili
LATP, Centre de Mathématiques et Informatique, 39, rue F. Joliot-Curie,
13453 Marseille Cedex, France
e-mail: nicolas@cmi.univ-mrs.fr

Y. Yuan (✉)
Department of Mathematics, University of Washington,
Box 354350, Seattle, WA 98195, USA
e-mail: yuan@math.washington.edu

metric so that its Gauss curvature is non-negative and it still admits no $C^{2,\beta}$ local isometric embedding in $\mathbb{R}^3$. Instead of (2.1), we take

$$u = \int_{\frac{a}{2}}^{r} \frac{-\left(s - \frac{a}{2}\right)^{2+\alpha}}{(2+\alpha)\, s} ds \quad \frac{a}{2} \leq r < a$$

and repeat Steps 1 to 4 in Sect. 2. Assuming the existence of $C^{2,\beta}$ isometric embedding, the contradiction inequality (2.4) becomes

$$h_{22}\,(t_*, b) \geq \frac{\eta\,(k)}{100 \cdot 2^{\beta} a^{\alpha} \, \|h\|_{C^{2,\beta}}} \frac{1}{c^{\beta-2\alpha}} \to \infty \quad \text{as } c \to 0.$$

On the other hand, Lin [5] showed that there exists a C^{k-8} isometric embedding of $\left(B_{r_k}, g\right)$ in $\mathbb{R}^3$ for the C^k metric g with $k \geq 12$ and $K_g \geq 0$; see also [2, Theorem 6.01]. For any sufficiently smooth metric with Gauss curvature changing sign cleanly, a sufficiently smooth local isometric embedding in $\mathbb{R}^3$ was obtained in Lin [6]; see also a simplified proof by Han [1]. For any smooth metric with Gauss curvature changing sign cleanly, a smooth local isometric embedding in $\mathbb{R}^3$ was derived by Nakamura and Maeda [7]. For smooth metrics of non-positive Gauss curvatures with certain non-degeneracy, Han, Hong, and Lin [3] derived a local smooth isometric embedding in $\mathbb{R}^3$.

Now one further problem is the existence of local smooth isometric embedding of any smooth Riemannian metric with non-negative curvature. More importantly, still remains the basic question of the existence in $\mathbb{R}^3$ of local C^∞, or even C^2 isometric embedding of arbitrary C^∞ Riemannian metrics. For a systematic study and references of the isometric embedding, we refer to the recent book by Han and Hong [2].

2 Construction

Step 1. We first define a metric $g = e^{2u} dx^2$ in $B_a\,(0)$ on $x = (x_1, x_2)$ plane for small positive a, where

$$u = \begin{cases} 0 & 0 \leq r < \frac{a}{2} \\ \int_{\frac{a}{2}}^{r} \frac{-\left(s-\frac{a}{2}\right)^2}{2s} ds & \frac{a}{2} \leq r < a \end{cases}, \tag{2.1}$$

and $r = |x|$. Then the metric is $C^{2,1}$ inside B_a and smooth away from $\partial B_{a/2}$. The Gauss curvature $K_g = -e^{-2u} \bigtriangleup u = -e^{-2u} \frac{(r u_r)_r}{r}$ satisfies

$$K_g = \begin{cases} \left[1 + O\left(a^2\right)\right] \frac{(r-a/2)}{r} & \frac{a}{2} \leq r < a \\ 0 & 0 \leq r < \frac{a}{2} \end{cases}. \tag{2.2}$$

Step 2. We next modify the metric g or rather u outside $B_{\frac{3}{4}a}$. Let Ω_a be a domain satisfying

$$\begin{cases} B_{\frac{3}{4}a} \subset \Omega_a \\ \Omega_a \subset [-a, a] \times [-1, 2] \\ \partial\Omega_a \cap \{x_2 = 2\} = \text{ a segment with positive length} \end{cases}.$$

We construct a new function v from u such that $v = u$ in $B_{3a/4}$, $v \equiv 0$ in $\left(\mathbb{R}^2 \backslash \Omega_a\right) \cap \{x_2 < 1\}$, and

$$\begin{cases} \triangle v = 0 & \text{in } \Omega_a \backslash B_{3a/4} \\ v = u & \text{on } \partial B_{3a/4} \\ v = 0 & \text{on } S \triangleq \partial\Omega_a \cap \{x_2 < 1\} \\ v = -N(a) & \text{on } \partial\Omega_a \cap \{x_2 = 2\} \end{cases}.$$

We choose $N(a) > 0$ large enough so that

$$\begin{cases} v_r < u_r & \text{on } \partial B_{3a/4} \\ v_\gamma < 0 & \text{on } S \end{cases}, \tag{2.3}$$

where γ is the inward normal of $\partial\Omega_a$.

We now mollify v in $\mathbb{R}^2 \backslash B_{5a/8}$ by the radially symmetric mollifier $\rho \in C_0^\infty(B_1)$ with the changing parameter $\epsilon(r)$, where the smooth function $\epsilon(r)$ satisfies

$$\epsilon(r) = \begin{cases} 0 & 0 \le r \le \frac{10}{16}a \\ \delta(a) & r > \frac{11}{16}a \end{cases},$$

with $0 < \delta(a) < \frac{1}{16}a$ being a small number to be determined later. Set

$$w(x) = \int v(x - \epsilon(|x|)\, y)\, \rho(y)\, dy.$$

This w is smooth away from $\partial B_{a/2}$, $w = u$ in $B_{5a/8,}$ $w = 0$ in

$$\left(R^2 \backslash \delta(a) \text{ neighborhood of } \Omega_a\right) \cap \{x_2 \le 1\}.$$

We know that $\triangle w \le 0$ in $\Omega_a \backslash B_{3a/4}$ and near the boundary S and $\partial B_{3a/4}$, because of $\triangle v = 0$ in $\Omega_a \backslash B_{3a/4}$ and (2.3).

Notice that in $B_{3a/4} \backslash B_{5a/8}$, $v = u$ and $\triangle u$ has a negative upper bound depending on a, we choose $\delta(a)$ small, then $\|\epsilon\|_{C^2}$ small enough depending on a and v so that $\triangle w \le 0$ in $B_{3a/4} \backslash B_{5a/8}$. Consequently, we have in $\{x_2 < 1\}$

$$K_g = -e^{-2w} \triangle w \ge 0.$$

Step 3. We finally construct the desired metric in $B_{1.}$ Pick a sequence of balls $B_{a_k}\left(\frac{1}{k}, 0\right)$ centered at $\left(\frac{1}{k}, 0\right)$ with radius $a_k = \frac{1}{3k^2}$, $k = 1, 2, 3, \ldots$ In each ball B_{a_k} and slice $\Pi_k = \left[\frac{1}{k} - a_k, \frac{1}{k} + a_k\right] \times [-1, 2]$, we repeat the procedure in Step 1 and Step 2 to construct w_k. Note that $w_k \equiv 0$ in $B_1(0) \backslash \Pi_k$, we construct a metric g in $B_1(0)$ satisfying $g = dx^2$ in $B_1(0) \backslash \Pi_k$ for $k = 1, 2, 3, \ldots$, and

$$g = e^{2\eta(k) w_k} dx^2 \quad \text{in } \Pi_k \cap B_1(0),$$

where $\eta(k) > 0$ goes to 0 fast enough depending on a_k as $k \to \infty$, so that the metric g is $C^{2,1}$ in $B_1(0)$. The Gauss curvature of the metric g is

$$K_g = \eta(k)\left[1 + O\left(a_k^2\right)\right] \frac{\left(\left|x - \left(\frac{1}{k}, 0\right)\right| - \frac{a_k}{2}\right)}{\left|x - \left(\frac{1}{k}, 0\right)\right|} \quad \text{for } \frac{a_k}{2} \le \left|x - \left(\frac{1}{k}, 0\right)\right| \le \frac{5a_k}{8},$$
$$K_g \ge 0 \quad \text{in} \quad B_1.$$

Step 4. For completeness, we recap Pogorelov's argument in [8] to show that the metric near 0 in Step 3 has no local isometric embedding in $\mathbb{R}^3$ with the Euclidean metric.

Suppose there is a C^2 isometric embedding i of $(B_r(0), g)$ in $\mathbb{R}^3$ for some $r > 0$. Then there is a large k such that

$$i : \left(B_{a_k}\left(\frac{1}{k}, 0\right), e^{2\eta(k)w_k} dx^2\right) \to \mathbb{R}^3.$$

For notation simplicity, we still denote this piece by $\left(B_a, e^{2\eta(k)w} dx^2\right)$.

We choose a even smaller if necessary so that $i\left(B_{5a/8}\right)$ can be represented as a graph in $\mathbb{R}^3$. Note that the Gauss curvature of surface $i\left(B_{a/2}\right)$ is 0 and that of $i\left(B_{5a/8} \backslash B_{a/2}\right)$ is positive, then we know $i\left(B_{5a/8}\right)$ is a convex surface in $\mathbb{R}^3$. Certainly $i\left(\partial B_{a/2}\right)$ is a strictly convex curve on $i\left(B_{5a/8}\right)$, so we take a straight generator R of the flat part $i\left(B_{a/2}\right)$ such that the end points of R on $i\left(\partial B_{a/2}\right)$ and the length of R small enough to be determined later. We parametrize $i\left(B_{5a/8}\right)$ as the following C^2 convex graph in $\mathbb{R}^3$

$$t_3 = h(t_1, t_2),$$

with the interval $[-c, c]$ on t_1–axis being the generator R and t_1–t_2 plane being tangent to $i\left(B_{5a/8}\right)$.

From a simple calculation, we have

$$h_{11}(t_*, b) = \min_{-c \le t_1 \le c} h_{11}(t_1, b) \le (M - m)\frac{b^2}{c^2},$$

where

$$M = \max_{\substack{-c \le t_1 \le c \\ 0 \le t_2 \le b}} h_{22}(t_1, t_2) \quad m = \min_{\substack{-c \le t_1 \le c \\ 0 \le t_2 \le b}} h_{22}(t_1, t_2).$$

We take $b = 10c^2/a$. For small enough (yet fixed) a, the geodesic distance from $P = (t_*, b, h(t_*, b))$ to the "circle" $i\left(\partial B_{a/2}\right)$ on $i\left(B_{5a/8}\right)$ is at least c^2/a. From (2.2), the Gauss curvature

$$K(P) \ge \eta(k)\frac{c^2}{a^2}.$$

On the other hand

$$h_{22}(t_*, b) \ge \frac{K(P)}{h_{11}(t_*, b)} \ge \frac{\eta(k)}{100(M - m)}. \tag{2.4}$$

For the fixed k, let $c \to 0$, then the oscillation $(M - m) \to 0$ because $h \in C^2$. Consequently $h_{22}(t_*, b) \to \infty$. We arrive at a contradiction.

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