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RESOLVING THE SINGULARITIES OF THE MINIMAL HOPF CONES*

Ding Weiyue
(Institute of Mathematics and LMAM Beijing University Beijing, 100082
and Institute of Mathematics, AMSS, Beijing, 100080, P. R. China)
(E-mail: dingwy@math.pku.edu.cn)
Yuan Yu
(Department of Mathematics, Box 354350 University of Washington
Seattle, WA 98195, USA)
(E-mail: yuan@math.washington.edu)

Dedicate to Professor K. C. Chang on his seventieth birthday

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Abstract We resolve the singularities of the minimal Hopf cones by families of regular minimal graphs.

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1. Introduction

In this paper, we resolve the singularities of the minimal Hopf cones found in Lawson and Osserman [1]. The Lipschitz yet non C^1 minimal graph cone in $\mathbb{R}^{2m} \times \mathbb{R}^{m+1}$ is

$$C_m = \left\{ \left(x, S_m \frac{H(x)}{r} \right) : x \in \mathbb{R}^{2m} \right\},$$

where $m = 2, 4, 8$, $S_m = \sqrt{\frac{2m+1}{4(m-1)}}$, $r = |x|$, and the Hopf map $H : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{m+1}$ is defined as follows. One identifies $\mathbb{R}^m$ with the normed algebra, complex numbers $\mathbb{C}$ ($m = 2$), quaternions $\mathbb{H}$ ($m = 4$), and octonions $\mathbb{O}$ ($m = 8$). Let $x = (u, v) \in \mathbb{R}^m \times \mathbb{R}^m$, then

$$H(x) = \left(|u|^2 - |v|^2, 2v\bar{u} \right).$$

For each of the minimal Hopf cones, we prove there exist a family of regular minimal graphs in $\mathbb{R}^{2m} \times \mathbb{R}^{m+1}$ whose tangent cone at ∞ are the minimal Hopf cone C_m. To be precise, we have

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Theorem 1.1 *There exist a family of analytic minimal graphs*

$$G_\mu = \left\{ \left(x, \mu^{-1} f(\mu r) \frac{H(x)}{r^2} \right) : \quad x \in \mathbb{R}^{2m} \right\}$$

for $m = 2,\ 4,\ 8,$ *where* $\mu > 0$ *and* f *satisfies*

$$\begin{aligned} &0 \le f(r) < S_m r, \\ &0 \le f_r(r); \end{aligned}$$

and for small r *near* 0

$$\begin{aligned} f(r) &= O\left(r^2\right), \\ f_r(r) &= O(r); \end{aligned}$$

while for large r

$$\begin{aligned} f(r) &= S_m r + O\left(\frac{1}{r^\delta}\right), \\ f_r(r) &= S_m + O\left(\frac{1}{r^{1+\delta}}\right) \end{aligned}$$

with $\delta = m - \sqrt{m^2 - 2m + \frac{1}{2m}} - 1 > 0.$

Further we have another family of minimal graphs which are "above" each of the minimal Hopf cones in the sense that $f(r) > S_m r$. Their tangent cones at ∞ are still the minimal Hopf cone C_m. This family of minimal graphs are only regular away from $0 \times \mathbb{R}^{m+1}$, but have finite area near the singular points.

Theorem 1.1. Theorem 1.2 *There exist a family of analytic minimal graphs*

$$G_\mu = \left\{ \left(x, \mu^{-1} f(\mu r) \frac{H(x)}{r^2} \right) : \quad x \in \mathbb{R}^{2m} \backslash \{0\}, \right\}$$

for $m = 2,\ 4,\ 8,$ *where* $\mu > 0$ *and* f *satisfies*

$$\begin{aligned} f(r) &> S_m r, \\ f_r(r) &\ge 0; \end{aligned}$$

for small r *near* 0

$$\begin{aligned} f(r) &= O(1), \\ f_r(r) &= O(r); \end{aligned}$$

for large r

$$\begin{aligned} f(r) &= S_m r + O\left(\frac{1}{r^\delta}\right), \\ f_r(r) &= S_m + O\left(\frac{1}{r^{1+\delta}}\right). \end{aligned}$$

Moreover, in the case $m = 2$*, one can take* $\delta = m + \sqrt{m^2 - 2m + \frac{1}{2m}} - 1 = \frac{3}{2}.$

Remark In fact, f satisfies $f\left(4f^2-5r^2\right)^2=1$ in the case $m=2$, see Harvey and Lawson [2, p.137].

Theorem 1.1 says there exist a family of global non-trivial minimal graphs of high co-dimension with bounded slope. By contrast, any global minimal graph of co-dimension one with bounded slope must be a hyperplane by the work of De Giorgi-Nash (cf. Moser [3]).

In the co-dimension one setting, the resolution of the singularities of minimal (Simons) cone by regular minimal graphs was first treated by Bombieri, De Giorgi and Giusti [4], as a key step to solve completely the famous Bernstein problem. See also Hardt and Simon [5] , McIntosh [6] , and Chan [7] for the resolution of the singularities of other minimal cones.

The proof of our theorems relies on a symmetry reduction which reduces the problem to solving a nonlinear ordinary differential equation of second order. By suitable transforms of both independent and dependent variables it turns out that the ode is equivalent to an autonomous one. Then by a phase plane analysis we are able to find the orbit which corresponds to the function f in our theorems.

We end the introduction by mentioning the area minimizing issue of the minimal Hopf cones. In the case $m=2$, Harvey and Lawson proved that C_2 is area minimizing by their calibration device. In fact, C_2 and the graph G_μ in Theorem 1.2 with $m=2$ are Sp_1 invariant fourfolds in $\mathbb{R}^7$ which are coassociative, hence area minimizing, see [2, Theorem 3.2]. It is tempting to guess that the other minimal Hopf cones and the minimal graphs G_μ in Theorem 1.1 and 1.2 are also area minimizing.

2. Proof of Theorem 1.1

By the equi-variance of the Hopf map, for any fixed $y\in\mathbb{R}^{2m}$, there exists an orthogonal transformation $T\in O\left(2m\right)$ and an induced orthogonal transformation $\mathcal{T}\in O\left(m+1\right)$ such that

$$y=\left(\left|y\right|,0,\cdots,0\right)T,$$

and

$$H\left(xT\right)=H\left(x\right)\mathcal{T}\quad\text{for all } x\in\mathbb{R}^{2m}$$

see the proposition in the appendix.

It follows that the minimal surface system

$$\triangle_g F=0,$$

where $F\left(x\right)=\left(x,f\left(r\right)\frac{H(x)}{r^2}\right)$, or equivalently (see Osserman [8, Theorem 2.2])

$$\sum_{i,j=1}^{2m}g^{ij}\frac{\partial^2}{\partial x_i\partial x_j}F=0,$$

holds if and only if it holds at $\left(r,0,\cdots,0\right)\in\mathbb{R}^{2m}$, where the induced metric $g=\nabla F\left(\nabla F\right)'$ and $\left(g^{ij}\right)=g^{-1}$.

Now at $(r, 0, \cdots, 0)$, the induced metric g takes the form

$$\begin{cases} g_{11} = 1 + f_r^2 \\ g_{ii} = 1 \quad 2 \le i \le m \\ g_{ii} = 1 + \frac{4f^2}{r^2} \quad m+1 \le i \le 2m \\ g_{ij} = 0 \quad i \ne j \end{cases},$$

and the double derivatives of F satisfy

$$\begin{cases} F_{11}^{2m+1} = f_{rr} \\ F_{ii}^{2m+1} = \frac{f_r}{r} \quad 2 \le i \le m \\ F_{ii}^{2m+1} = \frac{f_r}{r} - \frac{4f}{r^2} \quad m+1 \le i \le 2m \\ F_{ii}^{k} = 0 \quad k \ne 2m+1 \end{cases}.$$

Thus the minimal surface system reduces to

$$\frac{f_{rr}}{1+f_r^2} + (m-1)\frac{f_r}{r} + m\frac{1}{1+\frac{4f^2}{r^2}}\left(\frac{f_r}{r} - \frac{4f}{r^2}\right) = 0$$

or

$$\frac{f_{rr}}{1+f_r^2} + \left[m - 1 + \frac{mr^2}{r^2+4f^2}\right]\frac{f_r}{r} - \frac{4mf}{r^2+4f^2} = 0, \tag{2.1}$$

where $m = 2$ (complex), 4 (quaternion), or 8 (octonion).

Set $\varphi = f/r$, $t = \ln r$, then $f_r = r\varphi_r + \varphi = \varphi_t + \varphi$, and $f_{rr} = (\varphi_{tt} + \varphi_t)/r$. The ode (2.1) becomes

$$\frac{\varphi_{tt} + \varphi_t}{1 + (\varphi_t + \varphi)^2} + \left(m - 1 + \frac{m}{1+4\varphi^2}\right)\varphi_t + \left(m - 1 - \frac{3m}{1+4\varphi^2}\right)\varphi = 0.$$

Now we have the system

$$\begin{cases} \varphi_t = \psi \\ \psi_t = -\psi - \left[\left(m - 1 + \frac{m}{1+4\varphi^2}\right)\psi + \left(m - 1 - \frac{3m}{1+4\varphi^2}\right)\varphi\right]\left[1 + (\psi+\varphi)^2\right] \end{cases}$$

or

$$\begin{cases} \varphi_t = \psi \\ \psi_t = -\psi - \left[\left(m - 1 + \frac{m}{1+4\varphi^2}\right)\psi + 4(m-1)\frac{\varphi^2 - \frac{2m+1}{4(m-1)}}{1+4\varphi^2}\varphi\right]\left[1 + (\psi+\varphi)^2\right]. \end{cases} \tag{2.2}$$

At the saddle critical point $(0,0)$, the linearized system is

$$\begin{bmatrix} 0 & 1 \\ 2m+1 & -2m \end{bmatrix}$$

and $\lambda_1 = 1$, $V_1 = (1,1)$, $\lambda_2 = -(2m+1)$, $V_2 = (1, -(2m+1))$

At the stable critical point $(S_m, 0)$, the linearized system is

$$\begin{bmatrix} 0 & 1 \\ -2m + \frac{1}{2m} & -2m \end{bmatrix},$$

the two eigenvalues are both negative, $\lambda_{1,2} = -m \pm \sqrt{m^2 - 2m + \frac{1}{2m}} < -1$.

(2.2) is an autonomous system. We simply write it as

$$z_t = X(z),$$

where $z = (\varphi, \psi)$ and $X = (X_1, X_2)$. Let Δ be the (closed) triangle with the three vertices $P_0(0,0)$, $P_1(S_m, 0)$ and $P_2(0, mS_m)$. Then one can check that Δ is a positive invariant set of the system by verifying that the vector field X points inward at the three boundary segments except at the two singular points P_0 and P_1, where X vanishes. In other words, we need to show

(1) $X_2(\varphi, 0) > 0, \quad$ for $\quad \varphi \in (0, S_m)$;
(2) $X_1(0, \psi) > 0, \quad$ for $\quad \psi \in (0, mS_m]$;
(3) $X_2(\varphi, m(S_m - \varphi))/X_1(\varphi, m(S_m - \varphi)) < -m, \quad$ for $\quad \varphi \in [0, S_m)$.

(1) and (2) are obvious, (3) needs a little calculation.

Indeed along segment P_1P_2

$$\begin{aligned}
\frac{X_2}{X_1} &= -1 - \left[\left(m - 1 + \frac{m}{1+4\varphi^2}\right) + 4\left(m-1\right)\frac{\varphi + S_m}{1+4\varphi^2}\frac{\varphi}{-m}\right]\left[1 + \left(\psi + \varphi\right)^2\right] \\
&= -1 - \left[\left(m - 1 + \frac{m}{1+4\varphi^2}\right) - \left(\frac{m-1}{m}\right)\left(1 + \frac{4S_m\varphi - 1}{1+4\varphi^2}\right)\right]\left[1 + \left(\psi + \varphi\right)^2\right] \\
&= -1 - \left[m + \frac{1}{m} - 2 + \left(\frac{m-1}{m}\right)4S_m\frac{\frac{m^2+m-1}{(m-1)4S_m} - \varphi}{1+4\varphi^2}\right]\left[1 + \left(\psi + \varphi\right)^2\right].
\end{aligned}$$

Notice that the function $h = \frac{\frac{m^2+m-1}{(m-1)4S_m} - \varphi}{1+4\varphi^2} = \frac{c-\varphi}{1+4\varphi^2}$ has two critical points $c \pm \sqrt{c^2 + \frac{1}{4}}$, h is decreasing for $0 < \varphi < c + \sqrt{c^2 + \frac{1}{4}}$, and increasing for $\varphi > c + \sqrt{c^2 + \frac{1}{4}}$. Furthermore, $h'\left(S_m\right) < 0$, we see that h is decreasing on $\left[0, S_m\right]$. Then

$$\begin{aligned}
&\left[m + \frac{1}{m} - 2 + \left(\frac{m-1}{m}\right)4S_m\frac{\frac{m^2+m-1}{(m-1)4S_m} - \varphi}{1+4\varphi^2}\right] \\
&\geq \left(m + \frac{2}{m} - 3 + \frac{(m-1)\left(m^2 + 2m - 2\right)}{3m^2}\right) > 0.
\end{aligned}$$

Also $\psi + \varphi \geq S_m$ on the segment P_1P_2. Thus one checks (3) directly.

Since Δ is a positive invariant set, any orbit starting at a point in Δ will stay in Δ for all $t > 0$. Now, P_0 is a saddle point, and from the linearized system at P_0 one sees that exactly one orbit Γ in Δ goes to P_0 as $t \to -\infty$, see Hartman [9, p.217]. As $t \to \infty$, the ω-limit set of Γ can only be a limit cycle or a singular point inside the triangle Δ. Since there are no singular points in the interior of Δ, there cannot be limit cycles in Δ (cf. [9, p.150-151]). Also, the other separatrix to P_0 are outside the triangle Δ, so the orbit Γ cannot go back to P_0 as $t \to \infty$. It follows that Γ has to go to the singular point P_1, and we see that Γ is the *unique* orbit in Δ connecting P_0 and P_1.

Now from $\Gamma \subset \Delta$, it follows immediately that $0 \le f(r) = r\varphi < S_m r$, and $f_r(r) = \varphi + r\varphi_r = \varphi + \psi \ge 0$.

Next we know that $f(r)$ is smooth for $r > 0$. By Theorem 3.5 in [9]

$$\begin{aligned}\varphi(t) &= O\left(e^t\right)\\ \psi(t) &= O\left(e^t\right)\end{aligned}$$

as $t \to -\infty$, then as $r \to 0$,

$$\begin{aligned}f(r) &= r\varphi = O\left(r^2\right)\\ f_r(r) &= \varphi + \psi = O(r).\end{aligned}$$

It follows that $F(x)$ is C^1 near 0. Finally the minimal surface is analytic, see Morrey [10, Theorem 6.8.1].

Asymptotic behavior of f for large r follows from that the two eigenvalues of the linearized system at the attractor $(S_m, 0)$ are both less than -1. By Theorem 3.5 in chapter VIII of [9],

$$\begin{aligned}\varphi(t) &= S_m + O\left(e^{\ \sqrt{m^2-2m+\frac{1}{2m}}-m\ \ t}\right)\\ \psi(t) &= O\left(e^{\ \sqrt{m^2-2m+\frac{1}{2m}}-m\ \ t}\right)\end{aligned}$$

as $t \to +\infty$. Then for large r

$$\begin{aligned}f(r) = r\varphi &= r\left(S_m + O\left(r^{\sqrt{m^2-2m+\frac{1}{2m}}-m}\right)\right)\\ &= S_m r + O\left(\frac{1}{r^\delta}\right)\end{aligned}$$

and

$$f_r(r) = \varphi + \psi = S_m + O\left(\frac{1}{r^{1+\delta}}\right).$$

Once $\varphi(t)$ is a solution, $\varphi(t+c)$ is also a solution for any constant c. Correspondingly, we have a family of solutions $\frac{f(\mu r)}{\mu}$ with $\mu = e^c > 0$.

Remark In the complex case $m = 2$, one can rewrite the ode (2.1) in a different form.

$$\frac{\varphi_{tt}+\varphi_t}{1+(\varphi+\varphi_t)^2}+\frac{4\varphi_t}{1+4\varphi^2}+\frac{4\varphi^2-1}{4\varphi^2+1}\left(\varphi_t+\varphi-\frac{4\varphi}{4\varphi^2-1}\right)=0,$$

$$\frac{d}{dt}\left[\arctan(\varphi_t+\varphi)+2\arctan 2\varphi\right]+\frac{4\varphi^2-1}{4\varphi^2+1}\left(\varphi_t+\varphi-\frac{4\varphi}{4\varphi^2-1}\right)=0,$$

$$\frac{d}{dt}\left[\arctan(\varphi_t+\varphi)-\arctan\frac{4\varphi}{4\varphi^2-1}\right]+\frac{4\varphi^2-1}{4\varphi^2+1}\left(\varphi_t+\varphi-\frac{4\varphi}{4\varphi^2-1}\right)=0,$$

$$\frac{d}{dt}\left[\arctan\frac{\varphi_t+\varphi-\frac{4\varphi}{4\varphi^2-1}}{1+(\varphi_t+\varphi)\frac{4\varphi}{4\varphi^2-1}}\right]+\frac{4\varphi^2-1}{4\varphi^2+1}\left(\varphi_t+\varphi-\frac{4\varphi}{4\varphi^2-1}\right)=0.$$

Then solutions to the first order ode

$$\varphi_t+\varphi-\frac{4\varphi}{4\varphi^2-1}=0 \tag{2.3}$$

are also solutions to (2.1). In terms of f, the above equation reads

$$\frac{d}{dr}\left[\arctan\frac{f_r-\frac{4rf}{4f^2-r^2}}{1+\frac{4rff_r}{4f^2-r^2}}\right]+\frac{4f^2-r^2}{4f^2+r^2}\left(f_r-\frac{4rf}{4f^2-r^2}\right)=0,$$

the first order ode for f is $f_r-\frac{4rf}{4f^2-r^2}=0$, which was first found in [2, Lemma 3.7].

3. Proof of Theorem 1.2

We first handle the case $m = 4$ and 8. With the set up in the proof of Theorem 1.1, we start from (2.2). We look for solutions $f(r)$ to (2.1) satisfying in particular $f(r) > S_m r$ and $f_r(r) = O(r)$ for small r. Note that $\varphi = f/r$ and $\varphi + \psi = f_r$, we are led to the following transformation

$$\begin{cases}\eta=\frac{1}{\varphi}\\ \xi=\varphi+\psi\end{cases}.$$

With (2.2) and

$$\begin{cases}\varphi=\frac{1}{\eta}\\ \psi=\xi-\frac{1}{\eta}\end{cases}$$

we get the system for (η,ξ)

$$\begin{cases}\eta_t=\eta(1-\eta\xi)\\ \xi=\{4m\eta-[4(m-1)+(2m-1)\xi]\}\frac{1+\xi^2}{4+\eta^2}\end{cases}. \tag{3.1}$$

We analyze (3.1) in a similar way as for (2.2).

At the saddle critical point $(0,0)$, the linearized system is

$$\begin{bmatrix} 1 & 0 \\ m & 1-m \end{bmatrix}$$

and $\lambda_1 = 1$, $V_1 = (1,1)$, $\lambda_2 = 1-m$, $V_2 = (0,1)$.

At the stable critical point $(1/S_m, S_m)$, the linearized system is

$$\begin{bmatrix} -1 & -\frac{1}{S_m^2} \\ \frac{2m-1}{2m} S_m^2 & 1-2m \end{bmatrix}$$

and

$$\lambda_1 = -m + \sqrt{m^2 - 2m + \frac{1}{2m}} < -1,\ V_1 = \left(1, (-1-\lambda_1) S_m^2\right),$$
$$\lambda_2 = -m - \sqrt{m^2 - 2m + \frac{1}{2m}} < -1,\ V_2 = \left(1, (-1-\lambda_2) S_m^2\right).$$

(3.1) is again an autonomous system. We write it as

$$z_t = X(z),$$

where $z = (\eta, \xi)$ and $X = (X_1, X_2)$. Let Θ be the "trapped" region bounded by

$$\begin{aligned} \xi &= k\eta \quad \text{for large } k, \\ \xi &= S_m^2 \eta, \\ \eta\xi &= 1, \end{aligned}$$

we see that Θ is a positive invariant set of the system by verifying that the vector field X points inward along the boundary of Θ except the two singular points $(0,0)$ and $(1/S_m, S_m)$. Indeed along $\{\xi = k\eta\} \cap \Theta$ we have

$$\frac{X_2}{X_1} = \frac{4m - \left[4(m-1) + (2m+1)\eta^2\right] k}{1 - \eta\xi} \left(\frac{1 + k\eta^2}{4 + \eta^2}\right) < 0 < k,$$

for large k say $k > m/(m-1)$.

Along $\{\eta\xi = 1\} \cap \Theta$ we have

$$\begin{aligned} X_1 &= 0, \\ X_2 &= \left\{4m\eta^2 - \left[4(m-1) + (2m+1)\eta^2\right]\xi\eta\right\} \frac{1}{\eta}\frac{1+\xi^2}{4+\eta^2} \\ &= \left[4m\eta^2 - 4(m-1) - (2m+1)\eta^2\right] \frac{1}{\eta}\frac{1+\xi^2}{4+\eta^2} \\ &= (2m+1)\left[\eta^2 - \frac{1}{S_m^2}\right] \frac{1}{\eta}\frac{1+\xi^2}{4+\eta^2} \\ &< 0. \end{aligned}$$

Along $\{\xi = S_m^2 \eta\} \cap \Theta$ we have

$$\begin{aligned}
\frac{X_2}{X_1} &= \frac{4m - \left[4(m-1) + (2m+1)\eta^2\right] S_m^2}{1 - S_m^2 \eta^2} \frac{1 + S_m^2 \eta^2}{4 + \eta^2} \\
&= (2m-1) \frac{1 + S_m^2 \eta^2}{4 + \eta^2} \\
&> (2m-1) \frac{1}{4} > S_m^2 \quad \text{for } m = 4 \text{ and } 8.
\end{aligned}$$

The following argument is similar to the one in the proof of Theorem 1.1. Since Θ is a positive invariant set, any orbit starting at a point in Θ will stay in Θ for all $t > 0$. Now $(0,0)$ is a saddle point, and from the linearized system at $(0,0)$ one sees that exactly one orbit Γ in Θ goes to $(0,0)$ as $t \to -\infty$, see Hartman [9, p.217] . As $t \to \infty$, the ω-limit set of Γ can only be a limit cycle or a singular point inside Θ. Since there are no singular points in the interior of Θ, there cannot be limit cycles in Θ (cf. [9, p.150-151]). Also, since the other separatrix to $(0,0)$ are outside Θ, the orbit Γ cannot go back to $(0,0)$ as $t \to \infty$. It follows that Γ must go to the singular point $(1/S_m, S_m)$ as $t \to \infty$. Therefore, we see that Γ is the *unique* orbit in Θ connecting $(0,0)$ and $(1/S_m, S_m)$.

Now from $\Gamma \subset \Theta$, it follows immediately that $\frac{f(r)}{r} = \varphi = \frac{1}{\eta} > S_m$, and $f_r(r) = \varphi + \psi = \xi \geq 0$.

By Theorem 3.5 in chapter VIII of [9]

$$\begin{aligned}
\eta(t) &= O\left(e^t\right) \\
\xi(t) &= O\left(e^t\right)
\end{aligned}$$

as $t \to -\infty$, then as $r \to 0$,

$$\begin{aligned}
f(r) &= r\varphi = r\frac{1}{\eta} = O(1) \\
f_r(r) &= \varphi + \psi = \xi = O(r).
\end{aligned}$$

Also we know $f(r)$ is smooth for $r > 0$, the minimal surface F is analytic away from $0 \times \mathbb{R}^{m+1}$, see Morrey [10, Theorem 6.8.1].

By Theorem 3.5 in [9] again,

$$\begin{aligned}
\eta(t) &= \frac{1}{S_m} + O\left(e^{\left(\sqrt{m^2 - 2m + \frac{1}{2m}} - m\right) t}\right) \\
\xi(t) &= S_m + O\left(e^{\left(\sqrt{m^2 - 2m + \frac{1}{2m}} - m\right) t}\right)
\end{aligned}$$

as $t \to +\infty$. Then for large r

$$\begin{aligned}
f(r) &= r\varphi = r\frac{1}{\eta} = r\left(\frac{1}{S_m} + O\left(r^{\sqrt{m^2 - 2m + \frac{1}{2m}} - m}\right)\right) \\
&= \frac{r}{S_m} + O\left(\frac{1}{r^\delta}\right)
\end{aligned}$$

and

$$f_r(r) = \varphi + \psi = \xi = S_m + O\left(\frac{1}{r^{1+\delta}}\right).$$

Once $\varphi(t)$ is a solution, $\varphi(t+c)$ is also a solution for any constant c. Correspondingly, we have a family of solutions $\frac{f(\mu r)}{\mu}$ with $\mu = e^c > 0$.

Finally we deal with the case $m = 2$ separately, since the "trapped" region is not easy to find. However from the remark in section 2, we know $\varphi = f/r$ satisfies (2.3) which can be solved explicitly. In fact it was already done in [2, Lemma 3.7]. We include it here for completeness. Integrate (2.3) and use the original variable r, we have

$$r = \frac{1}{\mu}\left[\frac{1}{\varphi\left(4\varphi^2 - 5\right)^2}\right]^{1/5},$$

with constant $\mu > 0$. From this, one easily see the conclusion in Theorem 1.2. And the proof for Theorem 1.2 is complete.

Remark In the case $m = 2$, observe that the corresponding orbit (η, ξ) is still the *unique* one connecting $(0,0)$ and $(1/S_m, S_m)$ because $(0,0)$ is a saddle point, see [9, p.217].

4. Appendix: Cayley-Dickson Process for $\mathbb{B} = \mathbb{O} \oplus \mathbb{O}$

In this appendix, we present some simple algebra needed to justify the equi-variance of the Hopf maps in the complex, quaternionic, and octonionic cases. We mainly deal with the octonionic case, the former two cases follows easily along the same lines, because the alternativity and Moufang identity are already available for the normed quaternions and octonions. The complex case is straight forward, in fact it could be done by other means (see Harvey-Lawson [2, p.135]).

Given a normed algebra $\mathbb{A}$, one obtains a new algebra $\mathbb{B} = \mathbb{A} \oplus \mathbb{A}$ via the Cayley-Dickson process, where the product is defined by

$$(a, b)(c, d) = \left(ac - \bar{d}b, da + b\bar{c}\right),$$

and the conjugate of (a, b) is defined by

$$\overline{(a, b)} = (\bar{a}, -b).$$

This way, we have the complex numbers, quaternions (Hamilton numbers), and octonions (Cayley numbers)

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}, \quad \mathbb{H} = \mathbb{C} \oplus \mathbb{C}, \quad \mathbb{O} = \mathbb{H} \oplus \mathbb{H}.$$

However the next algebra $\mathbb{B} = \mathbb{O} \oplus \mathbb{O}$ is not normed any more by Hurwitz theorem (cf. [2, Theorem A.12]), that is, $|(a, b)(c, d)| = |(a, b)|\,|(c, d)|$ does not hold in general for all $a, b, c, d \in \mathbb{O}$. Yet $\left|(a, b)\,\overline{(a, b)}\right| = |(a, b)|^2$ is still valid. We start our presentation here.

1. Cayley-Dickson

For $x=(u,v)\in\mathbb{O}\oplus\mathbb{O}$, the Hopf map

$$H(x)=(\bar{u},v)(u,v)=\left(|u|^2-|v|^2,2v\bar{u}\right).$$

Set $(U,V)=(u,v)(\beta,b)$, then

$$\begin{aligned}(U,V)&=\left(u\beta-\bar{b}v,bu+v\bar{\beta}\right)=\left(u\beta-\bar{b}v,bu+v\beta\right),\\ \left(\bar{U},V\right)&=\left(\bar{\beta}\bar{u}-\bar{v}b,bu+v\bar{\beta}\right)=\left(\bar{\beta},b\right)(\bar{u},v)=(\beta,b)(\bar{u},v)\end{aligned}$$

for real β.

2. Alternativity

$$[(1,a)(c,d)](1,a)=(1,a)[(c,d)(1,a)].$$

In fact

$$\begin{aligned}LHS&=\left(c-\bar{d}a,d+a\bar{c}\right)(1,a)=\left(c-\bar{d}a-\bar{a}d-|a|^2\bar{c},ac-a\bar{d}a+d+a\bar{c}\right),\\ RHS&=(1,a)(c-\bar{a}d,ac+d)=\left(c-\bar{a}d-\bar{c}|a|^2-\bar{d}a,ac+d+a\bar{c}-a\bar{d}a\right),\end{aligned}$$

so $LHS=RHS$, where we use the alternativity of octonions, $\bar{a}(a\bar{c})=(\bar{a}a)c$, $(\bar{c}\bar{a})a=\bar{c}(\bar{a}a)$, and $a\left(\bar{d}a\right)=\left(a\bar{d}\right)a$, or Artin Theorem, which says the subalgebra with unit generated by any two octonions is associative (cf. [2, Theorem A.13]).

3. Moufang Identity

$$\begin{aligned}[(1,a)(\bar{c},d)][(c,d)(1,a)]&=(1,a)[(\bar{c},d)(c,d)](1,a)\\ &=(1,a)\left(|c|^2-|d|^2,2d\bar{c}\right)(1,a).\end{aligned}\tag{4.1}$$

By 2. the right hand sides make sense. We compute

$$\begin{aligned}&[(1,a)(\bar{c},d)][(c,d)(1,a)]\\ &\quad=\left(\bar{c}-\bar{d}a,d+ac\right)(c-\bar{a}d,ac+d)\\ &\quad=\left(\left(\bar{c}-\bar{d}a\right)(c-\bar{a}d)-\left(\bar{c}\bar{a}+\bar{d}\right)(d+ac),(ac+d)\left(\bar{c}-\bar{d}a\right)+(d+ac)\left(\bar{c}-\bar{d}a\right)\right)\\ &\quad=\left(\begin{array}{c}|c|^2-\bar{c}(\bar{a}d)-\left(\bar{d}a\right)c+|d|^2|a|^2\\ -(\bar{c}\bar{a})d-|c|^2|a|^2-|d|^2-\bar{d}(ac)\end{array},\begin{array}{c}a|c|^2-(ac)\left(\bar{d}a\right)+d\bar{c}-|d|^2a\\ +d\bar{c}-|d|^2a+a|c|^2-(ac)\left(\bar{d}a\right)\end{array}\right)\\ &\quad=\left(\begin{array}{c}\left(1-|a|^2\right)\left(|c|^2-|d|^2\right)\\ -\bar{c}(\bar{a}d)-\left(\bar{d}a\right)c-(\bar{c}\bar{a})d-\bar{d}(ac)\end{array},2\left(|c|^2-|d|^2\right)a-2a\left(c\bar{d}\right)a+2d\bar{c}\right),\end{aligned}$$

where again we use Artin Theorem and Moufang identity for Octonions. We also use them in the following

$$
\begin{aligned}
&(1,a)\left(|c|^2-|d|^2,2d\bar{c}\right)(1,a)\\
&=\left(|c|^2-|d|^2-2\left(c\bar{d}\right)a,2d\bar{c}+a\left(|c|^2-|d|^2\right)\right)(1,a)\\
&=\left(\begin{matrix}|c|^2-|d|^2-2\left(c\bar{d}\right)a\\-2\bar{a}\left(d\bar{c}\right)-|a|^2\left(|c|^2-|d|^2\right)\end{matrix},\begin{matrix}a\left(|c|^2-|d|^2\right)-2a\left(c\bar{d}\right)a\\+2d\bar{c}+a\left(|c|^2-|d|^2\right)\end{matrix}\right)\\
&=\left(\begin{matrix}\left(1-|a|^2\right)\left(|c|^2-|d|^2\right)\\-2\left(c\bar{d}\right)a-2\bar{a}\left(d\bar{c}\right)\end{matrix},2\left(|c|^2-|d|^2\right)a-2a\left(c\bar{d}\right)a+2d\bar{c}\right).
\end{aligned}
$$

Claim:

$$\bar{c}\left(\bar{a}d\right)+\left(\bar{d}a\right)c+\left(\bar{c}\bar{a}\right)d+\bar{d}\left(ac\right)=2\left(c\bar{d}\right)a+2\bar{a}\left(d\bar{c}\right).$$

Applying Artin Theorem, we have

$$
\begin{aligned}
&\bar{c}\left(\bar{a}d\right)+\left(\bar{d}a\right)c+\left(\bar{c}\bar{a}\right)d+\bar{d}\left(ac\right)\\
&\quad=\frac{c\left[\bar{c}\left(\bar{a}d\right)+\left(\bar{d}a\right)c\right]\bar{c}}{|c|^2}+\frac{d\left[\left(\bar{c}\bar{a}\right)d+\bar{d}\left(ac\right)\right]\bar{d}}{|d|^2}\\
&\quad=\left(\bar{a}d\right)\bar{c}+c\left(\bar{d}a\right)+d\left(\bar{c}\bar{a}\right)+\left(ac\right)\bar{d},
\end{aligned}
$$

and

$$
\begin{aligned}
2\left(c\bar{d}\right)a+2\bar{a}\left(d\bar{c}\right)&=\left(c\bar{d}\right)a+\bar{a}\left(d\bar{c}\right)+\left(c\bar{d}\right)a+\bar{a}\left(d\bar{c}\right)\\
&=\left(c\bar{d}\right)a+\bar{a}\left(d\bar{c}\right)+\frac{a\left[\left(c\bar{d}\right)a+\bar{a}\left(d\bar{c}\right)\right]\bar{a}}{|a|^2}\\
&=\left(c\bar{d}\right)a+\bar{a}\left(d\bar{c}\right)+a\left(c\bar{d}\right)+\left(d\bar{c}\right)\bar{a}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left[\left(\bar{a}d\right)\bar{c}+c\left(\bar{d}a\right)+d\left(\bar{c}\bar{a}\right)+\left(ac\right)\bar{d}\right]-\left[\left(c\bar{d}\right)a+\bar{a}\left(d\bar{c}\right)+a\left(c\bar{d}\right)+\left(d\bar{c}\right)\bar{a}\right]\\
&=\left[\bar{a},d,\bar{c}\right]-\left[c,\bar{d},a\right]+\left[a,c,\bar{d}\right]-\left[d,\bar{c},\bar{a}\right],
\end{aligned}
$$

where the associator $[a,b,c]\triangleq(ab)\,c-a\,(bc)$. Observe that if one of a,c,d is real, then all $[\ \ ]$ terms vanish. Without loss of generality, we assume a,c,d are all imaginary, then

$$
\begin{aligned}
&\left[\bar{a},d,\bar{c}\right]-\left[c,\bar{d},a\right]+\left[a,c,\bar{d}\right]-\left[d,\bar{c},\bar{a}\right]\\
&=[a,d,c]+[c,d,a]-[a,c,d]-[d,c,a]\\
&=[a,d,c]+[a,c,d]-[a,c,d]-[a,d,c]\\
&=0,
\end{aligned}
$$

where we use the alternativity of octonions (cf. [2, Lemma A.4]). The claim is proved, and we have the Moufang identity (4.1).

4. Partial Norm

$$
\begin{aligned}
|(1,a)(c,d)| &= |(1,a)|\,|(c,d)|\,,\\
|(c,d)(1,a)| &= |(c,d)|\,|(1,a)|\,.
\end{aligned}
$$

We show the first identity first.

$$
\begin{aligned}
|(1,a)(c,d)|^2 &= \left|\left(c-\bar{d}a, d+a\bar{c}\right)\right|^2\\
&= \left(c-\bar{d}a\right)\left(\bar{c}-\bar{a}d\right) + \left(d+a\bar{c}\right)\left(\bar{d}+c\bar{a}\right)\\
&= |c|^2 - c\left(\bar{a}d\right) - \left(\bar{d}a\right)\bar{c} + |d|^2|a|^2 + |d|^2 + d\left(c\bar{a}\right) + \left(a\bar{c}\right)\bar{d} + |a|^2|c|^2\\
&= \left(1+|a|^2\right)\left(|c|^2+|d|^2\right) - c\left(\bar{a}d\right) - \left(\bar{d}a\right)\bar{c} + \frac{\bar{d}\left[d\left(c\bar{a}\right)+\left(a\bar{c}\right)\bar{d}\right]d}{|d|^2}\\
&= \left(1+|a|^2\right)\left(|c|^2+|d|^2\right) - c\left(\bar{a}d\right) - \left(\bar{d}a\right)\bar{c} + \left(c\bar{a}\right)d + \bar{d}\left(a\bar{c}\right)\\
&= \left(1+|a|^2\right)\left(|c|^2+|d|^2\right) + \left[c,\bar{a},d\right] - \left[\bar{d},a,\bar{c}\right].
\end{aligned}
$$

We handle the two [ ] terms similarly as in 3. If one of a,c,d is real, then both [ ] terms vanish. Without loss of generality, we assume a,c,d are all imaginary, then

$$
\left[c,\bar{a},d\right] - \left[\bar{d},a,\bar{c}\right] = -\left[c,a,d\right] - \left[d,a,c\right] = \left[d,a,c\right] - \left[d,a,c\right] = 0.
$$

Thus we have $|(1,a)(c,d)| = |(1,a)|\,|(c,d)|$. Relying on this first identity, we show the second.

$$
\begin{aligned}
|(c,d)(1,a)| &= \left|\overline{(c,d)(1,a)}\right| = \left|\overline{(1,a)}\;\overline{(c,d)}\right| = |(1,-a)(\bar{c},-d)|\\
&= |(1,-a)|\,|(\bar{c},-d)| = |(1,a)|\,|(c,d)|\,.
\end{aligned}
$$

Proposition 4.1 *For any fixed* $y\in\mathbb{H}\cong\mathbb{R}^4$, $\mathbb{O}\cong\mathbb{R}^8$, *or* $\mathbb{B}=\mathbb{O}\oplus\mathbb{O}\cong\mathbb{R}^{16}$, *there exists an orthogonal transformation* $T\in O(2m)$ *and an induced orthogonal transformation* $\mathcal{T}\in O(m+1)$ *with* $m=2,4,$ *or* 8 *such that*

$$
\begin{aligned}
y &= (|y|,0,\cdots,0)\,T,\\
H(xT) &= H(x)\,\mathcal{T} \quad \text{for all } x\in\mathbb{R}^{2m},
\end{aligned}
$$

where we also identify T *and* $\mathcal{T}$ *with the orthogonal matrices.*

Proof First we deal with the octonionic case $\mathbb{B}=\mathbb{O}\oplus\mathbb{O}\cong R^{16}$. Let $x=(u,v)$, define $D(x) = xD = (u,v)(\beta,b)$ for $|(\beta,b)| = 1$ with real β. By 4. $|D(x)| = |(u,v)|\,|(\beta,b)| = |(u,v)|$, then D is an orthogonal transformation in $O(16)$. It follows from the above 3. and 2. that

$$
\begin{aligned}
H(D(x)) &= \left(\bar{U},V\right)(U,V)\\
&= \left[(\beta,b)(u,v)\right]\left[(u,v)(\beta,b)\right]\\
&= (\beta,b)\left(|u|^2-|v|^2, 2v\bar{u}\right)(\beta,b)\\
&\triangleq \left(|u|^2-|v|^2, 2v\bar{u}\right)\mathcal{D}.
\end{aligned}
$$

By the above 4., we have

$$\left|\left(|u|^2-|v|^2,2v\bar{u}\right)\mathcal{D}\right|=|(\beta,b)|\left|\left(|u|^2-|v|^2,2v\bar{u}\right)\right||(\beta,b)|=\left|\left(|u|^2-|v|^2,2v\bar{u}\right)\right|.$$

We see that the induced $\mathcal{D}$ belongs to orthogonal transformation $O(9)$.

Further, we define $E(x)=xE=(\bar{e}u,ev)$ for $e\in\mathbb{O}$ with $|e|=1$. Then $E\in O(16)$ and

$$\begin{aligned}H(E(x))&=\left(|u|^2-|v|^2,2(ev)(\bar{u}e)\right)\\&=\left(|u|^2-|v|^2,2e(v\bar{u})e\right)\\&\triangleq\left(|u|^2-|v|^2,2v\bar{u}\right)\mathcal{E},\end{aligned}$$

where we use Moufang identity for octonions (cf. [2, Lemma A. 16]). We have

$$\left|\left(|u|^2-|v|^2,2v\bar{u}\right)\mathcal{E}\right|=\left|\left(|u|^2-|v|^2,2v\bar{u}\right)\right|,$$

then the induced $\mathcal{E}$ is an orthogonal transformation in $O(9)$.

Now set $T=ED$ and $\mathcal{T}=\mathcal{E}\mathcal{D}$, note that $T(r,0,\cdots,0)=(\bar{e}r,0)(\beta,b)=(\bar{e}r\beta,b\bar{e}r)$ with r real runs over $\mathbb{B}=\mathbb{O}\oplus\mathbb{O}$, we thus prove the Proposition in the octonionic case.

Because the alternativity and Moufang identity are already available for octonions (cf. [2, Appendix IV.A]), the other two cases of the proposition follows along the same lines as above.

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