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LOCALLY MINIMIZING SMOOTH HARMONIC MAPS FROM ASYMPTOTICALLY FLAT MANIFOLDS INTO SPHERES

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Abstract By minimizing the so–called relative energy, we show that there exists a family of locally minimizing smooth harmonic maps from asymptotically flat manifolds into the standard sphere.

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1. Introduction

The existence and regularity of harmonic maps from noncompact domain with infinite energy are of interest, however we are short of this kind of examples. It was shown by Aviles–Choi–Micallef [1] that if M is complete and simply connected with sectional curvature K_M satisfying

$$-b^2 \leq K_M \leq -a^2 < 0, \tag{1.1}$$

then one can solve the Dirichlet problem with boundary at infinity for harmonic maps from M into $\overline{B_\tau(P)}$, which is a geodesic ball in the target manifolds N, where $\tau < \min\{\frac{\pi}{2}\sqrt{K}$, injectivity radius of N at $P\}, K \geq 0$ is an upper bound for the sectional curvature of N. Using heat flow and elliptic methods respectively, P.Li–L.Tam [2] and Ding–Wang [3] got some interesting existence results concerning proper harmonic maps from complete noncompact manifolds into nonpositive curvature targets.

Recently, Ding [4] introduced a so–called relative energy method for obtaining locally energy minimizing harmonic maps between noncompact manifolds. He proved that if M is simply connected with sectional curvature satisfying (1.1), then the Dirichlet problem with $C^\alpha(\alpha > 0)$ boundary data at infinity for harmonic maps from M into a compact target has a locally minimizing solution, which is smooth near infinity.

We would like to point out that the domain in the above cases has either negative curvature or positive first eigenvalue.

In this paper we discuss harmonic maps from asymptotically flat manifolds into spheres. Now that we deal with harmonic maps having infinite energy, we shall try to minimize the relative energy E_ϕ (cf. [4]) with respect to $\phi \in C^2(M^n, S^n)$ so that we can get a weakly harmonic map. Here we take a rotationally symmetric harmonic map from R^n into S^n as the approximate map ϕ near the infinity of M (cf. [5], [6]). By Hardy inequality (cf. [7], Theorem 330)

$$\inf_{u\in Y(R^n)} \frac{\int_{R^n} |\bigtriangledown u|^2}{\int_{R^n} \frac{u^2}{|x|^2}} = \left(\frac{n-2}{2}\right)^2 \tag{1.2}$$

where $Y(R^n) = \{u \in W^{1,2}_{loc}(R^n) : \|u\|^2_Y = \int_{R^n} |\bigtriangledown u|^2 dx + \int_{R^n} \frac{u^2}{|x|^2} dx < +\infty\}$, we overcome the difficulty in proving that E_ϕ satisfies the coercive condition for $n \geq 7$. We note that in [4], Ding assumed $\lambda_1(M) > 0$ and the tension field $T(\phi) \in L^2(M)$ then used Poincaré inequality to prove the coercive condition for E_ϕ . Furthermore, we prove that $Im(f)$, the image of minimizing f is contained in the open upper hemisphere, hence the regularity of f follows. Since $Im(f)$ can not be contained in any compact set of S^n_+, the methods of [1] fail in our case. The following is our main result.

Main Theorem *Let $(M^n, g), n \geq 7$, be a complete Riemannian manifold having finite asymptotically flat ends with the least order $k > n-2$, then there exists a sequence of locally minimizing smooth harmonic maps from M into S^n.*

Remark 1.1 For $n = 2$, the Hardy inequality is invalid, for $3 \leq n \leq 6$, the bad term in E_ϕ can not be controlled since ϕ oscillates near infinite and (3.12) does not hold, so we fail to have the coercive condition for E_ϕ.

Remark 1.2 We conjecture that there is no nonconstant locally minimizing smooth harmonic maps from M^n into S^n for $2 \leq n \leq 6$. It is true in the case that M^n coincides with R^n (cf. [6]).

In the next section, we first recall the concept of harmonic maps from noncompact domain and describe the relative energy method. The main theorem will be proved in Section 3.

2. Weakly Harmonic Maps from Noncompact Domain and the Relative Energy

Let (M^m, g) be a complete noncompact Riemannian manifold and (N^n, h) be a complete Riemannian manifold. By Nash's imbedding theorem, we may embed (N, h) isometrically into a Euclidean space R^K for some large K so that N is a submanifold of R^K and h is just the induced metric.

Taking any bounded domain Ω in M , we have the Sobolev space $W^{1,2}(\Omega, R^K)$ and its subset $W^{1,2}(\Omega, N) = \{f \in W^{1,2}(\Omega, R^k) : f(x) \in N \text{ for a.e. } x \in \Omega\}$. The energy of $f \in W^{1,2}(\Omega, R^k)$ on Ω is defined by

$$E(t, \Omega) = \frac{1}{2} \int_\Omega | \bigtriangledown f|^2 dv \tag{2.1}$$

where $|\bigtriangledown f|^2 = g^{\alpha\beta} \left\langle \frac{\partial f}{\partial x^\alpha}, \frac{\partial f}{\partial x^\beta} \right\rangle, dv = \sqrt{g} dx^1 \wedge \cdots \wedge dx^m$, in local coordinate of $M, \langle \cdot, \cdot \rangle$ is the inner product of R^k.

Definition 2.1 *For any bounded Ω, we call $v \in W_0^{1,2}(\Omega, R^k)$ an admissible variation for $f \in W_{loc}^{1,2}(M, N)$ if there exists a C^1 curve $f_t : [0, \varepsilon) \to W^{1,2}(\Omega, R^k)$ with $f_0 = f, f_t - f \in W_0^{1,2}(\Omega, R^k)$ and $f_t \in W^{1,2}(\Omega, N)$ for each t , such that*

$$v = \left. \frac{df_t}{dt} \right|_{t=0}.$$

Definition 2.2 *Any $f \in W_{loc}^{1,2}(M, N)$ is called weakly harmonic if it is critical for the energy w.r.t. admissible variations, i.e.*

$$\left. \frac{d}{dt}(E(f_t, \Omega)) \right|_{t=0} = \int_\Omega \langle \bigtriangledown f, \bigtriangledown v \rangle = 0$$

for all admissible variations v.

Remark 2.3 If $f \in W_{loc}^{1,2}(M, N)$ is a locally minimizing map, i.e. for any bounded $\Omega \subset M$ and any $g \in W_{loc}^{1,2}(M, N)$ s.t. $f - g \in W_0^{1,2}(\Omega, R^k)$,

$$\int_\Omega | \bigtriangledown f|^2 dv \leq \int_\Omega | \bigtriangledown g|^2 dv,$$

then f is weakly harmonic.

We may obtain the explicit form of the Euler–Lagrange equation satisfied by weakly harmonic maps from (2.1). Let $\eta \in C_0^\infty(\Omega, R^k)$ and $g_t = f + t\eta$. It is clear that g_t maps Ω into a small neighborhood U of N in R^k for small $t \geq 0$. We define the least distance mapping $\pi : U \to N$ s.t. $y \in U, |y - \pi(y)| = \text{dist}(y, N)$. Note that N is boundary–free. If N is compact, we can assume π is well–defined and smooth. In noncompact the case of N, we have to assume that U is an ε–neighborhood of Image $(g_{[0,\delta]})$ in R^k for some $\varepsilon, \delta > 0$ and π is well–defined, smooth on U.

For $y \in N, d\pi(y)$ is the orthogonal projection of R^k onto the tangent space of N at y, denote $d\pi(y)$ by $P(y) : R^k \to T_y N$.

Let $f_t = \pi(g_t)$, it is obvious that

$$\left. \frac{df_t}{dt} \right|_{t=0} = P(f)\eta$$

is an admissible variation. Since f is weakly harmonic, by (2.1) we have

$$\int \langle \bigtriangledown f, \bigtriangledown (P(f))\eta) \rangle = 0, \quad \forall \eta \in C_0^\infty(M, R^k). \tag{2.2}$$

If f is regular (cf. $f \in W_{loc}^{2,1}$), after integration by parts, (2.2) is equivalent to

$$\int \langle \triangle f, P(f)\eta \rangle = \int \langle P(f) \triangle f, \eta \rangle = 0$$

So f is a weak solution of the Euler–Lagrange equation

$$P(f) \triangle f = 0.$$

Definition 2.4 *For regular $f, \tau(f) = P(f) \triangle f$ is called th tension field of f.*

Remark 2.5 $\tau(f)$ can be expressed more explicitly,

$$\tau(f) \equiv \triangle f - g^{\alpha\beta} A(f) \left(\frac{\partial f}{\partial x^\alpha}, \frac{\partial f}{\partial x^\beta} \right)$$

where $A(y) = dP(y)$ is the second fundamental form of N in R^k at y. In fact we have $P(f)\frac{\partial f}{\partial x^\alpha} = \frac{\partial f}{\partial x^\alpha}$ since $\frac{\partial f}{\partial x^\alpha} \in T_f N$.

Hence,

$$\begin{aligned} \triangle f &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\beta} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} \right) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\beta} \left(\sqrt{g} g^{\alpha\beta} (P(f) \frac{\partial f}{\partial x^\alpha}) \right) \\ &= P(f) \triangle f + g^{\alpha\beta} dP(f) \left(\frac{\partial f}{\partial x^\alpha}, \frac{\partial f}{\partial x^\beta} \right), \end{aligned}$$

the desired form of $\tau(f)$ turns out.

In the following we recall the new relative energy method in [4]. Let $\phi \in C^2(M, N)$, we shall define the relative energy E_ϕ w.r.t. ϕ. Assume Y to be a reflexive Banach space for mappings from M into R^k with norm $\| \cdot \|_Y$. Taking $\| \bigtriangledown u\|_2 = \| \bigtriangledown u\|_{L^2(M)}$ for $u \in Y$, we require that $\| \bigtriangledown u\|_2$ is an equivalent norm for Y, i.e. $\exists C > 1$ such that

$$C^{-1} \| \bigtriangledown u\|_2 \leq \|u\|_Y \leq C \| \bigtriangledown u\|_2.$$

Furthermore, we assume that Y is continuously imbedded in $L^p(M, R^k)$ for some $p > 1$. Now, we define a subset X of Y by

$$X = X_\phi(Y) = \{u \in Y : \phi(x) + u(x) \in N \quad \text{a.e. for} \quad x \in M\},$$

and define a functional D_ϕ on X by

$$D_\phi(u) = \frac{1}{2} \int_M | \bigtriangledown u|^2 - \int_M \langle \triangle \phi, u \rangle.$$

For any map f from M into N with $f-\phi \in Y$, the relative energy of f w.r.t. ϕ is defined as,

$$E_\phi(f) = D_\phi(f-\phi).$$

Note that for $u \in X$ with compact support in a bounded domain Ω, using integration by parts, we have

$$E_\phi(\phi+u) = D_\phi(u) = E(\phi+u,\Omega) - E(\phi,\Omega).$$

It follows that $u \in X$ is critical for D_ϕ iff $f=\phi+u$ is critical for E , or, f is weakly harmonic.

In particular, if $u \in X$ is a minimizer of D_ϕ , i.e. $D_\phi(u) = \inf\limits_{v\in X} D_\phi(v)$, then $f=\phi+u$ is a locally minimizing harmonic map. More precisely, for any bounded domain $\Omega \subset M$ and any $v \in X$, let $g=\phi+v$, s.t. $f-g \in W_0^{1,2}(\Omega,R^k)$ or $u-v \in W_0^{1,2}(\Omega,R^k)$, we have $D_\phi(u)$ on Ω less than $D_\phi(V)$ on Ω, on the other hand,

$$\begin{aligned} D_\phi(u,\Omega) &= \frac{1}{2}\int_\Omega |\bigtriangledown u|^2 - \int_\Omega \langle \triangle\phi, u\rangle \\ &= \frac{1}{2}\int_\Omega |\bigtriangledown f|^2 - \frac{1}{2}\int_\Omega |\bigtriangledown \phi|^2 - \int_{\partial\Omega}\left\langle \frac{\partial\phi}{\partial n}, u\right\rangle, \\ D_\phi(v,\Omega) &= \frac{1}{2}\int_\Omega |\bigtriangledown v|^2 - \int_\Omega \langle \triangle\phi, v\rangle \\ &= \frac{1}{2}\int_\Omega |\bigtriangledown g|^2 - \frac{1}{2}\int_\Omega |\bigtriangledown \phi|^2 - \int_{\partial\Omega}\left\langle \frac{\partial\phi}{\partial n}, v\right\rangle \end{aligned}$$

where n is the exterior normal of $\partial\Omega$.

Note that $u=v$ on $\partial\Omega$, it follows that

$$E(f,\Omega) \le E(g,\Omega).$$

Lemma 2.6 [4] *Let $\phi \in C^2(M,N)$, the Banach space Y and its subset $X = X_\phi(Y)$ be as above. Suppose that the relative energy E_ϕ satisfies the coercive condition*

$$D_\phi(u) \ge a\| \bigtriangledown u\|_2^2 + b\| \bigtriangledown u\|_2 + c$$

for some constants $a>0, b, c$. Then there exists a locally minimizing harmonic map f such that $\|f-\phi\|_Y < \infty$.

Proof Note that the coercive condition guarantees that D_ϕ is bounded from below on X, and that if $\{u_k\}$ is a minimizing sequence for D_ϕ in X, then $\{u_k\}$ is bounded in Y since $\| \bigtriangledown u\|_2$ is equivalent to $\|u\|_Y$. Now that Y is reflexive, we may assume that $\{u_k\}$ converges weakly to some $u \in Y$. The embedding $Y \hookrightarrow L^p(M,R^k)$ implies that

$\{u_k\}$ converges to u in $L^1_{loc}(M, R^k)$, and hence converges almost everywhere to u. It follows that $u \in X$ since each $u_k \in X$. Finally, by lower semi–continuity of D_ϕ, u must achieve the infimum of D_ϕ on X. Clearly, $f = \phi + u$ is a locally minimizing harmonic map and $\|f - \phi\|_Y = \|u\|_Y < \infty$.

3. The Proof of Main Theorem

In this section we shall show the existence and regularity of the harmonic maps in two steps, but first we need the following definitions.

Definition 3.1 *Let (M^n, y) be a complete noncompact Riemannian manifold, we say that M has q ends $E_1, \cdots, E_q$, if there exists bounded open set U such that*

$$M \backslash \overline{U} = \bigcup_{i=1}^{q} E_q$$

with each E_i being a noncompact connected component of $M\backslash\overline{U}$.

Definition 3.2 *Let (M^n, g) be as above, $\tau > 0$, we say that M_∞ is an asymptotically flat end of order τ if there exists a decomposition $M^n = M_\circ \cup M_\infty$ and a diffeomorphism $M_\infty \cong R^n \backslash B_R$, where B_R is the ball of radius R with center at the origin in R^n, satisfying*

$$g_{ij} = \delta_{ij} + O(r^{-\tau}),$$

$$\partial g_{ij} = O(r^{-\tau-1}),$$

$$\partial\partial g_{ij} = O(r^{-\tau-2}),$$

as $r = |x| \to +\infty$, where x is the coordinate on M_∞ induced by the above diffeomorphism.

Remark 3.3 For such a g, we know $g^{ij} = \delta_{ij} + O(r^{-\tau}), \partial g^{ij} = O(r^{-\tau-1}), \partial\partial g^{ij} = O(r^{-\tau-2})$ and $g = 1 + O(r^{-\tau}), \partial g = O(r^{-\tau-1})$ where $g = \det(g_{ij})$.

In order to prove the theorem, we should find a suitable relative function $\phi \in C^2(M, N)$. We extend the following ϕ near the infinity of M smoothly to the whole M such that ϕ maps a compact set $\overline{U}$ as in Definition 3.1 to the north pole of S^n .

Let $\phi = \left(\frac{x}{|x|}\sin h, \cos h\right) \in S^n \subset R^{n+1}$ near the infinity of every asymptotically flat end in M, which is the smooth rotationally symmetric harmonic map from R^n into sphere S^n (cf. [5], [6]), ϕ satisfies

$$-\triangle_0 \phi = |\bigtriangledown_0 \phi|^2 \phi,$$

in particular, the function of $r = |x|, h : [0, +\infty) \to [0, \pi]$ satisfies

$$\begin{cases} h'' + \dfrac{n-1}{r}h' - \dfrac{n-1}{2r^2}\sin 2h = 0 \\ h(+\infty) = \dfrac{\pi}{2}, \quad h \not\equiv \dfrac{\pi}{2} \end{cases} \tag{3.1}$$

In the above formula, the $\triangle_0, \bigtriangledown_0$ denote the Laplace and gradient operator w.r.t. the standard metric g_0 in R^n. Our concern is the behavior of ϕ or h near infinity, hence the following lemma is needed.

Lemma 3.4 *The solution of (3.1), h can be taken such that*

$$h(r) = \frac{\pi}{2} - Cr^{\lambda_+}(1 + O(1)),$$

$$h'(r) = r^{\lambda_+ - 1}(1 + O(1)),$$

$$|h''(r)| \le Cr^{\lambda_+ - 2}, \quad \textit{for} \quad r \ge R,$$

where

$$\lambda_+ = \frac{-(n-2) + \sqrt{(n-2)^2 - 4(n-1)}}{2} < 0, \ n \ge 7.$$

Proof Set $t = lnr, \psi(t) = 2h(r) - \pi$, then $2h'(r) = \frac{1}{r}\psi'(lnr)$, (3.1) is transformed into an autonomous equation

$$\begin{cases} \psi'' + (n-2)\psi' + (n-1)\sin\psi = 0 \\ \psi(+\infty) = 0. \end{cases} \tag{3.2}$$

This equation can be analyzed by standard methods.

Put

$$q(t) = \psi(t), \quad p(t) = q'(t)$$

then (3.2) becomes the plane system

$$\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} p \\ -(n-2)p - (n-1)\sin q \end{bmatrix} \tag{3.3}$$

or

$$\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(n-1) & -(n-2) \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ (n-1)(q - \sin q) \end{bmatrix}.$$

The behavior of the system (3.3) in the neighborhood of the critical point $(0, 0)$ is determined by the linearized system

$$\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(n-1) & -(n-2) \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}. \tag{3.4}$$

The eigenvalues of the matrix of (3.4) are

$$\lambda_{\pm} = \frac{-(n-2) \pm \sqrt{(n-2)^2 - 4(n-1)}}{2} < 0,$$

since $n \geq 7$.

Noticing that

$$\frac{(n-1)(r - \sin r)}{r^{1+\varepsilon}} \to 0 \quad \text{as} \quad r \to 0$$

for some $\varepsilon > 0$, by the Theorem 3.5 of Chap. VIII in [8] we can make the solution $(q(t), p(t)) = e^{\lambda_+ t}(O(1), 2 + O(1))$, consequently we have

$$h'(r) = r^{\lambda_+ - 1}(1 + O(1))$$

$$h(r) = \frac{\pi}{2} + O(1)r^{\lambda_+} \quad \text{or}$$

$$h(r) = \frac{\pi}{2} - \int_r^{+\infty} h'(r)dr = \frac{\pi}{2} - Cr^{\lambda_+}(1 + O(1)).$$

Finally, from (3.1) we have

$$|h''(r)| \leq Cr^{\lambda_+ - 2}.$$

Proof of Main Theorem Throughout this proof, let $C(R)$ be any positive number depending only on R.

Let

$$Y = Y(M) = \left\{ u \in W^{1,2}_{loc}(M, R^{n+1}) : \|u\|_Y^2 = \| \bigtriangledown u\|_2^2 + \int_M \frac{u^2}{r^2} < +\infty \right\},$$

where $r(x)$ is a smooth function on M such that

$$r(x) = \begin{cases} R_0 & x \in B_{R_0}(x) \\ > R_0 & x \in B_{R_0+1}(x) \backslash B_{R_0}(x) \\ \text{dist}(x, x_0) & x \in M \backslash B_{R_0+1}(x) \end{cases}.$$

In the above definition $B_{R_0} = B_{R_0}(x)$ represents the geodesic ball in M with center at a fixed point $x_0 \in M$ and the radius R_0 sufficiently large. In order to apply Lemma 2.6 for getting our results, we need the following lemma.

Lemma 3.5 *Let $(M^n, g), n \geq 3$, be a complete Riemannian manifold having finite asymptotically flat ends with the least order $\tau > 0$, then*

$$\inf_{u \in Y(M)} \frac{\displaystyle\int_M |\bigtriangledown u|^2}{\displaystyle\int_M \frac{u^2}{r^2}} = \lambda > 0.$$

Consequently, $\| \bigtriangledown u\|_2$ *is an equivalent norm for* Y.

Proof Our argument is quite similar to that in Lemma 4.1 ([3]). Suppose $\lambda = 0$, let $Y(B_i) = \{u \in Y : \text{supp}(u) \subset B_i\}$, where $B_i = \{x \in M : \text{dist}(x, x_0) \leq i\}$, and let $\lambda_i = \inf\limits_{u\in Y(B_i)} \dfrac{\int_M |\bigtriangledown u|^2}{\int_M \dfrac{u^2}{r^2}}$, then $\lambda_i \to \lambda = 0$. Since $r(x)$ is regular in each B_i , we assume u_i be the eigenfunction on B_i corresponding to λ_i, satisfying $\triangle u_i + \lambda_i \dfrac{u_i}{r^2} = 0, u_i > 0$ in $B_i, u_i|_{\partial B_i} = 0$ and

$$u_i(x_0) = 1. \tag{3.5}$$

Let η_1 be a cut–off function such that $\eta_1 = 1$ on $B_{R_0}, \eta_1 = 0$ on $M\backslash B_{R_0+1}, 0 \leq \eta_1 \leq 1$ and $|\bigtriangledown \eta_1| \leq 2$ on M, let $\eta_2 = 1-\eta_1$. Extending $u_i = 0$ on $M\backslash B_i$, we have for $i > R_0+1$

$$\int_M |\bigtriangledown u_i|^2 = \lambda_i \int_M \frac{u_i^2}{r^2} \tag{3.6}$$

$$\begin{aligned} \int_M \frac{u_i^2}{r^2} &= \int_M \frac{(\eta_1 u_i + \eta_2 u_i)^2}{r^2} \leq 2\int_M \frac{\eta_1^2 u_i^2}{r^2} + \frac{\eta_2^2 u_i^2}{r^2} \\ &\leq 2\int_{B_{R_0+1}} \frac{\eta_1^2 u_i^2}{r^2} + 2\int_{M\backslash B_{R_0}} \frac{\eta_2^2 u_i^2}{r^2}. \end{aligned} \tag{3.7}$$

By Poincaré inequality on B_{R_0+1}, we have

$$\begin{aligned} 2\int_{B_{R_0+1}} \frac{\eta_1^2 u_i^2}{r^2} &\leq \frac{2}{R_0^2}\int_{B_{R_0+1}} (\eta_1 u_i)^2 \\ &\leq \frac{2}{R_0^2} C(R_0) \int_{B_{R_0+1}} |\bigtriangledown \eta_1 u_i|^2 \\ &\leq C(R_0) \int_{B_{R_0+1}} |\bigtriangledown \eta_1|^2 u_i^2 + \eta_1^2 |\bigtriangledown u_i|^2. \end{aligned}$$

Next, we estimate the second term in (3.7), taking the coordinate induced by the diffeomorphism on each end of $M\backslash B_{R_0}$. Since we assumed R_0 be sufficiently large, by Definition 3.2, we have

$$\| \bigtriangledown_0 u_i\|^2_{L^2(M\backslash B_{R_0})} \leq C(R_0) \| \bigtriangledown u_i\|^2_{L^2(M\backslash B_{R_0})} \tag{3.8}$$

$$\sqrt{g} \leq C(R_0) \quad \text{on} \quad M\backslash B_{R_0} \tag{3.9}$$

where $\bigtriangledown_0$ is the gradient operator w.r.t. the standard metric on R^n , then from (3.9),

(1.2) and (3.8) we have

$$\begin{aligned} 2\int_{M\setminus B_{R_0}} \frac{\eta_2^2 u_i^2}{r^2} dv &= 2\int_{M\setminus B_{R_0}} \frac{\eta_2^2 u_i^2}{r^2}\sqrt{g}dx \\ &\le C(R_0)\int_{M\setminus B_{R_0}} \frac{\eta_2^2 u_i^2}{|x|^2}dx \\ &\le C(R_0)\left(\frac{2}{n-2}\right)^2 \int_{M\setminus B_{R_0}} |\bigtriangledown_0 \eta_2 u_i|^2 dx \\ &\le C(R_0)\int_{M\setminus B_{R_0}} |\bigtriangledown \eta_2 u_i|^2 \\ &\le C(R_0)\int_{M\setminus B_{R_0}} |\bigtriangledown \eta_2|^2 u_i^2 + \eta_2^2 |\bigtriangledown u_i|^2. \end{aligned}$$

Combining this inequality with the estimate of $2\int_{B_{R_0}+1} \frac{\eta_1^2 u_i^2}{r^2}$, we have

$$\begin{aligned} \int_M \frac{u_i^2}{r^2} &\le C(R_0)\int_M (|\bigtriangledown \eta_1|^2 + |\bigtriangledown \eta_2|^2)u_i^2 + \int_M (\eta_1^2+\eta_2^2)|\bigtriangledown u_i|^2 \\ &\le C(R_0)\int_{B_{R_0+1}\setminus B_{R_0}} u_i^2 + C(R_0)\int_M |\bigtriangledown u_i|^2. \end{aligned}$$

It follows from (3.5) and Harnack inequality (cf. [9], [10]) that we can get the uniform bound of L^2 integrals of u_i on $B_{R_0+1}\setminus B_{R_0}$, also from (3.6) we have

$$\int_M \frac{u_i^2}{r^2} \le C(R_0) + C(R_0)\lambda_i \int_M \frac{u_i^2}{r^2}.$$

Since $\lambda_i \to 0$, there exists a constant $C(R_0)$. such that

$$\int_M \frac{u_i^2}{r^2} \le C(R_0). \tag{3.10}$$

However, the standard elliptic estimates and diagonal subsequence argument imply that we may assume some subsequence of $\{u_i\}$ converge in C^2 norm on any compact subset of M to a harmonic function u_∞. From (3.10) we know

$$\int_M \frac{u_\infty^2}{r^2} \le C. \tag{3.11}$$

By (3.6) and (3.10) we have

$$\int_M |\bigtriangledown u_\infty|^2 = \lim_{i\to+\infty} \lambda_i \int_M \frac{u_i^2}{r^2} = 0$$

hence $u_\infty \equiv$ constant, because of (3.5) $u_\infty \equiv 1$. But M is asymptotically flat,

$$\int_M \frac{u_\infty^2}{r^2} = \int_M \frac{1}{r^2} = +\infty,$$

this contradicts to (3.11), therefore the conclusion of the lemma follows.

Return to the proof of main theorem.

Let

$$X = \{u \in Y : \phi(x) + u(x) \in S^n \subset R^{n+1}, \quad \text{a.e.} \quad x \in M\},$$

we need to verify the coercive condition of the relative energy on X,

$$D_\phi(u) = \int_M \frac{1}{2} |\bigtriangledown u|^2 - \langle \triangle\phi, u\rangle.$$

Let B_R be a geodesic ball in M with center at $x_0 \in M$ and the radius R sufficiently large, on each end of $M\backslash B_R$. By Lemma 3.4, we have the estimates of $\bigtriangledown_0^2\phi, \bigtriangledown_0\phi$, then we have

$$\begin{aligned}
\triangle\phi &= \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j\phi) \\
&= g^{ij}\partial_i\partial_j\phi + \partial_i g^{ij}\partial_j\phi + g^{ij}\partial_j\phi\frac{\partial_i g}{2g} \\
&= \triangle_0\phi + O\left(\frac{1}{r^{k+2}}\right) \\
|\bigtriangledown\phi|^2 &= g^{ij}\partial_i\phi\partial_j\phi = |\bigtriangledown_0\phi|^2 + O\left(\frac{1}{r^{k+2}}\right).
\end{aligned}$$

Let η be the cut–off function on M s.t. $\eta \geq 0, \eta(x) = 0$ on $B_R, \eta(x) = 1$ on $M\backslash B_{R+1}$, then we have

$$\begin{aligned}
D_\phi(u) &= \frac{1}{2}\int_M |\bigtriangledown u|^2 - \int_M \langle\triangle\phi, \eta u + (1-\eta)u\rangle \\
&= \frac{1}{2}\int_M |\bigtriangledown u|^2 - \int_M \langle\triangle\phi, \eta u\rangle - \int_M \langle\triangle\phi, (1-\eta)u\rangle.
\end{aligned}$$

Because $|u+\phi|^2 = 1 = |\phi|^2$, we have $\langle u, \phi\rangle = -\frac{1}{2}|u|^2$ and $|u| \leq 2$. Hence the third part of the above is finite, we need to estimate the second term,

$$\begin{aligned}
\int_M \langle\triangle\phi, \eta u\rangle &= \int_{M\backslash B_R} \langle\triangle\phi, \eta u\rangle \\
&= \int_{M\backslash B_R} \langle -|\bigtriangledown_0\phi|^2\phi + O\left(\frac{1}{r^{k+2}}\right), \eta u\rangle\sqrt{g}dx \\
&\leq \frac{1}{2}\int_{M\backslash B_R} |\bigtriangledown_0\phi|^2 u^2\eta dx + C(R),
\end{aligned}$$

where $C(R) = \left|\int_{M\backslash B_R} < O\left(\frac{1}{r^{k+2}}\right), \eta u > \sqrt{g}dx\right| < +\infty$, since $\sqrt{g} = 1 + O\left(\frac{1}{r^k}\right)$, and $k > n-2$.

Note that $|\bigtriangledown_0 \phi|^2 = (h')^2 + \frac{n-1}{r^2}\sin^2 h$, then by Lemma 3.4,

$$\begin{aligned} & \frac{1}{2}\int_{M\setminus B_R} |\bigtriangledown_0 \phi|^2 u^2 \eta dx \\ \leq \ & \frac{1}{2}\int_{M\setminus B_R} \frac{n-1+\varepsilon}{r^2} u^2 \eta dx \end{aligned}$$

for some small $\varepsilon = \varepsilon(R) \to 0$ as $R \to +\infty$. We also have

$$C_\varepsilon = \frac{n-1+\varepsilon}{\left(\frac{n-2}{2}\right)^2} < 1. \tag{3.12}$$

By Hardy inequality (1.2)

$$\begin{aligned} & \frac{1}{2}\int_{M\setminus B_R} \frac{n-1+\varepsilon}{r^2} u^2 \eta dx \\ \leq \ & \frac{1}{2}\frac{n-1+\varepsilon}{\left(\frac{n-2}{2}\right)^2}\int_{M\setminus B_R} |\bigtriangledown_0 u\sqrt{\eta}|^2 dx \\ = \ & \frac{1}{2}C_\varepsilon \int_{M\setminus B_R} (\eta|\bigtriangledown_0 u|^2 + 2u\sqrt{\eta}\bigtriangledown_0\sqrt{\eta}\bigtriangledown_0 u + u^2|\bigtriangledown_0\sqrt{\eta}|^2) dx \\ \leq \ & \frac{1}{2}C_\varepsilon \int_{M\setminus B_R} |\bigtriangledown_0 u|^2 + C\left(\int_{B_{R+1}\setminus B_R} |\bigtriangledown_0 u|^2\right)^{1/2} + C(R) \\ \leq \ & \frac{1}{2}C'_\varepsilon \|\bigtriangledown u\|^2_{M\setminus B_R} + C\|\bigtriangledown u\|_{B_{R+1}\setminus B_R} + C(R) \end{aligned}$$

where $C'_\varepsilon < 1$ since $M\setminus B_R$ is asymptotically flat.

Therefore, we have

$$D_\phi(u) \geq \frac{1}{2}(1 - C'_\varepsilon)\|\bigtriangledown u\|_2^2 - C\|\bigtriangledown u\|_2 - C(R).$$

By Lemma 2.6 there exists a locally minimizing harmonic map $f = u+\phi$ s.t. $\|f-\phi\|_Y < +\infty$.

Let η_R be the cut off function on M such that $\eta_R(x) = 1$ in $B_{R-1}(x_0), \eta_R(x) = 0$ outside $B_R(x_0), 0 \leq \eta_R \leq 1$, then set $u_R = \eta_R u$. From the above proof of the coercive condition of E_ϕ, we also know

$$D_\phi(u_R) \to D_\phi(u) \quad \text{as} \quad R \to +\infty,$$

we define $f_R = u_R + \phi, \widetilde{f}_R = (f_R^1, \cdots, f_R^n, |f_R^{n+1}|)$, then

$$\begin{aligned} E_\phi(\widetilde{f}_R) &= \int_M \frac{1}{2} |\bigtriangledown (\widetilde{f}_R - \phi)|^2 - \langle \triangle\phi, \widetilde{f}_R - \phi\rangle \\ &= \int_{B_R} \frac{1}{2} |\bigtriangledown (\widetilde{f}_R - \phi)|^2 + \langle \bigtriangledown\phi, \bigtriangledown(\widetilde{f}_R - \phi)\rangle \\ &= \frac{1}{2}\int_{B_R} |\bigtriangledown \widetilde{f}_R|^2 - |\bigtriangledown \phi|^2 \\ &\leq \frac{1}{2}\int_{B_R} |\bigtriangledown f_R|^2 - |\bigtriangledown \phi|^2 \\ &= E_\phi(f_R), \end{aligned}$$

so we can take $\mathrm{Im}(f) \subset \overline{S^n_+}$.

In the following part, we prove the regularity of f. Noticing that

$$\int_M |\bigtriangledown (f - \phi)|^2 < +\infty,$$

we have

$$\int_{B_{\sigma(x)}} |\bigtriangledown (f - \phi)|^2 \to 0 \quad \text{as} \quad x \to +\infty.$$

However

$$\begin{aligned} |\bigtriangledown \phi|^2 &= |\bigtriangledown_0 \phi|^2 + O\left(\frac{1}{r^{k+2}}\right) \\ &= (h')^2 + \frac{n-1}{r^2}\sin^2 h + O\left(\frac{1}{r^{k+2}}\right) \\ &= O\left(\frac{1}{r^2}\right), \end{aligned}$$

so, for a fixed $\sigma > 0$, $\int_{B_{\sigma(x)}} |\bigtriangledown f|^2 \to 0$, as $x \to +\infty$.

It follows from the ε–regularity theorem for minimizing harmonic maps in [12] that f is smooth outside $B_R(x_0)$ for some $R > 0$.

In order to get the regularity of f in the whole M, we need to show that $\mathrm{Im}(f, B_R) \subset \overset{\circ}{S}_+$.

Outside B_R , we have

$$\begin{cases} \triangle f^{n+1} = -|\bigtriangledown f|^2 f^{n+1} \leq 0 \\ f^{n+1}|_{\partial B_R} \geq 0 \end{cases}.$$

By maximum principle

$$a) \quad f^{n+1} > 0 \quad \text{on} \quad M\backslash\overline{B}_R$$

or

$$b) \quad f^{n+1} \equiv 0 \quad \text{on} \quad M \backslash B_R$$

In the case b), we have

$$\begin{aligned} \int_{M\backslash B_R} |h'|^2 \sin^2 h &\leq \int_{M\backslash B_R} |\bigtriangledown (f^{n+1} - \phi^{n+1})|^2 \\ &\leq \int_{M\backslash B_R} |\bigtriangledown (f - \phi)|^2 < +\infty. \end{aligned}$$

But

$$\begin{aligned} & \int_{M\backslash B_R} |h'|^2 \sin^2 h \\ = & \int_{M\backslash B_R} r^{2(\lambda_+ - 1)}(1 + O(1)) \sin^2 h \left(1 + O\left(\frac{1}{r^k}\right)\right) r^{n-1} dr d\theta \\ = & \int_{M\backslash B_R} r^{\sqrt{(n-2)^2 - 4(n-1)} - 1}(1 + O(1)) \sin^2 h \left(1 + O\left(\frac{1}{r^k}\right)\right) dr d\theta \\ = & +\infty. \end{aligned}$$

This contradiction shows that $f^{n+1} > 0$ outside B_R. For large R, f is energy minimizing map in $W^{1,2}_{f|\partial B_R}(B_R)$,

$$f(\partial B_R) \subset B_\rho(N_0) \subset \overset{\circ}{B}_{\frac{\pi}{2}}(N_0),$$

where $B_\rho(N_0)$ is the geodesic ball with radius ρ and center at the north pole N_0 of S^n.

By the maximum principle for energy minimizing maps (cf. [11]),

$$\mathrm{Im}(f, B_R) \subset B_\rho(N) \subset S^n_+,$$

so the well–known regularity theorem shows that f is regular on B_R.

Up to now we have proved that there exists a locally minimizing smooth harmonic map from M into S^n. Finally, we can get a sequence of this kind of map only if we take the relative function as

$$\phi_\lambda(x) = \begin{cases} \phi(\lambda x) & outside B_R \\ \phi(x) & inside \overline{U} \subset \overset{\circ}{B}_R \end{cases}$$

for $\lambda > 1$.

Remark 3.6 Let f_λ, f_μ be the locally minimizing map w.r.t. ϕ_λ, ϕ_μ respectively, $\lambda \neq \mu$, then $E(f_\lambda - f_\mu) = +\infty$, since $E_{\phi_\lambda}(f_\lambda), E_{\phi_\mu}(f_\mu) < +\infty$, but $E_{\phi_\lambda}(\phi_\mu) = +\infty$.

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