A Priori Estimates for Solutions of Fully Nonlinear Equations with Convex Level Set

LUIS A. CAFFARELLI & YU YUAN

ABSTRACT. We derive an a priori $C^{2,\alpha}$ estimate for solutions of the fully non-linear elliptic equation $F(D^2u) = 0$, provided the level set $\Sigma = \{M \mid F(M) = 0\}$ satisfies: (a) $\Sigma \cap \{M \mid \operatorname{Tr} M = t\}$ is strictly convex for all constants t; (b) the angle between the identity matrix I and the normal F_{ij} to Σ is strictly positive on the non-convex part of Σ. Moreover, we do not need any convexity assumption on F in the course of the proof for the two dimensional case, as the classical result indicates.

1. INTRODUCTION

In this note we derive an *a priori* $C^{2,\alpha}$ estimate for solutions of the fully nonlinear equation

$$F(D^2u) = 0$$

under the assumption on F that allows part of the level set $\Sigma = \{M \in \mathbb{R}^{n\times n} \mid F(M) = 0\}$ to be non-convex. Heuristically, one principle curvature of Σ can be negative. To be more precise, when we represent the level set Σ as a Lipschitz graph in the direction of the Laplacian, that is, $\Sigma = \{M + f(M)I \mid M \text{ is traceless}\}$, we ask f to have convex level surfaces and to be strictly increasing (or decreasing) across the level surfaces.

When F itself is convex or concave, the level set Σ is convex, and the $C^{2,\alpha}$ estimate is the well known result of Evans [E] and Krylov [K]. When F is the

Indiana University Mathematics Journal ©, Vol. 49, No. 2 (2000)

minimum of a concave and a convex function, the $C^{2,\alpha}$ estimate is obtained in [CC2].

We assume F is uniformly elliptic, that is $0 < \lambda\|N\| \le F(M+N) - F(M) \le (1/\lambda)\|N\|$ for any positive matrix N, or $\lambda I \le (F_{ij}) \le (1/\lambda)I$, where $F_{ij} = \partial F(M)/\partial m_{ij}$, $M = (m_{ij})$.

Theorem 1. *Let u be a smooth solution of $F(D^2u) = 0$. Assume that the level set of F, $\Sigma = \{M \in \mathbb{R}^{n\times n} \mid F(M) = 0\}$ satisfies, for any $\Lambda > 0$*

(1) (*Convex part*) *The parallel cross sections along the identity matrix direction I, are strictly convex, i.e., the second fundamental form of $\Sigma \cap \{M \mid TrM = const.\}$*

$$II \ge \omega(\Lambda) > 0,$$

(2) (*Non-convex part, strict transversality*) *The angle between I and the normal (F_{ij}) to Σ,*

$$\measuredangle(I, F_{ij}) \ge \Theta(\Lambda) > 0,$$

provided that $\|M\| \le \Lambda$.

Then for $0 < \alpha < 1$,

$$\|D^2u\|_{C^\alpha(B_{1/2})} \le C(n, \lambda, \alpha, \omega, \Theta, \|\nabla^2 F\|, \|D^2u\|_{L^\infty(B_1)}).$$

Remark 1. The dependence on $\|D^2u\|$ is exponential in the estimate of Theorem 1. Theorem 1 still holds if we replace the identity matrix I with any other positive matrix in the assumption.

To complete the theory, if we assume that F is either concave, convex, or close to a linear function near infinity, in terms of the geometry of Σ, Σ is then convex , concave, or "close" to a hyperplane near infinity. Following the Bernstein techniques as in [CC1], we can then control the $C^{1,1}$ norm of u.

Theorem 2. *Let u be a smooth solution of $F(D^2u) = 0$. Assume either $\nabla^2 F(D^2u) \ge 0$, $\nabla^2 F(D^2u) \le 0$, or $\|\nabla^2 F(D^2u)\| \le \delta(n,\lambda)/\|D^2u\|$ for $\|D^2u\| \ge \chi > 0$ and small $\delta(n,\lambda)$. Then*

$$\|u\|_{C^{1,1}(B_{1/2})} \le C(n,\lambda)(\|u\|_{L^\infty(B_1)} + |F(0)| + \chi).$$

Example 1. $F(D^2u) = 0$ without any convexity assumption on F in the two dimensional case. We will show that the proof of Theorem 1 applies.

Example 2. Let λ_1, λ_2, λ_3 be the eigenvalues of $D^2u(x)$ for $x \in \mathbb{R}^3$, let

$$Z = \frac{1}{\sqrt{3}}(\lambda_1 + \lambda_2 + \lambda_3),$$
$$R = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - Z^2}.$$

Set the symmetric function F in terms of λ_1, λ_2, λ_3:

$$F(\lambda_1, \lambda_2, \lambda_3) = Z - f(R),$$

and

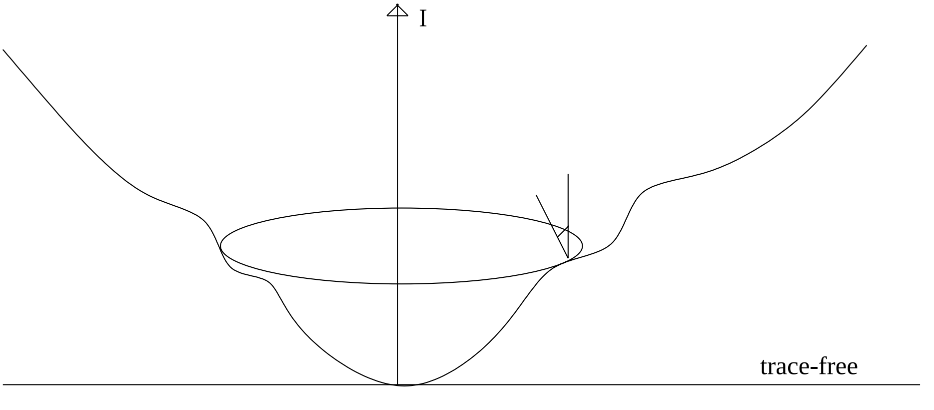
$$f(R) = \begin{cases} R^2 & \text{if } 0 \le R \le \dfrac{1}{4}, \\ \dfrac{1}{2}R - \dfrac{1}{16} + \varepsilon(R) & \text{if } R > \dfrac{1}{4}. \end{cases}$$

where ε, ε' are small (depending on λ, θ_1), the sign of $\varepsilon''(R)$ varies for $\frac{1}{4} < R \le 1$ (i.e. the level set Σ is neither convex nor concave), $\varepsilon''(R)$ becomes positive and small when $R > 1$.

The ellipticity of F is equivalent to $(\nabla f/|\nabla f|) \cdot (0,0,1) \ge \cos\theta(\lambda) > \sqrt{2/3}$, or $\tan\theta(\lambda) > f'$. The two conditions in the convexity assumptions of Theorem 1 become

(1) the second fundamental form $II = 1/R \ge \omega > 0$,
(2) $f' \ge \tan\Theta$.

We see that $F(\lambda_1, \lambda_2, \lambda_3) = Z - f(R)$ satisfies the assumptions in Theorem 1 and 2. In fact, we have a family of $F(\lambda_1, \lambda_2, \lambda_3) = Z - g(f(R))$ with $g'' \ge 0$, $\frac{1}{2} \le g' \le 1$ satisfying the assumptions for our theorems.



Notation

$$F_{ij}(A) = \frac{\partial F(A)}{\partial a_{ij}}, \quad F_{ij,k\ell}(A) = \frac{\partial^2 F(A)}{\partial a_{k\ell}\partial a_{ij}}, \quad \text{with } A = (a_{ij}).$$

We use the summation convention through out this note.

2. $C^{1,1}$ TO $C^{2,\alpha}$ ESTIMATES

In this section we prove that, for a classical solution u, we can deduce a $C^{2,\alpha}$ a priori estimate, controlled by the $C^{1,1}$ norm of u. The proof has two steps.

In step one, we show that "$e^{K\Delta u}$" is still, as in the convex case, a sub-solution of the linearized equation (Lemma 1). This forces the Hessian D^2u to concentrate (in measure) in one of the level surfaces of $f(M)$, that is, "Δu = constant" in a very large portion of (the normalized) unit ball $B_1(0)$.

The second step consists in showing that, if the assertion in step one happens, u is very close to a harmonic polynomial (Proposition 1), and that this "closedness" improves increasingly as we rescale (Proposition 2), thanks to the smoothness of $F(M)$ that makes it look, after rescaling, more and more like a linear operator around $M = D^2P_k$ (the k-th approximating polynomial).

Lemma 1. *Assume that $\|D^2u\| \le 1$, $F(D^2u) = 0$, and F satisfies the assumption of Theorem 1. Let $K = K(n, \lambda, \omega, \Theta, \|\nabla^2 F\|)$ sufficiently large. Then we have*

$$Le^{K\Delta u} = F_{ij}(D^2u)D_{ij}e^{K\Delta u} \ge 0.$$

Proof.

$$\begin{aligned} F_{ij}D_{ij}e^{K\Delta u} &= Ke^{K\Delta u}F_{ij}D_{ij}(\Delta u) + K^2e^{K\Delta u}F_{ij}D_i(\Delta u)D_j(\Delta u) \\ &= Ke^{K\Delta u}[\underbrace{-F_{ij,k\ell}D_{ij}u_\alpha D_{k\ell}u_\alpha}_{I} + \underbrace{KF_{ij}D_i(\Delta u)D_j(\Delta u)}_{II}], \end{aligned}$$

since

$$\begin{gathered} F_{ij}D_{ij}u_\alpha = 0, \\ F_{ij}D_{ij}u_{\alpha\alpha} + F_{ij,k\ell}D_{ij}u_\alpha D_{k\ell}u_\alpha = 0. \end{gathered}$$

Now F is elliptic, so the zero level set of F, $\Sigma = \{M \in \mathbb{R}^{n\times n} \mid F(M) = 0\}$ can be represented as a graph along the identity direction I in $\mathbb{R}^{n\times n}$, namely, $\Delta u - f(D^2u) = 0$, where f satisfies

$$\begin{gathered} f(D^2u + t\Delta uI) = f(D^2u) \quad \text{or} \quad f(M + tTrMI) = f(M) \\ \|\nabla^2 f\|_{L^\infty} \le C(n,\lambda)\|\nabla^2 F\|_{L^\infty}. \end{gathered}$$

Also we have

$$D_\alpha(\Delta u) = f_{ij}(D^2 u) D_{ij} u_\alpha,$$

then we decompose $D_{ij}u_\alpha = T_\alpha + [D_\alpha(\Delta u)](\nabla f/|\nabla f|^2) + \Delta u_\alpha I/\sqrt{n}$, with $T_\alpha \perp I$, $T_\alpha \perp \nabla f$, $\nabla f = (f_{ij})$.

$$\begin{aligned}
I &= f_{ij,k\ell} D_{ij} u_\alpha D_{k\ell} u_\alpha \\
&= \nabla^2 f\left(T_\alpha + D_\alpha(\Delta u)\frac{\nabla f}{|\nabla f|^2}, T_\alpha + D_\alpha(\Delta u)\frac{\nabla f}{|\nabla f|^2}\right) \\
&= \nabla^2 f(T_\alpha, T_\alpha) + 2\nabla^2 f\left(T_\alpha, D_\alpha(\Delta u)\frac{\nabla f}{|\nabla f|^2}\right) \\
&\qquad + \nabla^2 f\left(D_\alpha(\Delta u)\frac{\nabla f}{|\nabla f|^2}, D_\alpha(\Delta u)\frac{\nabla f}{|\nabla f|^2}\right) \\
&= |\nabla f|\, II(T_\alpha, T_\alpha) + 2\nabla^2 f\left(T_\alpha, D_\alpha(\Delta u)\frac{\nabla f}{|\nabla f|^2}\right) + \frac{|\nabla(\Delta u)|^2}{|\nabla f|^2} f_{\nabla f/|\nabla f|, \nabla f/|\nabla f|} \\
&\geq |\nabla f|\, II(T_\alpha, T_\alpha) - 2\|\nabla^2 f\|\, |T_\alpha|\, |D_\alpha(\Delta u)|\, \frac{1}{|\nabla f|} - \frac{\|\nabla^2 f\|}{|\nabla f|^2}\, |\nabla(\Delta u)|^2 \\
&\geq |\nabla f|\, II(T_\alpha, T_\alpha) - \omega(1)\, |\nabla f|\, |T_\alpha|\, |T_\alpha| \\
&\qquad - \frac{\|\nabla^2 f\|^2}{\omega(1)|\nabla f|^3} |\nabla(\Delta u)|^2 - \frac{\|\nabla^2 f\|}{|\nabla f|^2} |\nabla(\Delta u)|^2 \\
&\geq -\left(\frac{\|\nabla^2 f\|^2}{\omega(1)|\nabla f|^3} + \frac{\|\nabla^2 f\|}{|\nabla f|^2}\right) |\nabla(\Delta u)|^2.
\end{aligned}$$

Note that $|\nabla f| = \sqrt{n}\tan\theta$, where $\theta =$ the angle between the normal F_{ij} to the level set Σ and I. We know that $\theta \geq \Theta(1) > 0$, if we are away from the minimum point of f. Hence we have

$$I \geq -\left(\frac{\|\nabla^2 f\|^2}{\omega(1)\tan^3\Theta(1)} + \frac{\|\nabla^2 f\|}{\tan^2\Theta(1)}\right) |\nabla(\Delta u)|^2.$$

At the minimum point of f, $\nabla f = 0$, but f is convex there. So we have in the first place $I \geq 0$.

If we choose

$$K = K(n, \lambda, \omega, \Theta, \|\nabla^2 f\|) \geq \left(\frac{\|\nabla^2 f\|^2}{\omega(1)\tan^3\Theta(1)} + \frac{\|\nabla^2 f\|}{\tan^2\Theta(1)}\right)\frac{1}{\lambda},$$

then we have

$$I + II \geq \left[-\left(\frac{\|\nabla^2 f\|^2}{\omega(1)\tan^3\Theta(1)} + \frac{\|\nabla^2 f\|}{\tan^2\Theta(1)} \right) + K\lambda \right] |\nabla(\Delta u)|^2 \geq 0.$$

Therefore

$$Le^{K\Delta u} = F_{ij}(D^2u) D_{ij} e^{K\Delta u} \geq 0. \qquad \square$$

Remark 2. In the 2 dimensional case, i.e., $n = 2$, the conclusion of Lemma 1 is automatically true without any convexity assumption on F. This leads to a different proof of $C^{2,\alpha}$ estimates than that of Nirenberg [N].

Let $e \in \mathbb{R}^n$ with $|e| = 1$. Without loss of generality, we assume $u_{ee} = D_{11}u$. Now we represent $F(D^2u) = 0$ along $D_{22}u$, namely,

$$D_{22}u - f(D_{11}u, D_{12}u) = 0.$$

We have

$$\begin{aligned} Le^{Ku_{11}} &= Ke^{Ku_{11}} [f_{11,11} D_{11}u_1 D_{11}u_1 + 2f_{11,12} D_{11}u_1 D_{12}u_1 + f_{12,12} D_{12}u_1 D_{12}u_1 \\ &\qquad + KF_{ij} D_i u_{11} D_j u_{11}] \\ &\geq Ke^{Ku_{11}} [-\|\nabla^2 f\|(u_{111}^2 + u_{112}^2) + K\lambda(u_{111}^2 + u_{112}^2)]. \end{aligned}$$

Take $K = \|\nabla^2 f\|/\lambda$, consequently, we get

$$Le^{Ku_{11}} \geq 0.$$

Then a covering argument as in [CC1] leads from $C^{1,1}$ to a $C^{2,\alpha}$ estimate. The reader may also follow the lines in this note.

Lemma 2. *Let $F(D^2u) = 0$, then for any quadratic polynomial P, we can modify it to $\overline{P} = P + \frac{1}{2}sx^2$ so that*

$$F(D^2\overline{P}) = 0,$$
$$|s| \leq C(n, \lambda)\|u - P\|_{L^\infty(B_1)},$$
$$\|u - \overline{P}\|_{L^\infty(B_1)} \leq C(n, \lambda)\|u - P\|_{L^\infty(B_1)}.$$

Proof. Note that $F(D^2P) - F(D^2u) = F(D^2P)$, by the maximum principle, we have

$$|F(D^2P)| \leq C(n, \lambda)\|u - P\|_{L^\infty(B_1)}.$$

By the ellipticity of F, there exists s with

$$|s| \leq C(n, \lambda)\|u - P\|_{L^\infty(B_1)}$$

so that

$$F(D^2 P + sI) = 0.$$

Now let $\overline{P} = P + \frac{1}{2}sx^2$, we arrive at the conclusion of the above lemma. □

The next proposition shows how $D^2 u$ concentrates on a level surface $\{\Delta u = \text{const.}\}$, and u gets close to a polynomial.

Proposition 1. *Assume that* $\|D^2 u\|_{L^\infty(B_1)} \le 1$, $F(D^2 u) = 0$ *in* B_1, *and* F *satisfies the assumptions of Theorem 1. Then for any* $\varepsilon > 0$, *there exist* $\eta = \eta(n, \lambda, K, \varepsilon)$ *and a quadratic polynomial* P *so that*

$$\left| \frac{1}{\eta^2} u(\eta x) - P(x) \right| \le \varepsilon \quad \text{for } x \in B_1,$$
$$F(D^2 P) = 0.$$

Proof. Take ρ, ξ, δ, $k_0 > 0$ to be chosen later. Set

$$s_k = \sup_{x \in B_{1/2^k}} \Delta u(x), \quad 1 \le k \le k_0.$$

Since $\|D^2 u\|_{L^\infty(B_1)} \le 1$, we know $|\Delta u| \le \sqrt{n}$, then $|s_k| \le \sqrt{n}$. Consider the set

$$\{x \in B_{1/2^k} | \Delta u(x) \le s_k - \xi\},$$

one of the two cases happens.

CASE 1. There is $1 \le \ell \le k_0$ such that

$$|\{x \in B_{1/2^\ell} | \Delta u(x) \le s_\ell - \xi\}| \le \delta |B_{1/2^\ell}|.$$

CASE 2. For all $1 \le k \le k_0$

$$|\{x \in B_{1/2^k} | \Delta u(x) \le s_k - \xi\}| > \delta |B_{1/2^k}|.$$

CASE 1 IS EASY. Let

$$w_\ell(x) = 2^{2\ell} u\left(\frac{x}{2^\ell}\right),$$

solve

$$\begin{cases} \Delta v(x) = s_\ell & \text{in } B_1, \\ (x) w_\ell & \text{on } \partial B_1. \end{cases}$$

By the Alexandrov-Bakelman-Pucci maximum principle, we have

$$\|w_\ell - v\|_{L^\infty(B_1)} \le C(n)\|\Delta w_\ell - \Delta v\|_{L^n(B_1)} \le C(n)(\xi^n + \delta)^{1/n}.$$

Also

$$\begin{aligned} \|D^3 v\|_{L^\infty(B_{1/2})} &\le C(n) \sup_{x\in\partial B_1} \left| w_\ell(x) - w_\ell(0) - \langle \nabla w_\ell(0), x\rangle - \frac{1}{2} s_\ell |x|^2 \right| \\ &\le C(n)\left(\|D^2 u\|_{L^\infty(B_1)} + \frac{\sqrt{n}}{2} \right) \le C(n). \end{aligned}$$

If we take the quadratic part $\overline{P}$ of v at the origin, then

$$|w_\ell(x) - \overline{P}(x)| \le C(n)(\xi^n + \delta)^{1/n} + C(n)|x|^3.$$

CASE 2 INVOLVES THE WEAK HARNACK INEQUALITY. Applying the key Lemma 1 to $w_k(x) = 2^{2k} u(x/2^k)$ in B_1,

$$L[1 - e^{K(\Delta w_k - s_k)}] \le 0 \quad \text{in } B_1.$$

By the weak Harnack inequality which can be derived from [KS], see also [CC1, Theorem 4.8], we get

$$1 - e^{K(\Delta w_k - s_k)} \ge c(n,\lambda)\|1 - e^{K(\Delta w_k - s_k)}\|_{L^{p_0(n,\lambda)}(B_1)} \quad \text{in } B_{1/2}.$$

Then

$$\begin{aligned} \Delta w_k &\le s_k + \frac{1}{K}\ln[1 - c(n,\lambda)(1 - e^{-K\xi})\delta^{1/p_0}] \\ &\le s_k - c(n,\lambda)\frac{1}{K}(1 - e^{-K\xi})\delta^{1/p_0}. \end{aligned}$$

That is, Δw_k decreases by $\theta = c(n,\lambda)(1/K)(1-e^{-K\xi})\delta^{1/p_0}$ each time. After $k_0 = (\sup \Delta u - \inf \Delta u)/\theta = C(n)/\theta$ steps,

$$s_{k_0} - \xi \le \Delta w_{k_0}(x) \le s_{k_0} \quad \text{for } x \in B_1.$$

Solve

$$\begin{cases} \Delta v(x) = s_{k_0} & \text{in } B_1, \\ v(x) = w_{k_0} & \text{on } \partial B_1. \end{cases}$$

Then

$$\|w_{k_0} - v\|_{L^\infty(B_1)} \le C(n)\xi$$

by the maximum principle. Also

$$\begin{aligned} \|D^3 v\|_{L^\infty(B_{1/2})} &\le C(n) \sup_{x\in\partial B_1} \left| w_{k_0}(x) - w_{k_0}(0) - \langle \nabla w_{k_0}(0), x\rangle - \frac{1}{2} s_{k_0}|x|^2 \right| \\ &\le C(n)\left(\|D^2 u\|_{L^\infty(B_1)} + \frac{\sqrt{n}}{2}\right) \le C(n). \end{aligned}$$

We take the quadratic part $\overline{P}$ of v at the origin, then

$$|w_{k_0}(x) - \overline{P}(x)| \le C(n)(\xi + |x|^3).$$

Now let $x = \rho y$, $\overline{\overline{P}}(y) = (1/\rho^2)\overline{P}(\rho y)$. For $|y| \le 1$, from case 1 or case 2, we either have

$$\left| \frac{1}{\rho^2} w_\ell(\rho y) - \overline{\overline{P}}(y) \right| \le \frac{C(n)(\xi^n + \delta)^{1/n}}{\rho^2} + C(n)\rho$$

or

$$\left| \frac{1}{\rho^2} w_{k_0}(\rho y) - \overline{\overline{P}}(y) \right| \le C(n,\lambda)\left(\frac{\xi}{\rho^2} + \rho \right).$$

Since both $(1/\rho^2)w_\ell(\rho y)$ and $(1/\rho^2)w_{k_0}(\rho y)$ satisfy $F(D^2u) = 0$, by the modifying Lemma 2, we perturb $\overline{\overline{P}}(y)$ to another quadratic polynomial $P(y)$ so that

$$F(D^2 P) = 0,$$

with

$$\left|\frac{1}{\rho^2}w_\ell(\rho y)-P(y)\right|\le C(n,\lambda)\left\{\frac{[\xi^n+\delta]^{1/n}}{\rho^2}+\rho\right\}$$

or

$$\left|\frac{1}{\rho^2}w_{k_0}(\rho y)-P(y)\right|\le C(n,\lambda)\left(\frac{\xi}{\rho^2}+\rho\right).$$

Finally we choose ρ, then ξ, δ, k_0 depending on n, λ, K, ε so that

$$\left|\frac{1}{\eta^2}u(\eta y)-P(y)\right|\le\varepsilon$$

where $\eta=\eta(n,\lambda,K,\varepsilon)=\rho/2^\ell$ or $\rho/2^{k_0}$. ◻

Remark 3. The weak point of Proposition 1 is that η depends on K. Recall that K depends on $\|\nabla^2 f\|$. So this is a drawback!

Finally, Proposition 2 indicates the inductive process by which, once u is close to a polynomial, it becomes $C^{2,\alpha}$.

Proposition 2. *There exist positive μ, m depending on n, λ, $\|\nabla^2 f\|_{L^\infty}$, and α so that, if $\|u-P\|_{L^\infty(B_1)}\le\mu^{2+\alpha+m}$ and $F(D^2u)=F(D^2P)=0$ in B_1, then we have a family of polynomials $P_k=a^k+\langle b^k,x\rangle+\frac{1}{2}x^tC^kx$ satisfying*

(i) $\|u-P_k\|_{L^\infty(B_{\mu^k})}\le\mu^{k(2+\alpha)+m}$,
(ii) $|a^k-a^{k+1}|$, $\mu^k|b^k-b^{k+1}|$, $\mu^{2k}|C^k-C^{k+1}|\le C(n,\lambda)\mu^{k(2+\alpha)+m}$,
(iii) $F(C^k)=0$.

Proof. We follow [C]. Let $P_1=P$, we prove this proposition by induction. Set

$$w(x)=\frac{(u-P_k)(\mu^kx)}{\mu^{k(2+\alpha)+m}}\quad\text{for } x\in B_1,$$

then $|w(x)|\le 1$ and for $F^k(M)=(1/\mu^{k\alpha+m})F(\mu^{k\alpha+m}M+C^k)$,

$$F^k(D^2w)=\frac{1}{\mu^{k\alpha+m}}F(D^2u(\mu^kx))=0.$$

Let v be the solution of

$$\begin{cases}F^k_{ij}(0)D_{ij}v=0 & \text{in } B_{3/4}\\ v=w & \text{on } \partial B_{3/4}\end{cases}$$

We see

$$\begin{aligned}F^k(D^2v) &= F^k(0) + F^k_{ij}(0)D_{ij}v + O(\|\nabla^2 f\|\,\|D^2v\|^2\mu^{k\alpha+m})\\&= O(\|\nabla^2 f\|\,\|D^2v\|^2\mu^{k\alpha+m}).\end{aligned}$$

Using the interior Hölder estimate on w with $\beta = \beta(n,\lambda)$, for example, Proposition 4.10 in [CC1],

$$\|w\|_{C^\beta(\bar{B}_{3/4})} \leq C(n,\lambda).$$

By the global Hölder estimate on v,

$$\|v\|_{C^{\beta/2}(\bar{B}_{3/4})} \leq C(n,\lambda)\|w\|_{C^\beta(\partial B_{3/4})} \leq C(n,\lambda).$$

Applying the interior estimate on v, we have

$$\begin{aligned}\|v\|_{C^3(B_{1/2})} &\leq C(n,\lambda),\\ \|D^2v\|_{L^\infty(B_{3/4-\delta})} &\leq C(n,\lambda)\delta^{-2}.\end{aligned}$$

By the maximum principle, we have

$$\begin{aligned}\|w-v\|_{L^\infty(B_{3/4-\delta})} &\leq \sup_{\partial B_{3/4-\delta}}|w-v| + C(n,\lambda)\|F^k(D^2w)-F^k(D^2v)\|_{L^\infty(B_{3/4-\delta})}\\&\leq C(n,\lambda)[(\delta^\beta+\delta^{\beta/2}) + \|\nabla^2 f\|\mu^{k\alpha+m}\delta^{-4}].\end{aligned}$$

Now take $\overline{P}$ to be the quadratic part of v at the origin, we have

$$\|w-\overline{P}\|_{L^\infty(B_\mu)} \leq C(n,\lambda)[\mu^3 + (\delta^\beta+\delta^{\beta/2}) + \|\nabla^2 f\|\mu^{k\alpha+m}\delta^{-4}].$$

Since $F^k(D^2w) = 0$, by the modifying Lemma 2 properly scaled, we perturb $\overline{P}$ to another quadratic $\overline{\overline{P}}$ so that $F^k(D^2\overline{\overline{P}}) = 0$ with

$$\|w-\overline{\overline{P}}\|_{L^\infty(B_\mu)} \leq C(n,\lambda)[\mu^3 + (\delta^\beta+\delta^{\beta/2}) + \|\nabla^2 f\|\mu^{k\alpha+m}\delta^{-4}].$$

We finally choose μ, then δ and m, depending on n, λ, $\|\nabla^2 f\|$, and α so that

$$\|w-\overline{\overline{P}}\|_{L^\infty(B_\mu)} \leq \mu^{2+\alpha}.$$

Rescaling back, we get

$$\|u - P_k - \mu^{k(2+\alpha)+m}\overline{\overline{P}}(\mu^{-k}x)\|_{L^\infty(B_{\mu^{k+1}})} \leq \mu^{(k+1)(2+\alpha)+m}.$$

Let $P_{k+1} = P_k + \mu^{k(2+\alpha)+m}\overline{\overline{P}}(\mu^{-k}x)$. We see that (i), (ii), and (iii) hold. □

Proof of Theorem 1. We apply Proposition 1 and Proposition 2 to $\bar{u} = u/\|D^2u\|$ and $\bar{F} = (1/\|D^2u\|)F(\|D^2u\|\cdot)$. From (i)-(ii) in Proposition 2, we see that the family of polynomials $\{P_k\}$ converges uniformly to a polynomial $Q(x)$ satisfying

$$\left|\frac{1}{\eta^2\|D^2u\|}u(\eta x) - Q(x)\right| \le C(n,\lambda)|x|^{2+\alpha}.$$

Let $y = \eta x$; we get

$$\left|u(y) - \|D^2u\|\eta^2 Q\left(\frac{y}{\eta}\right)\right| \le C(n,\lambda)\frac{\|D^2u\|}{\eta^\alpha}|y|^{2+\alpha}.$$

Hence D^2u is C^α at the origin. Similarly, one proves D^2u is C^a at every point inside $B_{1/2}$. Therefore

$$\|D^2u\|_{C^\alpha(B_{1/2})} \le C(n,\lambda,\alpha,\omega,\Theta,\|\nabla^2F\|,\|D^2u\|_{L^\infty(B_1)}).$$

If we keep track the constants, we see that the C^α norm of D^2u depends on $\|D^2u\|_{L^\infty}$ exponentially. ◻

3. $C^{1,1}$ ESTIMATES

To complete our work, we indicate some cases in which, from the behavior of F at infinity we can deduce $C^{1,1}$ a priori estimates for bounded solution of $F(D^2u) = 0$.

Proof of Theorem 2. We first discuss the case $\nabla^2 f \ge 0$ near infinity, the case $\nabla^2 f \le 0$ is similar. We take a function $\varphi \in C_0^\infty(B_{3/4})$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ in $B_{1/2}$. Consider for $e \in \mathbb{R}^n$ with $|e| = 1$ and γ large to be chosen later

$$g(x) = \gamma u_e^2 + \varphi^2 u_{ee}^2$$

in $\Omega = \{x \in B_{3/4} \mid u_{ee} > \chi\}$.

We know that $F_{ij}(D^2u)D_{ij}u_e = 0$ and

$$F_{ij}(D^2u)D_{ij}u_{ee} = -F_{ij,k\ell}D_{ij}u_eD_{k\ell}u_e \ge 0$$

for $\|D^2u\| \geq \chi$, since $F(D^2u)$ is concave when $\|D^2u\|$ large. We compute $F_{ij}D_{ij}g(x)$ in Ω.

$$\begin{aligned} F_{ij}D_{ij}g(x) &= 2\gamma u_e F_{ij}D_{ij}u_e + 2\gamma F_{ij}D_iu_eD_ju_e + 2\varphi^2 u_{ee}F_{ij}D_{ij}u_{ee} \\ &\quad + 2\varphi^2 F_{ij}D_iu_{ee}D_ju_{ee} + u_{ee}^2F_{ij}D_{ij}\varphi^2 + 4\varphi u_{ee}F_{ij}D_iu_{ee}D_j\varphi \\ &\geq 2\gamma F_{ij}D_iu_eD_ju_e + 2\varphi^2F_{ij}D_iu_{ee}D_ju_{ee} + u_{ee}^2F_{ij}D_{ij}\varphi^2 \\ &\quad - 4u_{ee}^2F_{ij}D_i\varphi D_j\varphi - \varphi^2F_{ij}D_iu_{ee}D_ju_{ee} \\ &\geq 2\gamma\lambda|\nabla u_e|^2 + (F_{ij}D_{ij}\varphi^2 - 4F_{ij}D_i\varphi D_j\varphi)u_{ee}^2 \geq 0 \end{aligned}$$

if we choose $\gamma = \gamma(\lambda) > 0$ large enough.

Note that $\partial\Omega \subset \partial B_{3/4} \cup \{x \in B_{3/4} \mid u_{ee}(x) = \chi\}$, applying the maximum principle, we get

$$\sup_{\Omega} g \leq \sup_{\partial\Omega} g \leq \gamma \sup_{B_{3/4}} u_e^2 + \chi^2.$$

So

$$\sup_{B_{1/2}} (u_{ee}^+)^2 = \sup_{\Omega\cap B_{1/2}} u_{ee}^2 \leq \sup_{\Omega} g \leq \gamma \sup_{B_{3/4}} u_e^2 + \chi^2.$$

In order to bound u_e^2 in terms of $\|u\|_{L^\infty}$, we can employ a similar Bernstein technique as in [CC1, p.87], or use Corollary 5.7 in [CC1] without any concavity assumption on F,

$$\|\nabla u\|_{L^\infty(B_{3/4})} \leq C(n,\lambda)(\|u\|_{L^\infty(B_1)} + |F(0)|).$$

Hence

$$\sup_{B_{1/2}} u_{ee}^+ \leq C(n,\lambda)(\|u\|_{L^\infty(B_1)} + |F(0)| + \chi).$$

By Lemma 6.4 in [CC1], we have

$$\|D^2u\|_{L^\infty(B_{1/2})} \leq C(n,\lambda)(\|u\|_{L^\infty(B_1)} + |F(0)| + \chi).$$

Therefore,

$$\|u\|_{C^{1,1}(B_{1/2})} \leq C(n,\lambda)(\|u\|_{L^\infty(B_1)} + |F(0)| + \chi).$$

Next we discuss the case $\|\nabla^2 f(D^2u)\| \le \delta(n,\lambda)/\|D^2u\|$ for $\|D^2u\| \ge \chi$. We still have $F_{ij}(D^2u)D_{ij}u_e = 0$ and $F_{ij}(D^2u)D_{ij}u_{ee} = -F_{ij,k\ell}D_{ij}u_eD_{k\ell}u_e$. Consider

$$h(x) = \gamma \sum_{e\in\Gamma} u_e^2 + \varphi^2 \sum_{e\in\Gamma} u_{ee}^2$$

in $\Omega = \{x \in B_{3/4} \,|\, \|D^2u\| > \chi\}$, where γ, φ are the same as above, and

$$\Gamma = \left\{e_1,\ e_2,\ \dots,\ e_n,\ \frac{1}{\sqrt{2}}(e_i \pm e_j),\ 1 \le i < j \le n\right\}.$$

We compute $F_{ij}D_{ij}h(x)$ in Ω.

$$\begin{aligned}
&F_{ij}D_{ij}h(x)\\
&\quad= 2\gamma \sum_{e\in\Gamma}(u_eF_{ij}D_{ij}u_e + F_{ij}D_iu_eD_ju_e) + \sum_{e\in\Gamma}\Big[2\varphi^2u_{ee}F_{ij}D_{ij}u_{ee}\\
&\qquad\qquad + 2\varphi^2F_{ij}D_iu_{ee}D_ju_{ee} + u_{ee}^2F_{ij}D_{ij}\varphi^2 + 4\varphi u_{ee}F_{ij}D_iu_{ee}D_j\varphi\Big]\\
&\quad\ge 2\gamma\lambda \sum_{e\in\Gamma}|\nabla u_e|^2 + \sum_{e\in\Gamma}\Big[-2\varphi^2u_{ee}F_{ij,k\ell}(D^2u)D_{ij}u_eD_{k\ell}u_e + 2\varphi^2F_{ij}D_iu_{ee}D_ju_{ee}\\
&\qquad\qquad + u_{ee}^2F_{ij}D_{ij}\varphi^2 - 4u_{ee}^2F_{ij}D_i\varphi D_j\varphi - \varphi^2F_{ij}D_iu_{ee}D_ju_{ee}\Big]\\
&\quad= 2\gamma\lambda \sum_{e\in\Gamma}|\nabla u_e|^2 + \sum_{e\in\Gamma}u_{ee}^2(F_{ij}D_{ij}\varphi^2 - 4F_{ij}D_i\varphi D_j\varphi)\\
&\qquad\qquad + \varphi^2\Big[\sum_{e\in\Gamma}F_{ij}D_iu_{ee}D_ju_{ee} - \sum_{e\in\Gamma}2u_{ee}F_{ij,k\ell}(D^2u)D_{ij}u_eD_{k\ell}u_e\Big]\\
&\quad\ge 2\gamma\lambda \sum_{e\in\Gamma}|\nabla u_e|^2 + \sum_{e\in\Gamma}u_{ee}^2(F_{ij}D_{ij}\varphi^2 - 4F_{ij}D_i\varphi D_j\varphi)\\
&\qquad\qquad + \varphi^2\Big[\lambda\sum_{e\in\Gamma}|\nabla u_{ee}|^2 - 2\delta(n,\lambda)\sum_{e\in\Gamma}\|D^2u_e\|^2\Big] \ge 0
\end{aligned}$$

if we choose $\gamma = \gamma(\lambda) > 0$ large enough and $\delta(n,\lambda) > 0$ small enough. Again by the maximum principle, we have

$$\sup_{\Omega} h \le \sup_{\partial\Omega} h \le \gamma n^2 \sup_{B_{3/4}} |\nabla u|^2 + n^2\chi^2.$$

So

$$\sup_{B_{1/2}} \|D^2u\| \le \gamma n \sup_{B_{3/4}} |\nabla u| + n\chi \le C(n,\lambda)(\|u\|_{L^\infty(B_1)} + |F(0)| + \chi),$$

where we again use Corollary 5.7 in [CC1]. $\square$

Acknowledgement. Both authors were supported, in part, by grants from the National Science Foundation.

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Department of Mathematics
The University of Texas at Austin
Austin, TX 78712, U. S. A.
EMAIL: caffarel@math.utexas.edu
EMAIL: yyuan@math.utexas.edu

SUBJECT CLASSIFICATION: Primary 35J
KEYWORDS: fully nonlinear equation, Hölder continuity, convexity

Submitted: November 17th, 1999.