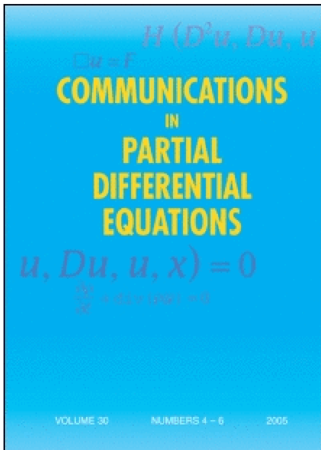


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Micah Warren [a]; Yu Yuan [a]
[a] Department of Mathematics, University of Washington, Seattle, Washington, USA

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# A Liouville Type Theorem for Special Lagrangian Equations with Constraints

MICAH WARREN AND YU YUAN

Department of Mathematics, University of Washington, Seattle, Washington, USA

*We derive a Liouville type result for special Lagrangian equations with certain "convexity" and restricted linear growth assumptions on the solutions.*

**Keywords** Liouville type theorems; Special Lagrangian equations.

**Mathematics Subject Classification** Primary 35J60; Secondary 53C38.

## 1. Introduction

In this note, we show the following

**Theorem 1.1.** *Let $u$ be a smooth solution to the special Lagrangian equation*

$$\sum_{i=1}^{n} \arctan \lambda_i = c \quad \text{on } \mathbb{R}^n, \tag{1.1}$$

*where $\lambda_i$s are the eigenvalues of the Hessian $D^2u(x)$. Suppose that*

$$3 + (1-\varepsilon)\lambda_i^2(x) + 2\lambda_i(x)\lambda_j(x) \geq 0 \tag{1.2}$$

*for all $i, j, x$ and any small fixed $\varepsilon > 0$; and the gradient $\nabla u(x)$ satisfies*

$$|\nabla u(x)| \leq \delta(n)|x| \tag{1.3}$$

*for large $|x|$ and any fixed $\delta(n) < 1/\sqrt{n-1}$. Then $u$ must be a quadratic polynomial.*

The special Lagrangian equation (1.1) arises in the calibrated geometry (Harvey and Lawson, 1982). A Lagrangian graph $M = (x, \nabla u(x)) \subset \mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$ is called

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Address correspondence to Yu Yuan, Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195, USA; E-mail: yuan@math.washington.edu

*special* when the calibrating $n$-form

$$\Omega_c = \mathrm{Re}(e^{-\sqrt{-1}c} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n)$$

is equal to the induced volume form along $M$; equivalently, $u$ satisfies (1.1). The equation (1.1) holds if and only if the gradient graph $(x, \nabla u(x)) \subset \mathbb{C}^n$ is a (volume minimizing) minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$ (Harvey and Lawson, 1982, Theorem 2.3, Proposition 2.17).

By Fu's classification result (Fu, 1998), any global solution to (1.1) on $\mathbb{R}^2$ is either quadratic or harmonic; a harmonic function with any linear growth condition on the gradient is certainly quadratic; see also Yuan (2006) for a uniqueness result for the global solutions to (1.1) with $|c| > (n-2)\frac{\pi}{2}$. In the case $n = 3$, other Liouville–Bernstein type results hold true for (1.1) under the following conditions respectively: $\lambda_i \geq -K$ (Yuan, 2002); $\lambda_i\lambda_j \geq -K$ (Yuan, unpublished); or $c = \pi$ and the solution is strictly convex with quadratic growth (Bao et al., 2003). While boundedness of the Hessian alone is sufficient in dimension three, certain boundedness and convexity are both needed for Liouville–Bernstein type results to be valid for (1.1) in the general dimension ($n \geq 4$). The results hold with the assumptions that $c = k\pi$ and the solution is convex with linear growth (Borisenko, 1992); with the almost convex assumption $\lambda_i \geq -\varepsilon(n)$ (Yuan, 2002); with the semi-convex assumption $\lambda_i \geq -\frac{1}{\sqrt{3}} + \gamma$ everywhere, or with the ("equivalent") assumption $|\lambda_i| \leq \sqrt{3} - \gamma'$ everywhere (Yuan, unpublished); or with the assumption $\lambda_i\lambda_j \geq -1 - \varepsilon(n)$ (Yuan, unpublished). (It is straightforward that any convex solution with a bounded Hessian to (1.1) is a quadratic polynomial, by the well-known $C^\alpha$ Hessian estimate of Krylov–Evans for now the convex elliptic equation (1.1); see also Xin (2003, pp. 217–218), for a different approach via the iteration argument of Hildebrandt et al. (1980/81).) A Liouville–Bernstein type result with the assumption $|\lambda_i| \leq K$ and $\lambda_i\lambda_j \geq const > -\frac{3}{2}$ was stated in Tsui and Wang (2002).

The more general "convexity" condition (1.2) does not alone lead to any Hessian bound for the solutions to (1.1), but does guarantees that the volume element $V$, which is a geometric combination of the eigenvalues, is subharmonic. Better yet, the Laplacian of $V$ bounds its gradient; see Lemma 2.1, which is a key piece in our proof of Lemma 2.2 on our Hessian estimates.

In fact, this paper grows out of our attempts towards deriving a Hessian estimate in terms of the gradient, for solutions to the special Lagrangian equation (1.1). The unpleasant technical assumption $\delta(n) < 1/\sqrt{n-1}$ in (1.3) reflects the limitation of our current arguments; the assumption is necessary for us to push the Bernstein–Pogorelov–Korevaar technique to obtain a Hessian estimate for special Lagrangian equations; see Lemma 2.2.

Once a Hessian bound for solutions to (1.1) is available, the "standard" blow-down process from the geometric measure theory will show that the global solution is a quadratic polynomial, provided certain convexity conditions like (1.2) or others are available in the *whole* process (for $n \geq 4$). (Unlike Jost and Xin, 1999, we could not generalize the iteration argument in Hildebrandt et al. (1980/81) to get a Liouville type result for now the larger image set (1.2) of the corresponding harmonic Gauss map to the Lagrangian Grassmanian.) The simple constraints $|\lambda_i| \leq K$ like $|\lambda_i| \leq 1$ or $|\lambda_i| \leq \sqrt{3} - \gamma$ are easily shown to be available in the blow-down process. An *extra* effort is needed to justify that the nonlinear constraints (1.2) or others like $\lambda_i\lambda_j \geq const$ are preserved under the $C^{1,\alpha}$ convergence of the scaling process $u_k(x) = u(kx)/k^2$. Taking advantage of the single elliptic equation (1.1), we apply the $W^{2,\delta}$ estimates for solutions in terms of the supreme norm of the solution to extract a $W^{2,\delta}$ sub-convergent sequence, as in Yuan (2001). Then we extract another subsequence with the Hessians converging almost everywhere. This justifies that the constraints (1.2) are preserved in the above blow-down process. Another route of the justification is through Allard's regularity result (cf. Simon, 1983, Section 36).

Actually, Theorem 1.1 holds true for $n = 3$ without any growth condition like (1.3). The condition (1.2) implies $\lambda_i\lambda_j \geq -K$, so as in Yuan (unpublished) we can find a bound on the Hessian (possibly for a new potential), and then draw the conclusion. Note that the boundedness on the Hessian alone for $n = 3$ is enough for one to run the blow-down process to obtain a Liouville type result; see Fischer-Colbrie (1980, Theorem 5.4). In general dimension $n \geq 4$, we derive yet another Liouville–Bernstein type result for the solutions to (1.1) with the bounded Hessian satisfying weaker constraints (3.1); see Theorem 3.1 in the Appendix. One consequence of Theorem 3.1 coupled with the De Giorgi-Allard $\varepsilon$-regularity theory is an improvement of the above mentioned Liouville–Bernstein type result in Yuan (unpublished), namely, any global solution to (1.1) with $\lambda_i \geq -\frac{1}{\sqrt{3}} - \varepsilon(n)$ everywhere or $|\lambda_i| \leq \sqrt{3} + \varepsilon'(n)$ everywhere is a quadratic polynomial (for $n \geq 4$). The argument is identical to the one in Yuan (2002) with Proposition 2.1 there replaced by Proposition 3.1 here.

The desired Hessian estimate for special Lagrangian equations in the two dimensional case follows from the gradient estimates in terms of the heights of the two dimensional minimal graphs with any codimension by Gregori (1994), where some Jacobian estimates of Heinz were employed. For higher dimensional and codimensional minimal graphs with the assumption that the product of any two slopes is between $-1$ and 1, the gradient estimates were obtained in Wang (2004), using an integral method developed for codimension one minimal graphs. The gradient estimate for codimension one minimal graphs is by now a classical result.

The general Hessian estimate for special Lagrangian equations is still a puzzling issue to us.

**Notation.** $\partial_i = \frac{\partial}{\partial x_i}$, $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$, $u_i = \partial_i u$, $u_{ji} = \partial_{ij} u$, etc.

## 2. Proof of Theorem 1.1

Taking the gradient of both sides of the special Lagrangian equation (1.1), we have

$$\sum_{i,j=1}^{n} g^{ij}\partial_{ij}(x, \nabla u(x)) = 0, \tag{2.1}$$

where $(g^{ij})$ is the inverse of the induced metric $g = (g_{ij}) = I + D^2uD^2u$ on the surface $(x, \nabla u(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$. Simple geometric manipulation of (2.1) yields the usual form of the minimal surface equation

$$\triangle_g(x, \nabla u(x)) = 0,$$

where the Laplace–Beltrami operator of the metric $g$ is given by

$$\triangle_g = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \partial_i\left(\sqrt{\det g} g^{ij} \partial_j\right).$$

Because we are using harmonic coordinates $\triangle_g x = 0$, we see that $\triangle_g$ also equals the linearized operator of the special Lagrangian equation (1.1) at $u$,

$$\triangle_g = \sum_{i,j=1}^{n} g^{ij} \partial_{ij}.$$

The gradient and inner product with respect to the metric $g$ are

$$\nabla_g v = \left(\sum_{k=1}^{n} g^{1k} v_k, \ldots, \sum_{k=1}^{n} g^{nk} v_k\right)$$

$$\langle \nabla_g v, \nabla_g w\rangle_g = \sum_{i,j=1}^{n} g^{ij} v_i w_j, \quad \text{in particular } |\nabla_g v|_g^2 = \langle \nabla_g v, \nabla_g v\rangle_g.$$

We begin by demonstrating a Jacobi inequality for the volume element

$$V = \sqrt{\det g} = \prod_{i=1}^{n} (1+\lambda_i^2)^{\frac{1}{2}}.$$

**Lemma 2.1.** *Suppose that $u$ is a smooth solution to* (1.1) *satisfying* (1.2). *Then*

$$\triangle_g \ln V \geq \frac{\varepsilon}{n} |\nabla_g \ln V|_g^2$$

*or equivalently*

$$\triangle_g V^{\frac{\varepsilon}{n}} \geq 2 \frac{|\nabla_g V^{\frac{\varepsilon}{n}}|_g^2}{V^{\frac{\varepsilon}{n}}}. \tag{2.2}$$

*Proof.* By differentiating the minimal surface equation (2.1) again and performing some long and tedious computation, one gets the standard formula for $\triangle_g \ln V$; see for example (Yuan, 2002, Lemma 2.1). (The general formula for minimal submanifolds of any dimension or codimension originates in Simons (1968, p. 90)). At any fixed point, we assume that $D^2 u$ is diagonalized, then

$$\triangle_g \ln V = \sum_{i,j,k=1}^{n} (1+\lambda_i \lambda_j) h_{ijk}^2,$$

where $h_{ijk} = \sqrt{g^{ii}}\sqrt{g^{jj}}\sqrt{g^{kk}} u_{ijk}$. Gathering all terms containing $h_{ijj}^2 = h_{jij}^2 = h_{jji}^2$ for a fixed $i$, we have

$$\begin{aligned}(1+\lambda_i^2) h_{iii}^2 &+ \sum_{j\neq i} (1+\lambda_j^2) h_{jji}^2 + \sum_{j\neq i} (1+\lambda_i\lambda_j) h_{ijj}^2 + \sum_{j\neq i} (1+\lambda_j\lambda_i) h_{jij}^2 \\ &= (1+\lambda_i^2) h_{iii}^2 + \sum_{j\neq i} (3+\lambda_j^2 + 2\lambda_i\lambda_j) h_{jji}^2.\end{aligned}$$

Thus

$$\triangle_g \ln V = \sum_{i=1}^{n}\left[(1+\lambda_i^2)h_{iii}^2 + \sum_{j\neq i}(3+\lambda_j^2+2\lambda_i\lambda_j)h_{jji}^2\right] + 2\sum_{i<j<k}(3+\lambda_i\lambda_j+\lambda_j\lambda_k+\lambda_k\lambda_i)h_{ijk}^2. \tag{2.3}$$

Condition (1.2) gives that

$$\underline{3} + (1-\varepsilon)\lambda_i^2 + \lambda_i\lambda_j + \underline{\lambda_i\lambda_j + \lambda_k(\lambda_j+\lambda_i)} - \lambda_k(\lambda_j+\lambda_i) \geq 0$$

that is

$$S_{ijk} = \underline{3+\lambda_i\lambda_j+\lambda_j\lambda_k+\lambda_k\lambda_i} \geq (\lambda_k-\lambda_i)(\lambda_i+\lambda_j) + \varepsilon\lambda_i^2.$$

Switching $\lambda_i$ and $\lambda_{j,}$ we also have

$$S_{ijk} = S_{jik} \geq (\lambda_k-\lambda_j)(\lambda_j+\lambda_i) + \varepsilon\lambda_j^2.$$

By symmetry of $S_{ijk}$, we may assume

$$\lambda_i \geq \lambda_k \geq \lambda_j, \tag{2.4}$$

then either $(\lambda_k-\lambda_i)(\lambda_i+\lambda_j)$ or $(\lambda_k-\lambda_j)(\lambda_j+\lambda_i)$ has to be non-negative, thus

$$S_{ijk} \geq \varepsilon\min\{\lambda_i^2, \lambda_j^2\}. \tag{2.5}$$

We conclude that

$$\triangle_g \ln V \geq \sum_{i=1}^{n}\left[(1+\lambda_i^2)h_{iii}^2 + \sum_{j\neq i}(3+\lambda_j^2+2\lambda_i\lambda_j)h_{jji}^2\right]. \tag{2.6}$$

To bound the gradient, we compute, (still at the same fixed point with $D^2u$ diagonalized)

$$\partial_i \ln V = \sum_{j=1}^{n} g^{jj}\lambda_j u_{jji},$$

then

$$|\nabla_g \ln V|_g^2 = \sum_{i=1}^{n} g^{ii}\left(\sum_{j=1}^{n} g^{jj}\lambda_j u_{jji}\right)^2 = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\lambda_j h_{jji}\right)^2 \leq n\sum_{i,j=1}^{n}\lambda_j^2 h_{jji}^2. \tag{2.7}$$

Combining (1.2) with (2.6) and (2.7) we have

$$\triangle_g \ln V - \frac{\varepsilon}{n}|\nabla_g \ln V|^2$$

$$\geq \sum_{i=1}^{n}[1+(1-\varepsilon)\lambda_i^2]h_{iii}^2 + \sum_{j\neq i}[3+(1-\varepsilon)\lambda_j^2+2\lambda_i\lambda_j]h_{jji}^2 \geq 0. \tag{2.8}$$

The proof of Lemma 2.1 is complete. □

**Lemma 2.2.** *Suppose that $u$ is a smooth solution to* (1.1) *on $B_1(0)$ satisfying condition* (1.2) *and*

$$|\nabla u| \leq \delta < \frac{1}{\sqrt{n-1}}.$$

*Then*

$$|D^2u(0)| \leq C(n, \delta, \varepsilon).$$

*Proof.* Set

$$v = u + \alpha\frac{1}{|\nabla u(0)|}\langle \nabla u(0), x\rangle \quad \text{or} \quad u+\alpha x_1 \quad \text{if } \nabla u(0)=0,$$

where $\alpha = (\frac{1}{\sqrt{n-1}} - \delta)/2$. Now $v$ satisfies in $B_1$ the following

$$D^2v = D^2u, \quad |\nabla v(0)| \geq \alpha, \quad \text{and} \quad |\nabla v| \leq \alpha+\delta < \frac{1}{\sqrt{n-1}}.$$

Set $b = V^{\frac{\varepsilon}{n}}$, and consider the function

$$w = \eta b = \left[|\nabla v|^2 - (\alpha+\delta)^2|x|^2\right]^+ b \geq 0.$$

A positive maximum for $w$ will be attained at a point $p$ on the interior, since $w(0) > 0$ and $w(x)$ vanishes on the boundary $\partial B_1$. At this point $p$,

$$\begin{aligned}\nabla_g(\eta b) = 0 \quad &\text{or} \quad \nabla_g\eta = -\frac{\eta}{b}\nabla_g b,\\ 0 \geq \triangle_g(\eta b) &= \eta\triangle_g b + 2\langle\nabla_g\eta, \nabla_g b\rangle_g + b\triangle_g\eta\\ &= \eta\left(\triangle_g b - 2\frac{|\nabla_g b|_g^2}{b}\right) + b\triangle_g\eta\\ &\geq b\triangle_g\eta,\end{aligned}$$

by the inequality (2.2) in Lemma 2.1. This last inequality implies a bound on $|D^2v(p)|$ as the following. We have

$$\begin{aligned}0 \geq \triangle_g\eta &= \triangle_g[|\nabla v|^2 - (\alpha+\delta)^2|x|^2]\\ &= \sum_{i,j=1}^{n} g^{ij}\left[2\sum_{k=1}^{n}(v_{ki}v_{kj} + v_k\partial_{ij}v_k) - (\alpha+\delta)^2\partial_{ij}|x|^2\right]\\ &= 2\sum_{i=1}^{n}\frac{\lambda_i^2 - (\alpha+\delta)^2}{1+\lambda_i^2} \geq 2\left[\frac{\lambda_1^2-(\alpha+\delta)^2}{1+\lambda_1^2} - (n-1)(\alpha+\delta)^2\right],\end{aligned}$$

using the minimal surface equation (2.1) and assuming $|\lambda_1| \geq |\lambda_i|$ for all $i$. It follows that

$$1 + \lambda_1^2(p) \leq \frac{1 + (\alpha + \delta)^2}{1 - (n-1)(\alpha + \delta)^2}.$$

We get

$$\alpha^2 b(0) \leq |\nabla v(0)|^2 b(0) \leq \eta(p) b(p) \leq (\alpha + \delta)^2 \left[\frac{1 + (\alpha + \delta)^2}{1 - (n-1)(\alpha + \delta)^2}\right]^{\frac{\varepsilon}{2}},$$

then

$$1 + \lambda_i^2(0) \leq \left(1 + \frac{\delta}{\alpha}\right)^{\frac{4n}{\varepsilon}} \left[\frac{1 + (\alpha + \delta)^2}{1 - (n-1)(\alpha + \delta)^2}\right]^n. \tag{2.9}$$

Therefore, we conclude the estimate $|D^2 u(0)| \leq C(n, \delta, \varepsilon)$ in Lemma 2.2. □

**Lemma 2.3.** *Let* $u \in C^\infty(\mathbb{R}^n \backslash \{0\})$ *be a solution to the special Lagrangian equation* (1.1) *and homogeneous of order* 2; *that is,* $u(x) = |x|^2 u(x/|x|)$. *Suppose that the eigenvalues* $\lambda_i$ *of the Hessian* $D^2 u(x)$ *satisfy* (1.2). *Then* $u$ *must be quadratic.*

*Proof.* Lemma 2.3 follows from Proposition 3.1; nonetheless we give a direct proof in the following. Considering (1.2), (2.4), and (2.5), we observe that the coefficients of $h_{ijk}^2$ in (2.3) are strictly positive. Accordingly,

$$\triangle_g \ln V \geq c(\lambda) \sum_{i,j,k=1}^{n} h_{ijk}^2 \tag{2.10}$$

with $c(\lambda) > 0$.

Since $u$ is homogeneous of order 2, the homogeneous order 0 function $\ln V$ attains its maximum along a ray. We infer from the strong maximum principle that $\ln V \equiv const$. It follows from (2.10) that $D^3 u \equiv 0$. Therefore, $u$ must be quadratic, as claimed in Lemma 2.3. □

*Proof of Theorem* 1.1. Now the Hessian bound is available by Lemma 2.2. We run the "routine" blow-down procedure "in detail" to finish the proof of Theorem 1.1, as in Yuan (2002).

*Step* 1. From the assumption that $|\nabla u(x)| \leq \delta |x|$ for large $x$, we have on the ball $B_R(p)$ with any fixed $p \in \mathbb{R}^n$

$$|\nabla u(x)| \leq \delta(|p| + R) = \left(\delta + \frac{\delta |p|}{R}\right) R.$$

A rescaled version of Lemma 2.2 with $R$ going to $\infty$ then leads to a Hessian bound, $|D^2 u(p)| \leq C(n, \delta, \varepsilon) \triangleq K$, which must hold at each point $p \in \mathbb{R}^n$.

*Step* 2. Repeating verbatim the argument in Yuan (2001, pp. 263–264), we show that we can find a tangent cone of the special Lagrangian graph $(x, \nabla u(x))$

at $\infty$ whose potential function is $C^{1,1}$, homogenous order 2, and still satisfies the "convexity" condition (1.2).

Without loss of generality, we assume $u(0) = 0$, $\nabla u(0) = 0$. We "blow down" $u$ at $\infty$.

Set

$$u_k(x) = \frac{u(kx)}{k^2}, \quad k = 1, 2, 3, \ldots .$$

We see that

$$\|u_k\|_{C^{1,1}(B_R)} \leq C(K, R),$$

so there exists a subsequence, still denoted by $\{u_k\}$ and a function $u_R \in C^{1,1}(B_R)$ such that $u_k \to u_R$ in $C^{1,\alpha}(B_R)$ as $k \to \infty$, and $|D^2 u_R| \leq K$. By the fact that the family of viscosity solution is closed under $C^0$ uniform limit, we know that $u_R$ is also a viscosity solution of

$$F(D^2 u) = \sum_{i=1}^{n} \arctan \lambda_i = c \quad \text{on } B_R.$$

Applying the $W^{2,\delta}$ estimate (cf. Caffarelli and Cabré, 1995, Proposition 7.4) to the difference $u_k - u_R$, we have

$$\|D^2 u_k - D^2 u_R\|_{L^\delta(B_{R/2})} \leq C(K, R)\|u_k - u_R\|_{L^\infty(B_R)} \to 0 \quad \text{as } k \to \infty.$$

Note that $|D^2 u_k|, |D^2 u_R| \leq K$, so also

$$\|D^2 u_k - D^2 u_R\|_{L^n(B_{R/2})} \to 0 \quad \text{as } k \to \infty.$$

By a standard fact from real analysis, there exists another subsequence and $C^{1,1}$ function on $B_R$, still denoted by $\{u_k\}$ and $u_{R/2}$ such that $D^2 u_k \to D^2 u_{R/2}$ almost everywhere as $k \to \infty$. So $D^2 u_R$ still satisfies (1.2) almost everywhere on $B_{R/2}$.

The diagonalizing process yields yet another subsequence, again denoted by $\{u_k\}$ and $v \in C^{1,1}(\mathbb{R}^n)$ such that $u_k \to v$ in $W^{2,n}_{loc}(\mathbb{R}^n)$ as $k \to \infty$; $v$ is a viscosity solution of (1.1) on $\mathbb{R}^n$; $|D^2 v| \leq K$; and $D^2 v$ still satisfies (1.2) almost everywhere on $\mathbb{R}^n$.

The surfaces $(x, \nabla u_k(x))$ are minimal in $\mathbb{R}^n \times \mathbb{R}^n$ and their potentials $u_k$ converge to $v$ in $W^{2,n}_{loc}(\mathbb{R}^n)$, so by the monotonicity formula (cf. Simon, 1983, Theorem 19.3, p. 84), we conclude that $M_v = (x, \nabla v(x))$ is a cone.

*Step* 3. We claim that $M_v$ is smooth away from the vertex. Suppose $M_v$ is singular at $P$ away from the vertex. We blow up $M_v$ at $P$ to get a tangent cone, which is a lower dimensional special Lagrangian cone crossing a line; repeat the procedure if the resulting cone is still singular away from the vertex. Finally we get a special Lagrangian cone which is smooth away from the vertex, and the bounded eigenvalues of the Hessian of the potential function satisfies (1.2), by a similar $W^{2,\delta}$ argument as in Step 2. By Lemma 2.3, the cone is flat. This is a contradiction to Allard's regularity result (cf. Simon, 1983, Theorem 24.2).

Applying Lemma 2.3 to $M_v$, we see that $M_v$ is flat.

*Step* 4. Now with the flatness of $M_v$, a final application of the monotonicity formula yields that the original gradient graph $(x, \nabla u(x))$ is also a plane (cf. Yuan, 2002, p. 123). Therefore, $u$ is a quadratic polynomial. □

## 3. Appendix

We include here a uniqueness result for global solutions to the special Lagrangian equation (1.1) with bounded Hessian satisfying certain "convexity" constraints (3.1). The constraints are only needed for $n \geq 4$.

**Theorem 3.1.** *Let $u$ be a smooth solution to the special Lagrangian equation* (1.1). *Suppose that the eigenvalues $\lambda_i$ of the Hessian $D^2u(x)$ are bounded $|\lambda_i(x)| \leq K$ and satisfy*

$$3 + \lambda_i^2(x) + 2\lambda_i(x)\lambda_j(x) \geq 0 \tag{3.1}$$

*for all $i$, $j$, and $x$. Then $u$ must be a quadratic polynomial.*

*Proof.* The proof is identical to the one of Theorem 1.1 with Lemma 2.3 replaced by the following proposition. □

**Proposition 3.1.** *Let $u \in C^\infty(\mathbb{R}^n \backslash \{0\})$ be a solution to the special Lagrangian equation* (1.1) *and homogeneous of order* 2, *that is $u(x) = |x|^2 u(x/|x|)$. Suppose that the eigenvalues $\lambda_i$ of the Hessian $D^2u(x)$ satisfy* (3.1) *for all $i$, $j$, and $x \neq 0$. Then $u$ must be quadratic.*

*Proof.* By (3.1), we certainly have (2.8) with $\varepsilon = 0$ in Lemma 2.1, that is

$$\begin{aligned}\triangle_g \ln V &\geq \sum_{i=1}^n (1+\lambda_i^2) h_{iii}^2 + \sum_{j\neq i} (3 + \lambda_j^2 + 2\lambda_i\lambda_j) h_{jji}^2 \\ &= \sum_{i=1}^n \frac{1}{(1+\lambda_i^2)^2} u_{iii}^2 + \sum_{j \neq i} \frac{(3+\lambda_j^2 + 2\lambda_i\lambda_j)}{(1+\lambda_j^2)^2(1+\lambda_i^2)} u_{jji}^2 \geq 0.\end{aligned} \tag{3.2}$$

Since $u$ is homogeneous of order 2, the Hessian $D^2u(x)$ is homogeneous of order 0, hence $\ln V$ must attain its maximum along a ray. The strong maximum principle yields that $\ln V$ is constant, so in fact

$$0 = \triangle_g \ln V. \tag{3.3}$$

We claim now that

$$\triangle u = const \tag{3.4}$$

on $\mathbb{R}^n \backslash \{0\}$. At any point $p$ compute the derivative

$$\partial_i(\triangle u) = \sum_{j=1}^n u_{jji} \tag{3.5}$$

for all $i$. Still assuming that $D^2u$ is diagonalized at $p$, an inspection of (3.2), together with (3.3) shows that for all $j$ with $u_{jji} \neq 0$,

$$3 + \lambda_j^2 + 2\lambda_i\lambda_j = 0. \tag{3.6}$$

From $3 + \lambda_i^2 + 2\lambda_i\lambda_j \geq 0$, we see that $\lambda_i^2 \geq \lambda_j^2$. Solving (3.6) for $\lambda_j$ we get

$$\lambda_j = -\lambda_i - \sqrt{\lambda_i^2 - 3}, \quad \text{if } \lambda_i < 0,$$
$$\lambda_j = -\lambda_i + \sqrt{\lambda_i^2 - 3}, \quad \text{if } \lambda_i > 0.$$

The minimal surface equation (2.1) at $p$ then reads

$$0 = \triangle_g u_i \stackrel{p}{=} \sum_{j=1} \frac{1}{1 + \lambda_j^2} u_{jji} = \frac{1}{1 + (-\lambda_i \pm \sqrt{\lambda_i^2 - 3})^2} \sum_j u_{jji}.$$

Hence $\partial_i(\triangle u) = 0$ and $\triangle u$ is constant.

Differentiating (3.4), we see that each $u_{ij}$ satisfies

$$\triangle u_{ij} = 0.$$

Applying the strong maximum principle once again to each (homogeneous order 0) function $u_{ij}$, we have immediately

$$u_{ij} = const;$$

that is, $u$ is quadratic. □

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## References


Bao, J., Chen, J., Guan, B., Ji, M. (2003). Liouville property and regularity of a Hessian quotient equation. *Amer. J. Math.* 125:301–316.

Borisenko, A. A. (1992). On a Liouville-type theorem for the equation of special Lagrangian submanifolds. (Russian) *Mat. Zametki* 52:22–25; English translation in Math. Notes 52:1094–1096 (1993).

Caffarelli, L. A., Cabré, X. (1995). *Fully Nonlinear Elliptic Equations.* American Mathematical Society Colloquium Publications, 43. Providence, RI: American Mathematical Society.

Fischer-Colbrie, D. (1980). Some rigidity theorems for minimal submanifolds of the sphere. *Acta Math.* 145(1–2):29–46.

Fu, L. (1998). An analogue of Bernstein's theorem. *Houston J. Math.* 24:415–419.

Gregori, G. (1994). Compactness and gradient bounds for solutions of the mean curvature system in two independent variables. *J. Geom. Anal.* 4(3):327–360.

Harvey, R., Lawson, H. B., Jr. (1982). Calibrated geometry. *Acta Math.* 148:47–157.

Hildebrandt, S., Jost, J., Widman, K. (1980/81). Harmonic mappings and minimal submanifolds. *Invent. Math.* 62(2):269–298.

Jost, J., Xin, Y. (1999). Berstein type theorems for higher codimension. *Calc. Var. Partial Differential Equations* 9(4):277–296.

Simon, L. (1983). *Lectures on Geometric Measure Theory*. Proceedings of the Centre for Mathematical Analysis, Australian National University, 3. Canberra: Australian National University, Centre for Mathematical Analysis.

Simons, J. (1968). Minimal varieties in Riemannian manifolds. *Ann. of Math.* 88(2):62–105.

Tsui, M.-P., Wang, M.-T. (2002). A Bernstein type result for special Lagrangian submanifolds. *Math. Res. Lett.* 9:529–535.

Wang, M.-T. (2004). Interior gradient bounds for solutions to the minimal surface system. *Amer. J. Math.* 126(4):921–934.

Xin, Y. (2003). *Minimal Submanifolds and Related Topics*. Nankai Tracts in Mathematics 8. River Edge, NJ: World Scientific Publishing Co., Inc.

Yuan, Y. (2001). A priori estimates for solutions of fully nonlinear special Lagrangian equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 18:261–270.

Yuan, Y. (2002). A Bernstein problem for special Lagrangian equations. *Invent. Math.* 150:117–125.

Yuan, Y. (2006). Global solutions to special Lagrangian equations. *Proc. Amer. Math. Soc.* 134(5):1355–1358.

Yuan, Y. (unpublished). *Notes on Special Lagrangian Equations.*