

Lecture 7 Calderon-Zygmund for Δ || Schauder for Δ

- egs
- L^2 weak 2-2
- Weak 1-1
- interpolation to L^p
- L^p for $a_{ij}D_{ij}$ w/ $a_{ij} \in C^0$, $b, c \in L^\infty$
- Bdry/Global L^p
- L^2 for $\sum_{i,j}^2 a_{ij}D_{ij}$ w/ $a_{ij} \in L^\infty$ & elliptic

Question: $\Delta u = f \in L^p \stackrel{?}{\Rightarrow} D^2u \in L^p$

eg1. $p = 2$ $\widehat{D_{ij}u} = -\xi_i \xi_j \widehat{u} = \frac{\xi_i \xi_j}{|\xi|^2} \widehat{f}$ implies $\|D_{ij}u\|_{L^2} \leq \left\| \widehat{f} \right\|_{L^2} = \|f\|_{L^2}$.

eg2. $p = \infty$ Let $f = \chi_{\{x_1 x_2 > 0\} \cap B_1}$, $u = \Gamma * f$, recall $D_{12}u \notin L^\infty$.

eg3. $p = 1$ Let $f_\varepsilon = \varphi_\varepsilon(x) = \varphi(x/\varepsilon)/\varepsilon^n$ with $\varphi(x) = \varphi(|x|) \in C_0^\infty(R^n)$ and $\int \varphi = 1$. Then the convolution with fundamental solution

$$u_\varepsilon = \Gamma * f_\varepsilon = \begin{cases} \Gamma & |x| \geq \varepsilon \\ \sim & |x| < \varepsilon \end{cases}.$$

Then

$$D_{ij}u_\varepsilon = \frac{x_i x_j}{|x|^{n+2}} \text{ for } |x| \geq \varepsilon.$$

It follows

$$C \|f_\varepsilon\|_{L^1} \not\leq \|D_{ij}u_\varepsilon\|_{L^1(B_1)} > \int_{B_1/B_\varepsilon} \frac{|x_i x_j|}{|x|^{n+2}} > \int_{B_1/B_\varepsilon} \frac{1}{|x|^n} > \ln \varepsilon \rightarrow \infty.$$

Recall for $\varphi \in C_0^\infty(R^n)$, $u = \Gamma * \varphi$

$$D_{ij}u = \begin{cases} c_n \int_{R^n} \frac{y_i y_j}{|y|^{n+2}} \varphi(x-y) dy & i \neq j \\ c_n \int_{R^n} \frac{y_i^2 - n^{-1}|y|^2}{|y|^{n+2}} \varphi(x-y) dy + \frac{1}{n} \varphi(x) & i = j \end{cases}.$$

Next the above convolution $u = \Gamma * f$ and $D_{ij}u$ also make sense for $f \in L^p$. Indeed by Young

$$\|\Gamma * f\|_p \leq \|\Gamma\|_1 \|f\|_p.$$

We define $D_{ij}u$ by approximating f with $f_\varepsilon \in C_0^\infty(R^n)$ in L^p , provided the singular integrals have the right L^p estimates. For example, $p = 2$ is fine as in eg1 $\|D_{ij}u\|_{L^2} \leq \left\| \widehat{f} \right\|_{L^2} = \|f\|_{L^2}$.

Theorem. Let $u = \Gamma * f$ and $f \in L^p(\Omega)$ with Ω bounded and also $1 < p < \infty$. Then we have

- a) $\|D^2u\|_{L^2(R^n)} \leq \|f\|_{L^2(\Omega)}$
- b) $|\{ |D^2u| > t \} \cap \Omega| \leq \frac{C}{t} \|f\|_{L^1(\Omega)} \quad (\text{weak 1-1})$
- c) $\|D^2u\|_{L^p(\Omega)} \leq C_p \|f\|_{L^p(\Omega)}$

Proof. a)

First way. Fourier transform.

Second way. We approximate $f \in L^2(\Omega)$ by $f \in C_0^\infty(R^n)$, then as before $\Delta u = \Delta \Gamma * f = f$. By Green's identity

$$\begin{aligned}
\int_{B_R} \Delta u \Delta u &= \int_{\partial B_R} \Delta u u_\gamma - \int_{B_R} \nabla \Delta u \cdot \nabla u \\
&= \overbrace{\int_{\partial B_R} f u_\gamma}^0 - \int_{B_R} \Delta u_k u_k \\
&= - \int_{\partial B_R} \partial_\gamma u_k u_k + \int_{B_R} \nabla u_k \cdot \nabla u_k \\
&= \sim \frac{1}{R^n} \frac{1}{R^{n-1}} R^{n-1} + \int_{B_R} |D^2 u|^2 \\
&\rightarrow \int_{R^n} |D^2 u|^2 \quad \text{as } R \rightarrow \infty,
\end{aligned}$$

where we used the fact for f with bounded support

$$\begin{aligned}
u_k(x) &= \int_{\Omega} \frac{x_k - y_k}{|x - y|^n} f(y) dy \sim \frac{1}{R^{n-1}} \\
u_{kj}(x) &= \int_{\Omega} \frac{(x_k - y_k)(x_j - y_j)}{|x - y|^{n+2}} f(y) dy \sim \frac{1}{R^n} \\
u_{ii} &= \text{similar.}
\end{aligned}$$

b)

Idea. W.L.O.G. Assume $f \geq 0$.

Warm up:

$$\begin{aligned}
\int_{\Omega} f dx &= \int_0^\infty \underbrace{|\{f > t\}|}_{\sim 1/t} dt \\
|\{f > t\}| &\leq \int_{\Omega} \frac{f}{t} = \frac{1}{t} \|f\|_{L^1(\Omega)} \\
|\{f > t\}| &\leq \int_{\Omega} \frac{f^p}{t^p} = \frac{1}{t^p} \|f\|_{L^p(\Omega)}^p
\end{aligned}$$

We split f in two parts: f itself where $f \leq t$ and its average where $f > t$.

figure f itself where $f \leq t$ and its average where $f > t$

In $\{f \leq t\}$, the distribution of $K * f = D_{ij} \Gamma * f$ can be controlled by L^2 estimate of L^2 lower part in a);

In $\{f > t\}$, the distribution of $K * f$ can be taken care of by the cancellation effect of the average and the integrability for the gradient of singular kernel DK near ∞ .

Now Calderon-Zygmund decomposition.

Step0. Fix t , choose large enough cube Q_0 (possibly larger than Ω) s.t.

$$\frac{1}{|Q_0|} \int_{Q_0} f < t.$$

Step1. Split Q_0 in half along all axis directions and check $\oint_Q f$ on each subcube Q :

figure cubes collected and cubes splitting

$$\begin{aligned} &\text{If } \oint_Q f \geq t, \text{ collect } Q; \\ &\text{if } \oint_Q f < t, \text{ keep splitting.} \end{aligned}$$

For the collection of cubes Q_j and their predecessor $\tilde{Q}_j$ we know

$$t \leq \frac{1}{|Q_j|} \int_{Q_j} f \leq \frac{2^n}{2^n |Q_j|} \int_{\tilde{Q}_j} f = 2^n \oint_{\tilde{Q}_j} f < 2^n t.$$

Set (good) set $G = \Omega \setminus \cup_j Q_j$. For a.e. $x \in G$, there exist a sequence of cubes Q shrinking to x such that $\oint_Q f < t$, thus by Lebesgue $f(x) \leq t$ a.e. in G . Then (good) function

$$g(x) = \begin{cases} f(x) & x \in G \\ \oint_{Q_j} f & x \in Q_j \end{cases} \leq 2^n t \quad \text{a.e. in } \Omega.$$

Its square integral

$$\int_{\Omega} g^2 \leq 2^n t \int_{\Omega} g = 2^n t \int_{\Omega} f = 2^n t \|f\|_{L^1}.$$

Step2. Let (bad) function $b = f - g = \begin{cases} 0 & x \in G \\ f - \oint_{Q_j} f & x \in Q_j \end{cases}$. Then

$$K * f = K * (g + b) = K * g + K * b$$

and

$$\{K * f > t\} \subset \left\{K * g > \frac{t}{2}\right\} \cup \left\{K * b > \frac{t}{2}\right\}.$$

From a) and Step1

$$\left| \left\{K * g > \frac{t}{2}\right\} \right| \leq \frac{1}{\left(\frac{t}{2}\right)^2} \|K * g\|_{L^2(\Omega)}^2 \leq \frac{1}{\left(\frac{t}{2}\right)^2} \|g\|_{L^2(\Omega)}^2 \leq \frac{2^{n+1}}{t} \|f\|_{L^1(\Omega)}.$$

Next we cover the latter set (go with $2Q_j$ for easy calculation of following improper integrals)

$$\left\{ K * b > \frac{t}{2} \right\} \subset \bigcup_j 2Q_j \cup \left(\left\{ K * b > \frac{t}{2} \right\} \setminus \bigcup_j 2Q_j \right).$$

For $\bigcup_j 2Q_j$, summing the collection criterion

$$\sum_j : |Q_j| < \frac{1}{t} \int_{Q_j} f$$

we get

$$\left| \bigcup_j 2Q_j \right| \leq 2^n \left| \bigcup_j Q_j \right| < \frac{2^n}{t} \|f\|_{L^1(\Omega)}.$$

For $x \in \Omega \setminus 2Q_j$, the convolution

$$\begin{aligned} K * b(x) &= \int_{\Omega} K(x-y) b(y) dy = \int_{Q_j} K(x-y) b(y) dy \\ &= \int_{Q_j} K(x-y) b(y) dy \\ &= \int_{Q_j} [K(x-y) - K(x-y_j)] b(y) dy \quad \text{as } \int_{Q_j(y_j)} b = 0 \\ &\leq \int_{Q_j} \frac{C_n \text{diam} Q_j}{|x-y_j|^{n+1}} b(y) dy \\ &\leq \frac{C_n \text{diam} Q_j}{|x-y_j|^{n+1}} 2 \|f\|_{L^1(Q_j)} \quad \text{since } \left| \int_{Q_j} b(y) dy \right| \leq 2 \int_{Q_j} |f| dy. \end{aligned}$$

Integrate

figure: 2 concentric cubes nesting a concentric sphere

$$\begin{aligned} \int_{\Omega \setminus \bigcup_j 2Q_j} |K * b(x)| dx &\leq \sum_j \int_{\Omega \setminus 2Q_j} \frac{C_n \text{diam} Q_j}{|x-y_j|^{n+1}} 2 \|f\|_{L^1(Q_j)} dx \\ &\leq \sum_j 2 \|f\|_{L^1(Q_j)} \int_{|x-y_j| \geq \text{diam} Q_j / 2} \frac{C_n \text{diam} Q_j}{|x-y_j|^{n+1}} dx \\ &= \sum_j 2 \|f\|_{L^1(Q_j)} \frac{C_n \text{diam} Q_j}{r} \Big|_{\infty}^{\text{diam} Q_j / 2} \\ &= \sum_j C_n \|f\|_{L^1(Q_j)} \leq C_n \|f\|_{L^1(\Omega)}. \end{aligned}$$

It follows that

$$\left| \left\{ K * b > \frac{t}{2} \right\} \setminus \bigcup_j 2Q_j \right| \leq \frac{C_n}{t/2} \|f\|_{L^1(\Omega)}$$

and then

$$\left| \left\{ K * b > \frac{t}{2} \right\} \right| \leq \frac{C_n}{t} \|f\|_{L^1(\Omega)}.$$

Therefore from this and the distribution estimate for $K * g$ in the beginning, we obtain

$$\begin{aligned} |\{K * f > t\}| &\leq \left| \left\{ K * g > \frac{t}{2} \right\} \right| + \left| \left\{ K * b > \frac{t}{2} \right\} \right| \\ &\leq \frac{C_n}{t} \|f\|_{L^1(\Omega)}. \end{aligned}$$

c) Case $1 < p < 2$.

Again, w.l.o.g. assume $L^p f \geq 0$ a.e. First

$$\begin{aligned} \|K * f\|_{L^p(\Omega)}^p &= \int_{\Omega} |K * f|^p dx = \int_0^\infty |\{ |K * f|^p > s \}| ds \\ &\stackrel{s=t^p}{=} \int_0^\infty |\{ |K * f| > t \}| p t^{p-1} dt. \end{aligned}$$

Split f into two parts along t line

figure above t f_1 below t f_2

$$f_1 = \begin{cases} f & f \geq t \\ 0 & f < t \end{cases} \quad \text{and} \quad f_2 = \begin{cases} 0 & f \geq t \\ f & f < t \end{cases}.$$

By weak 1-1 b) and L^2 estimate a), we have

$$\begin{aligned} |\{ |K * f| > t \}| &\leq \left| \left\{ |K * f_1| > \frac{t}{2} \right\} \right| + \left| \left\{ |K * f_2| > \frac{t}{2} \right\} \right| \\ &\leq \frac{2C_1}{t} \|f_1\|_{L^1} + \frac{2C_2}{t^2} \|f_2\|_{L^2}^2 \\ &= \frac{2C_1}{t} \int_{\{f \geq t\}} f dx + \frac{2C_2}{t^2} \int_{\{f < t\}} f^2 dx. \end{aligned}$$

Integrating, we have

$$\begin{aligned} \int_0^\infty \left(\frac{2C_1}{t} \int_{\{f \geq t\}} f dx \right) p t^{p-1} dt &= 2C_1 p \int_{\Omega} f \int_0^f t^{p-2} dt dx = 2C_1 p \int_{\Omega} f \frac{f^{p-1}}{p-1} dx \\ &= \frac{2C_1 p}{p-1} \int_{\Omega} f^p dx \end{aligned}$$

and

$$\begin{aligned} \int_0^\infty \left(\frac{2C_2}{t^2} \int_{\{f < t\}} f^2 dx \right) p t^{p-1} dt &= 2C_2 p \int_{\Omega} f^2 \int_f^\infty t^{p-3} dt dx = 2C_2 p \int_{\Omega} f^2 \frac{f^{p-2}}{2-p} dx \\ &= \frac{2C_2 p}{2-p} \int_{\Omega} f^p dx. \end{aligned}$$

Thus

$$\|K * f\|_{L^p(\Omega)}^p \leq p \left(\frac{2C_1}{p-1} + \frac{4C_2}{2-p} \right) \|f\|_{L^p(\Omega)}^p.$$

RMK. The coefficient $C(p) \sim \frac{1}{p-1}$ blows up for p near 1 as expected. But $C(p) \sim \frac{1}{2-p}$ for p near 2 is bad.

Question/Exercise. Make $C(p)$ bounded near 2. Hint: as above interpolate between L^1 and L^4 .

Case $2 < p < \infty$

Let $g \in L^{p'}(\Omega)$ with $\frac{1}{p'} + \frac{1}{p} = 1$. Let us do a duality argument,

$$\begin{aligned} \int_{\Omega} K * f(x) g(x) dx &= \int_{\Omega} \int_{\Omega} K(x-y) f(y) dy g(x) dx \\ &= \int_{\Omega} \int_{\Omega} K(x-y) g(x) dx f(y) dy \\ &= \int_{\Omega} \int_{\Omega} \bar{K}(y-x) g(x) dx f(y) dy \quad \text{with } \bar{K}(x) = K(-x) \\ &\leq \|\bar{K} * g\|_{L^{p'}} \|f\|_{L^p} \\ &\leq C(p') \|g\|_{L^{p'}} \|f\|_{L^p}. \end{aligned}$$

Thus

$$\|K * f\|_{L^p(\Omega)} \leq C(p') \|f\|_{L^p}.$$

RMK. To get bounded $C(p)$ for p near 2, one can also do a strong interpolation between say $L^{1.5}$ and L^4 by Riesz convexity theorem: Given $\|K * f\|_p \leq C_p \|f\|_p$ and $\|K * f\|_q \leq C_q \|f\|_q$. Then

$$\|K * f\|_r \leq C_p^\alpha C_q^{1-\alpha} \|f\|_r \quad \text{with } \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}.$$

Or the coefficient C_r is log convex in terms of $\frac{1}{r}$.

Model equation: $\Delta u = f \in L^p$ in B_1 :

$$\Delta(u - \Gamma * f) = 0$$

then the the interior estimate for harmonic function and C-Z, we have

$$\begin{aligned} \|u - \Gamma * f\|_{W^{2,p}(B_{1/2})} &\leq C_n \|u - \Gamma * f\|_{L^\infty(B_{3/4})} \leq C_n \|u - \Gamma * f\|_{L^p(B_1)} \\ &\leq C_n \|u\|_{L^p(B_1)} + \|\Gamma * f\|_{L^p(B_1)} \\ &\leq C(n, p) \left(\|u\|_{L^p(B_1)} + \|f\|_{L^p(B_1)} \right). \end{aligned}$$

Note that $\frac{1}{|x|^{n-2}}$ and $\frac{1}{|x|^{n-1}}$ are in $L^1(\Omega)$, by Young's inequality

$$\begin{aligned} \|\Gamma * f\|_{L^p(\Omega)} &\leq \|\Gamma\|_{L^1(\Omega)} \|f\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)} \\ \|D\Gamma * f\|_{L^p(\Omega)} &\leq \|D\Gamma\|_{L^1(\Omega)} \|f\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}. \end{aligned}$$

Therefore we obtain

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C(n, p) \left(\|u\|_{L^p(B_1)} + \|f\|_{L^p(B_1)} \right).$$

Perturbation.

$$\sum a_{ij}(0) D_{ij}u = \sum [a_{ij}(0) - a_{ij}(x)] D_{ij}u - b_i(x) D_i u - c(x) u + f$$

we have (heuristically)

$$\|D^2 u\|_{L^p(B_{1/2})} \leq \|u\|_{L^p(B_1)} + \|f\|_{L^p(B_1)} + \varepsilon \|D^2 u\|_{L^p(B_1)}.$$

As for interior Schauder estimate (for $p > n$; different route for $1 < p \leq n$, because of different scaling), via weighted norm, we move ε -term to the left hand side to obtain $W^{2,p}$ estimates for

$$\sum a_{ij}(x) D_{ij}u + b_i(x) D_i u + c(x) u = f$$

with coefficients $a_{ij}(x) \in C^0$ $\lambda I \leq (a_{ij}) \leq \lambda^{-1} I$, $|b|, |c| \leq \Lambda$

$$\|D^2 u\|_{L^p(B_{1/2})} \leq C(\|a\|_{C^0}, \lambda, \Lambda, n, p) \left[\|u\|_{L^p(B_1)} + \|f\|_{L^p(B_1)} \right].$$

For boundary L^p estimates on $C^{1,1}$ domains, again it is parallel to boundary Schauder estimates (Poisson convolution is fine, Dirichlet to Neumann singular convolution is handled as in the interior case but with one dimension less, still use weighted norm device ...). Skip.

Instead study 2-d situation: For $u \in W_0^{2,2}(\Omega)$ or $u \in C^2$ satisfying

$$\begin{cases} \sum_{i,j=1}^2 a_{ij}(x) D_{ij}u = f & \text{in } \Omega \text{ convex} \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with L^∞ coefficients $\lambda I \leq (a_{ij}) \leq \lambda^{-1} I$. Then

$$\|D^2 u\|_{L^2(\Omega)} \leq C(\lambda) \|f\|_{L^2(\Omega)}.$$

Proof. At each point, rotate coordinates so that

$$(a_{ij}(x)) = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}.$$

Then the equation becomes

$$(\lambda_1 u_{11} + \lambda_2 u_{22})^2 = f^2$$

or

$$\lambda_1^2 u_{11}^2 + \lambda_2^2 u_{22}^2 + 2\lambda_1 \lambda_2 u_{12}^2 + 2\lambda_1 \lambda_2 (u_{11} u_{22} - u_{11}^2) = f^2.$$

Use ellipticity $\lambda \leq \lambda_i \leq \lambda^{-1}$, we have

$$\lambda^4 |D^2 u(x)|^2 + 2 \det D^2 u(x) \leq \frac{1}{\lambda^2} f^2(x).$$

This coordinate invariant inequality is in fact true everywhere under one same coordinate. Integrate, we have

$$\int_{\Omega} \lambda^4 |D^2 u|^2 + \int_{\Omega} 2 \det D^2 u \leq \int_{\Omega} \frac{1}{\lambda^2} f^2.$$

Next we show that the integral of the determinant with u vanishing on the bounded convex boundary is positive. Note

$$2 \det D^2 u = (\Delta u)^2 - |D^2 u|^2.$$

Exercise (exactly as in the proof of $W^{2,2}$ estimate in C-Z a), by Green's identity)

$$\int_{\Omega} (\Delta u)^2 - |D^2 u|^2 = \int_{\partial\Omega} \Delta u \partial_{\gamma} u - u_k \partial_{\gamma} u_k.$$

We finish the proof by proving that the integrand in the boundary integral is positive pointwise. Fix a point on the boundary, say this point at the origin, x-axis is tangent to the boundary, and y-axis is along γ direction.

figure x-axis is tangent to the boundary, and y-axis is along γ direction

As $u = 0$ on $\partial\Omega$, $u_x(0) = 0$. Also $u_{\gamma}(0) = u_y(0)$. We have

$$\text{integrand} \stackrel{0}{=} \Delta u u_y - u_y u_{yy} = u_y u_{xx}.$$

W.l.o.g. assume $u_y(0) = |Du(0)| > 0$, otherwise, consider $-u$. Observe near 0, u becomes negative entering the domain Ω from 0 boundary. By double finite difference to approximate $u_{xx}(0)$, one sees that

$$u_{xx}(0) \geq 0.$$

Thus the integrand is positive.

Therefore

$$\int_{\Omega} 2 \det D^2 u \geq 0$$

and

$$\int_{\Omega} |D^2 u|^2 \leq \frac{1}{\lambda^4} \int_{\Omega} f^2.$$

RMK. In general n-d

$$\int_{\Omega} \det D^2 u = \int_{\Omega} d(u_1 \wedge du_2 \wedge \cdots \wedge du_n) = \int_{\partial\Omega} u_1 du_2 \wedge \cdots \wedge du_n.$$

Is there a sign restriction on the boundary integrand for $C^2 u$ vanishing on the convex boundary? (Positive in even-dim and negative in odd-dim)