Lecture 2  Harmonic functions

- invariance
- mean value
- maximum principle,
  (higher order) derivative estimates,
  Harnack
- weak formulations
  mean value
  weak/Weyl
  viscosity

Invariance for Harmonic functions, solutions to $\Delta u = 0$

- $u (x + x_0)$
- $u (Rx)$
- $u (tx)$

RMK. Equations don’t know/care which coordinates they are in.

- $u + v, au$, where $\Delta v = 0$
- $\int u (x - y) \varphi (y) \, dy$
- $\frac{u (x + \varepsilon x) - u (x)}{\varepsilon} \rightarrow Du$, so is $D^k u$
- $\frac{u (Rx) - u (x)}{\varepsilon} \rightarrow D_\theta u = x_i u_j - x_j u_i$, where $R_\varepsilon = \exp \left( \varepsilon \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \right)$, in polar form

$R_{\varepsilon z} = z e^{\frac{-\varepsilon}{1 - \varepsilon}} = r (\cos (\theta + \varepsilon), \sin (\theta + \varepsilon))$.

- $\frac{u (\frac{1 + \varepsilon x}{1 - \varepsilon} - u (x)}{\varepsilon} \rightarrow D u (x) \cdot x = ru_r$, so are $ru_r, ru_{rr}, ru_{rrr}, \cdots$
- $|x|^2 - n u \left( \frac{x}{|x|^2} \right)$ Kelvin transformation

RMK. “Kelvin” transformation for the heat equation $u_t - \Delta u = 0, \frac{1}{r^{n+2}} e^{-\frac{4}{n-2}} u \left( \frac{x}{r^{n+2}}, \frac{t}{r^{n+2}} \right)$; the wave equation $u_{tt} - \Delta u = 0, \left( |x|^2 - t^2 \right)^{\frac{2-n}{2}} u \left( \frac{x}{|x|^2}, \frac{t}{|x|^2} \right)$.

More harmonic functions.

eg1.

$$D_1 r^{2-n} = \left( 2 - n \right) r^{1-n} \frac{x_1}{r} = (2 - n) r^{-n} x_1 = (2 - n) \frac{x_1}{r^n}$$

$$D_{11} r^{2-n} = (2 - n) \left[ -n r^{-n-1} \frac{x_1}{r} x_1 + r^{-n} \right] = (2 - n) \frac{r^2 - nx_1^2}{r^{n+2}}$$

$$D_{12} r^{2-n} = (2 - n) \frac{-nx_1 x_2}{r^{n+2}}$$

Let $P_k (x)$ be any homogeneous polynomial of degree $k$, $P_k (D) r^{2-n} = \frac{H_k (t)}{r^{n+2}}$. For example, $\sigma_k (D) r^{2-n} = \frac{\sigma_k (t)}{r^{n+2}}$. Note $H_k \neq P_k$ in general, but $H_k (x) = r^{2-n} \frac{H_k (\frac{x}{r^{n+2}})}{r^{n+2}}$ is the Kelvin transform of harmonic function $P_k (D) r^{2-n}$, thus harmonic.

Exercise: $H_k (x)$ are ALL harmonic polynomials of degree $k$. 

\(^0\)April 17, 2019
Harmonic function

\[ |x - x_0|^{2-n} - |x|^{2-n} \frac{x}{|x|^2} x_0^{2-n} - |x - x_0|^{2-n} - |x - x_0|^{2-n} = 0, \]

is Green's function (up to a multiple) for the unit ball.

Mean value equality

Recall the divergence formula (the fundamental theorem of calculus)

\[ \int_{\Omega} \text{div} \left( \tilde{V} \right) \, dx = \int_{\partial \Omega} \langle \tilde{V}, \gamma \rangle \, dA. \]

If \( \tilde{V} = Du \), then \( 0 = \int_{\partial \Omega} u \gamma \, dA. \)

If \( \tilde{V} = v Du \), then \( \int_{\Omega} \langle Dv, Du \rangle + v \Delta u = \int_{\partial \Omega} vu \gamma \, dA. \)

If \( \tilde{V} = u Dv \), then \( \int_{\Omega} \langle Du, Dv \rangle + u \Delta v = \int_{\partial \Omega} uv \gamma \, dA. \)

\[ \int_{\Omega} v \Delta u - u \Delta v = \int_{\partial \Omega} vu \gamma - uv \gamma \, dA. \]

Mean value case. Now \( \Delta u = 0 \) in \( B_1 \), take \( v = |x|^{2-n} \), \( \Omega = B_1 \setminus B_{\varepsilon}, \)

\[ B_1 \setminus B_{\varepsilon} \text{ figure} \]

we then have \( 0 = \int_{\partial \Omega} vu \gamma - uv \gamma \, dA \), or

\[ \int_{\partial (B_1 \setminus B_{\varepsilon})} vu \gamma \, dA = \int_{\partial (B_1 \setminus B_{\varepsilon})} uv \gamma \, dA = \int_{\partial B_1} u^{(2-n)} \frac{1}{r^{n-1}} \, dA - \int_{\partial B_{\varepsilon}} u^{(2-n)} \frac{1}{r^{n-1}} \, dA. \]  

\[ (*) \]

We get \( \int_{\partial B_1} u \, dA = \int_{\partial B_\varepsilon} u \, dA \frac{1}{\varepsilon^{n-1}} \varepsilon^{2-n} |\partial B_1| u (0) \). So \( u (0) = \frac{1}{|\partial B_1|} \int_{\partial B_1} u \, dA. \)

RMK. In hindsight one just takes \( v = \frac{1}{(n-2)} \frac{1}{|\partial B_1| \, |x|^{2-n}} \equiv \Gamma. \)

Also

\[ u (0) = \frac{1}{|\partial B_1|} \int_{\partial B_\varepsilon} u \, dA. \]

Take a weight function \( |\partial B_r|, \) \( u (0) |\partial B_1| = \int_0^1 u (0) |\partial B_r| \, dr = \int_0^1 \int_{\partial B_r} u \, dA \, dr = \int_{B_1} u \, dx. \) So \( u (0) = \frac{1}{|B_1|} \int_{B_1 (0)} u \, dx. \)

Also

\[ u (0) = \frac{1}{|B_r|} \int_{B_r (0)} u \, dx. \]
RMK1. Tracing the sign of $\Delta u$, one gets mean value inequalities for superharmonic functions $\Delta u \leq 0 : u (0) \leq \bar{u}$ and subharmonic functions $\Delta u \geq 0 : u (0) \leq \underline{u}$.

RMK2. “... all the women are strong, all the men are good-looking, and all the children are above average.” –A Prairie Home Companion with Garrison Keillor.

RMK3. Newtonian case. Recover $u \in C_0^\infty (\mathbb{R}^n)$ from $\Delta u$

$$\begin{align*}
u (x) &= \int_{\mathbb{R}^n} \frac{-1}{(n-2)n |B_1|} \frac{1}{|x-y|^{n-2}} \Delta u (y) dy = \Gamma \ast \Delta u. \\
\text{Note the already interesting 1d situation } &\Gamma = \frac{1}{2} |x|; 2d \Gamma = \frac{1}{2\pi} \ln |x| \text{ with } \ln |x| = \lim_{n \to 2} \frac{|x|^{2-n} - 1}{n^{2-n}}.
\end{align*}$$

Green case. Still $\Delta u = 0$ in $B_1$, but

$$\begin{align*}
u &= G (x, x_0) = \frac{-1}{(n-2)|\partial B_1|} \left( |x-x_0|^{2-n} - |x|^2 - x_0 \right) \\
\Omega &= B_1 \setminus B_\varepsilon (x_0). \\
\text{Taking limits on two ends of } (*) &\text{, we get}
\end{align*}$$

$$\begin{align*}
u (x_0) &= \int_{\partial B_1} \frac{\partial G (x, x_0)}{\partial \gamma_x} u (x) dA = \frac{1}{|\partial B_1|} \int_{\partial B_1} \frac{1 - |x_0|^2}{|x-x_0|^n} u (x) dA.
\end{align*}$$

Note $u (x) = \int_{\partial B_1} \frac{\partial G (y, x_0)}{\partial \gamma_y} \varphi (y) dA_y$, as sum of harmonic functions $\frac{1 - |y|^2}{|y-x|^n}$, is harmonic-smooth, analytic in terms of regularity—for $\varphi \in C^0, L^1, \cdots$.

Application 1. Strong maximum principle (No toughing).

$$\begin{align*}
\Delta u_1 &= \Delta u_2 = 0 \\
u_1 &= u_2, \text{ "=} \text{ at } 0
\end{align*}$$

then

$$0 = u_1 (0) - u_2 (0) = \frac{1}{|B_r|} \int_{B_r} (u_1 - u_2) dx \geq 0.$$

It follows that $u_1 \equiv u_2$.

Application 2. Smooth effect and derivative estimates.

Take radial weight $\varphi (y) = \varphi (|y|) \in C_0^\infty (\mathbb{R}^n)$ such that $1 = \int \varphi (y) dy = \int_0^\infty \varphi (r) |\partial B_r| dr$.

Then

$$\begin{align*}
\int_{\mathbb{R}^n} u (y) \varphi (x-y) dy &= \int_0^\infty \int_{\partial B_r (x)} u (y) \varphi (x-y) dA dr \\
&= \int_0^\infty u (r) \varphi (r) |\partial B_r| dr = u (x) \int \varphi (y) dy \\
&= u (x).
\end{align*}$$
Consequence $u(x) = \int_{\mathbb{R}^n} u(y) \varphi(x-y) dy$ is smooth for continuous initial $u(y)$, and

$$D^k u(0) = \int u(y) D^k_x \varphi(x-y) dy = (-1)^k \int u(y) D^k_y \varphi(x-y) dy.$$ 

Thus, (say support of $\varphi$ is in $B_1$)

$$|D^k u(0)| \leq C(k,n,\varphi) \|u\|_{L^1(B_1)}$$

and also the point-to-point version for $u \geq 0$

$$|D^k u(0)| \leq C(k,n,\varphi) \int_{B_1} u dx = C(k,n,\varphi) u(0)$$

Scaled version

$$|D^k u(0)| \leq \left\{ \begin{array}{l}
\frac{C(k,n,\varphi)\|u\|_{L^1(B_R)}}{C(k,n,\varphi)\|u\|_{L^\infty(B_R)}} \\
\frac{R^n}{R^2}
\end{array} \right.$$

and also

$$|D^k u(0)| \leq \frac{C(k,n,\varphi)}{R^2} u(0) \text{ provided } u \geq 0.$$

That is the larger the domain, the flatter the harmonic graph.

**Liouville.** Every bounded or even one side bounded entire harmonic function in $\mathbb{R}^n$ is constant. Similarly every entire polynomial growth harmonic function is a polynomial.

**Application 3.** **Harnack inequality**--a quantitative version of the strong maximum principle.

eg. Consider positive harmonic functions $r^{2-n}$, $x_1 r^{-n}$ on $\{x_1 > 0\}$.

![Figure 1: $r^{2-n}$ and $x_1 r^{-n}$](image)

eg. In general for $\Delta u = 0$, $u > 0$ in $B_1(0)$, we have

$$u(x) = \frac{1}{|B_1-|x||} \int_{B_1-|x|} u dx \leq \frac{1}{|B_1-|x||} \int_{B_1(0)} u dx = \frac{|B_1|}{|B_1-|x||} u(0) = \frac{1}{(1-|x|)^n} u(0).$$
RMK. As those two examples suggest, from estimating the kernel of Poisson representation, we have a sharper comparison

\[
\frac{(1 - |x|)}{2^{n-1}} u(0) \leq u(x) \leq \frac{2}{(1 - |x|)^{n-1}} u(0).
\]

**Harnack.** Suppose \( \triangle u = 0, \ u > 0 \) in \( B_r(x_0) \). Then we have

\[
\sup_{B_{r/4}(x_0)} u \leq 3^n \inf_{B_{r/4}(x_0)} u.
\]

Discrete way. In fact

![Figure 2: Four circles](image)

\[
\max_{B_{1/4}} u = u(x_{\text{max}}) = \frac{1}{|B_{1/4}|} \int_{B_{1/4}(x_{\text{max}})} u \, dx \\
\leq \frac{1}{|B_{1/4}|} \int_{B_{3/4}(x_{\min})} u \, dx \\
= 3^n u(x_{\min}) = 3^n \min_{B_{1/4}} u.
\]

Continuous way. Suppose \( \triangle u = 0, \ u > 0 \) in \( B_2(0) \). Then we have

\[
\max_{B_1(0)} u \leq C(n) \min_{B_1(0)} u.
\]

Indeed by the point-to-point version of gradient estimate, \( |Du(x)|/u(x) \leq C(n) \) in \( B_1(0) \). We measure the ratio between \( u_{\text{max}} \) and \( u_{\text{min}} \) in \( B_1(0) \) by integration

\[
\log \frac{u(x_{\text{max}})}{u(x_{\text{min}})} = \int_{x_{\text{min}}}^{x_{\text{max}}} d \log u(x) = \int_{x_{\text{min}}}^{x_{\text{max}}} \frac{Du}{u} (x_{\min} + t (x_{\text{max}} - x_{\text{min}})) dt \\
\leq \int_{x_{\text{min}}}^{x_{\text{max}}} \frac{|Du|}{u} (x_{\min} + t (x_{\text{max}} - x_{\text{min}})) dt \leq |x_{\text{max}} - x_{\text{min}}| \cdot C(n).
\]
Then
\[
\max_{B_1(0)} u \leq e^{2 \cdot C(n)} \min_{B_1(0)} u.
\]

Consequences \ldots, for example one sided Liouville for entire harmonic functions.
RMK. Harnack inequality is in fact a quantitative version of the strong maximum principle. It measures how much the minimum leaps when moving inside, or flipping around how much the maximum drops when moving inside. For example, to move inside \(B_{1/4}\) from \(B_1\),
\[
\min_{B_{1/4}} (u - m_1) \geq 3^{-n} \max_{B_{1/4}} (u - m_1)
\]
or
\[
m_{1/4} \geq m_1 + 3^{-n} (M_{1/4} - m_1).
\]
The flip version is
\[
\min_{B_{1/4}} (M_1 - u) \geq 3^{-n} \max_{B_{1/4}} (M_1 - u)
\]
or
\[
M_{1/4} \leq M_1 - 3^{-n} (M_1 - m_{1/4}).
\]
(This should be Moser’s observation: subtracting the leap from the drop, one has oscillation decay of the “harmonic” function.)

Weak formulation for Laplace equation: \(\Delta u = 0\).
Mean value formulation.
Suppose \(u \in L^1\) satisfy \(u(x) = \int_{B_r(x)} u(y) \, dy\) for all \(x\) and \(r\).
Exercise. Then \(u\) is continuous, since
\[
u(x) - u(x_0) = \int_{B_1(x)} u(y) \, dy - \int_{B_1(x_0)} u(y) \, dy \overset{x \to x_0}{\to} 0.
\]
2 minor overlap circle figure

In turn, we have \(u(x) = \int_{\partial B_r(x)} u(y) \, dy\). In fact
\[
\frac{d}{dr} : r^n |B_1| u(x_0) = \int_{B_r(x_0)} u(y) \, dy
\]
\[
rm^{n-1} |B_1| u(x_0) = \int_{\partial B_r(x_0)} u(y) \, dy
\]
\[
|\partial B_r| u(x_0) = \int_{\partial B_r(x_0)} u(y) \, dy.
\]
Then

\[ u(x) = \int_{R^n} \varphi(x - y) u(y) \, dy \in C^\infty \]

for \( \varphi(x) = \varphi(|x|) \) with \( \int_{R^n} \varphi(|x|) \, dx = 1 \). Let us check \( \Delta u = 0 \).

\[
\int_{\partial B_\varepsilon(0)} u dA = \int_{\partial B_\varepsilon(0)} u(0) + Du(0) \cdot x + \frac{1}{2} \sum_{ij} D_{ij} u(0) x_i x_j + \varepsilon^3 dA
\]

\[
|\partial B_\varepsilon| u(0) = |\partial B_\varepsilon| u(0) + 0 + \frac{1}{2} \left( \lambda_1 \varepsilon^2 + \cdots + \lambda_n \varepsilon^2 \right) |\partial B_\varepsilon| + O(\varepsilon^3) |\partial B_\varepsilon|
\]

\[
\Rightarrow \frac{1}{2n} \Delta u(0) = 0.
\]

Integration by parts formulation.

For \( u \in C^0 / L^1 / \text{distribution} \int u \Delta \varphi = 0 \) for any \( \varphi \in C_0^\infty \). How to move to mean value formulation?

Q. How to find \( \varphi \in C_0^\infty \) such that

\[
\Delta \varphi = \frac{1}{|B_2|} \chi_{B_2} - \frac{1}{|B_1|} \chi_{B_1},
\]

\( C^{1,1} \) approach. \( \varphi \sim \frac{|x|^2}{2|B_1|} \chi_{B_2} - \frac{|x|^2}{2|B_2|} \chi_{B_1} \).

Analytic way. We just look for those radial ones by solving

\[
\varphi_{rr} + \frac{n - 1}{r} \varphi_r = \frac{1}{|B_1|} \chi_{B_1} \quad \text{or} \quad \frac{1}{|B_2|} \chi_{B_2}.
\]

For \( r \leq 1 \)

\[
\varphi = \frac{1}{|B_1|} \frac{r^2}{2n} \chi_{[0,1]} + c_1.
\]

For \( r > 1 \)

\[
\varphi = c_2 r^{2-n} + c_3.
\]

After \( C^{1,1} \) matching at \( r = 1 \), we have

\[
\varphi_1 = \begin{cases} 
\frac{1}{|B_1|} \frac{r^2}{2n} \chi_{B_1} - \frac{1}{|B_1|} \frac{1}{2n} - \frac{1}{|B_1|} \frac{1}{(n-2)n} & \text{for } |x| \leq 1 \\
\frac{1}{|B_1|} \frac{1}{(n-2)n} \frac{1}{r^{n-2}} & \text{for } |x| > 1
\end{cases}
\]

Similarly

\[
\varphi_2 = \begin{cases} 
\frac{1}{|B_2|} \frac{r^2}{2n} \chi_{B_2} - \frac{1}{|B_2|} \frac{2^2}{2n} - \frac{1}{|B_2|} \frac{1}{(n-2)n} & \text{for } |x| \leq 2 \\
\frac{1}{|B_2|} \frac{1}{(n-2)n} \frac{1}{r^{n-2}} & \text{for } |x| > 2
\end{cases}
\]

“Incidentally” the gradient matching coefficient \( c_2 \) leads exactly the coefficient for the fundamental solution \( \Gamma = \frac{1}{|B_1|} \frac{1}{(n-2)n} \frac{1}{|x|^{n-2}} \) for \( n = 1, 3, 4, \cdots \) and \( \frac{1}{2n} \ln |x| \) for \( n = 2 \).

Geometric way (Caffarelli).

quadratics drop down to fundamental figure
This requires \( \varphi_2 = \frac{|x|^2}{2n|B_2|} - A \) to touch \( r^{2-n} \), in fact \( \frac{-1}{2^{2n-2}} \) at \( |x| = 2 \). We have a system

\[
\begin{cases}
\frac{2^2}{2n|B_2|} - A = \frac{-1}{2^{2n-2}} \\
\frac{2^2}{2n|B_2|} = \frac{(n-2)}{2^{2n-2}}
\end{cases}
\]

which implies \( ? = n(n-2)|B_1| \) and \( A = \frac{2(n-1)}{n(n-2)|B_1|} \). Similarly we get \( \varphi_1 = \frac{|x|^2}{2n|B_1|} - A' \) touching \( \frac{-1}{2^{2n-2}} = \frac{1}{n(n-2)|B_1|^{2n-2}} \) at \( |x| = 1 \). Thus \( \varphi = \varphi_2 - \varphi_1 \in C_0^{1,1} \) answers the above question.

For \( u \in L^1 \), \( \int u \Delta \varphi = 0 \Rightarrow \int_{B_2} u = \int_{B_1} u \).

Therefore (exercise)

\[
u(x) = \lim_{r \to 0} \int_{B_r(x)} u \text{ a.e. at Lebesgue point of } L^1 u.
\]

Cor. (Weyl) \( u \in L^1/C^0 \) satisfying \( \int u \Delta \varphi = 0 \) for any \( \varphi \in C_0^\infty \). Then \( u \in C^\infty \) and \( \Delta u = 0 \).

Warning:

\[
\int \frac{1}{|x|^{n-2}} \Delta \varphi = c_n \varphi(0) \neq 0 !
\]

\( C^\infty \) approach (Weyl, works for distribution \( u \))

\[
\psi(x) = \psi(|x|) \in C_0^\infty \text{ with } \int \psi = 1
\]

\[
\psi_\varepsilon(x) = \frac{1}{\varepsilon^n} \psi \left( \frac{x}{\varepsilon} \right)
\]

\( \Gamma \ast \psi_\varepsilon \) graph figure

Step 1. \( \varphi_\varepsilon = \Gamma \ast \psi_\varepsilon = \left\{ \begin{array}{ll} 
\Gamma & \text{for } |x| \geq \varepsilon \\
\text{smooth for } |x| \leq \varepsilon \end{array} \right. \). Recall \( \Gamma = \frac{-1}{(n-2)|\partial B_1|} |x|^{1-n} \).

Step 2. \( \Delta \Gamma \ast \psi = \Gamma \ast \Delta \psi = \psi \).

Step 3. \( \varphi_{\varepsilon_2} - \varphi_{\varepsilon_1} \in C_0^\infty \)

\[
\int \mathbb{R}^n u \Delta (\varphi_{\varepsilon_2} - \varphi_{\varepsilon_1}) = 0 \Rightarrow \int \mathbb{R}^n u \psi_{\varepsilon_2} = \int \mathbb{R}^n u \psi_{\varepsilon_1}
\]

- \( u \ast \psi_\varepsilon \) is independent of \( \varepsilon \)
- \( u \ast \psi_\varepsilon \in C^\infty \) (Review distribution theory, try it; [Hörmander, vol 1 Thm 2.1.3])
- \( u \ast \psi_\varepsilon = u \) as a distribution (Exercise).

Michael-Simon (mysterious, works for min/max surfaces)
\[ \varphi (r) = \int_r^\infty t \chi (t / \rho) \, dt = \begin{cases} \frac{\rho^2 - r^2}{2} & \text{for } r < \rho \\ 0 & \text{for } r > \rho \end{cases} \] where \( \chi (t) \) is a smooth approximation of the characteristic function \( \chi_{[-\infty, 1]} \). What’s mysterious is

\[ \Delta \varphi = \rho^{n+1} \frac{d}{d\rho} \left[ \rho^{-n} \chi (r / \rho) \right]. \]

Indeed,

\[ \Delta \varphi = \varphi_{rr} + \frac{n-1}{r} \varphi_r = [-r \chi (r / \rho)]_r - \frac{(n-1)}{r} r \chi (r / \rho) \]

\[ = -n \chi (r / \rho) - \frac{r}{\rho} \chi (r / \rho) \] \[ \implies \rho^{n+1} \frac{d}{d\rho} \left[ \rho^{-n} \chi (r / \rho) \right]. \]

Testing harmonic function by this smooth, compactly supported \( \varphi \), we have

\[ 0 = \int u \Delta \varphi \, dx = \rho^{n+1} \frac{d}{d\rho} \left[ \rho^{-n} \int \chi (r / \rho) \, u \, dx \right] \rightarrow \rho^{n+1} \frac{d}{d\rho} \left[ \rho^{-n} \int_{B_\rho} u \, dx \right]. \]

It follows that \( u (0) = \frac{1}{|B_\rho|} \int_{B_\rho} u \, dx \). Next, differentiate \( |B_\rho| u (0) = \int_{B_\rho} u \, dx \) w.r.t. \( \rho \), we obtain \( u (0) = \frac{1}{|B_\rho|} \int_{\partial B_\rho} u \, dA \).

Pointwise (viscosity) formulation.
Definition: \( u \in C^0 \) is a viscosity solution to \( \Delta u = 0 \) if for any quadratic \( P \geq u \leq \text{near an interior point } x_0 \) and \( "=\" \) at \( x_0 \), then \( \Delta P \geq 0 \).

RMK. If there is no quadratic touching \( u \) from above or below at \( x_0 \), then one checks nothing. No touching, no checking!

RMK. We can replace those quadratics by equivalent \( C^2 / C^\infty \) testing functions. Certainly \( C^2 \) harmonic functions satisfy this definition. We do have \( C^0 \) but non \( C^2 \) solutions to (fully nonlinear) elliptic equations such as Monge-Ampere/Special Lagrangian equations.

We verify \( C^0 \) harmonic functions in the viscosity sense are in fact smooth and satisfy the “harmonic” equation by Poisson representation formula. Note explicitly representation for solutions to nonlinear equations are NOT available in general.

Claim: Let \( u \in C^0 (\bar{B}_1) \) be a viscosity sol to \( \Delta u = 0 \) in \( B_1 \) and

\[ h = \int_{\partial B_1} P (x, y) \, u (y)|_{B_1} \, dA_y. \]

THEN
\[ \cdot h = u \text{ on } \partial B_1. \]
\[ \cdot \Delta h = 0 \text{ in } B_1. \]
\[ \cdot h = u \text{ in } \bar{B}_1 \]

Indeed if \( \text{subsolution } u > h \) somewhere at \( x_0 \in \bar{B}_1 \), say \( (u - h) (x_0) = \max_{B_1} (u - h) > 0 \)

u,h graph figure
\[ h + \max \geq u \text{ in } B_1, \quad "=\text{" at } x_0. \]

Also \( h + \max - \varepsilon |x|^2 \geq u, \quad "=\text{" at } x'_0 \in \overline{B}_1, \) yes we can replace.

But \( \triangle \left( -2n \varepsilon < 0. \right. \) This contradiction shows \( u \leq h. \)

Similarly, if supersolution \( u < h \) somewhere at \( x_0 \in \overline{B}_1, \) say \( (u - h)(x_0) = \min_{B_1}(u - h) < 0 \quad u, h \text{ graph figure}\)

\[ h + \min \leq u \text{ in } B_1, \quad "\text{" at } x_0. \]

Also \( h + \min + \varepsilon |x|^2 < u, \quad "\text{" at } x'_0 \in \overline{B}_1, \) yes we can replace.

But \( \triangle \left( 2n \varepsilon > 0. \right. \) This contradiction shows \( u \geq h. \)

Thus \( u \equiv h. \)