Lecture 2 Harmonic functions

○ invariance
○ mean value
 maximum principle,
 (higher order) derivative estimates,
 Harnack
○ weak formulations
mean value
weak/Weyl
viscosity

<u>Invariance for Harmonic functions, solutions to</u> $\triangle u = 0$
· $u(x + x_0)$
· $u(Rx)$
· $u(tx)$
RMK. Equations don't know/care which coordinates they are in.
· $u + v$, au, where $\triangle v = 0$
· $\int u(x-y)\varphi(y)\,dy$
·$\frac{u(x+\varepsilon e)-u(x)}{\varepsilon} \to D_e u$, so is $D^k u$
·$\frac{u(R\varepsilon x)-u(x)}{\varepsilon} \to D_\theta u = x_i u_j - x_j u_i$
· $\frac{u((1+\varepsilon)x)-u(x)}{\varepsilon} \to Du(x)\cdot x = ru_r$, so are $r\partial_r(ru_r) = ru_r + \underline{r^2 u_{rr}}$, $r^3 u_{rrr}, \cdots$
· $|x|^{2-n} u\left(\frac{x}{|x|^2}\right)$ Kelvin transformation

Rmk. "Kelvin" transformation for the heat equation $u_t - \triangle u = 0$, $\frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}} u\left(\frac{x}{t}, \frac{-1}{t}\right)$.
More harmonic functions.
eg1.

$$\begin{aligned} D_1 r^{2-n} &= (2-n)\, r^{1-n}\frac{x_1}{r} = (2-n)\, r^{-n}x_1 = (2-n)\frac{x_1}{r^n} \\ D_{11} r^{2-n} &= (2-n)\left[-nr^{-n-1}\frac{x_1}{r}x_1 + r^{-n}\right] = (2-n)\frac{r^2 - nx_1^2}{r^{n+2}} \\ D_{12} r^{2-n} &= (2-n)\frac{-nx_1x_2}{r^{n+2}} \end{aligned}$$

Let $P_k(x)$ be any homogeneous polynomial of degree k, $P_k(D)\, r^{2-n} = \frac{H_k(x)}{r^{n-2-2k}}$. For example, $\sigma_k(D)\, r^{2-n} = \frac{\sigma_k(x)}{r^{n-2-2k}}$. Note $H_k \neq P_k$ in general, but $H_k(x) = r^{2-n}\frac{H_k\left(\frac{x}{r^2}\right)}{\left|\frac{x}{r^2}\right|^{n-2-2k}}$ is the Kelvin transform of harmonic function $P_k(D)\, r^{2-n}$, thus harmonic.

Exercise: $H_k(x)$ are ALL harmonic polynomials of degree k.

eg2. Harmonic function

$$|x-x_0|^{2-n} - |x|^{2-n}\left|\frac{x}{|x|^2} - x_0\right|^{2-n} \overset{|x|=1}{=} |x-x_0|^{2-n} - |x-x_0|^{2-n} = 0,$$

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is Green's function (up to a multiple) for the unit ball.

Mean value equality

Recall the divergence formula (the fundamental theorem of calculus)

$$\int_{\Omega} div\left(\vec{V}\right)dx = \int_{\partial\Omega}\left\langle \vec{V},\gamma\right\rangle dA.$$

$\vec{V} = Du$, then $0 = \int_{\partial\Omega} u_{\gamma} dA$.
$\vec{V} = vDu$, then $\int_{\Omega}\langle Dv, Du\rangle + v \triangle u = \int_{\partial\Omega} vu_{\gamma}dA$.
$\vec{V} = uDv$, then $\int_{\Omega}\langle Du, Dv\rangle + u \triangle v = \int_{\partial\Omega} uv_{\gamma}dA$.

$$\int_{\Omega} v \triangle u - u \triangle v = \int_{\partial\Omega} vu_{\gamma} - uv_{\gamma}dA.$$

Mean value case. Now $\triangle u = 0$ in B_1, $v = |x|^{2-n}$, $\Omega = B_1 \backslash B_{\varepsilon}$

figure

$0 = \int_{\partial\Omega} vu_{\gamma} - uv_{\gamma}dA$, or

$$\overbrace{\int_{\partial(B_1\backslash B_\varepsilon)} vu_{\gamma}dA}^{0} = \int_{\partial(B_1\backslash B_\varepsilon)} uv_{\gamma}dA = \int_{\partial B_1} u\frac{(2-n)}{r^{n-1}}dA - \int_{\partial B_\varepsilon} u\frac{(2-n)}{r^{n-1}}dA. \qquad (*)$$

We get $\int_{\partial B_1} udA = \int_{\partial B_\varepsilon} u\frac{1}{\varepsilon^{n-1}}dA \overset{\varepsilon\to 0}{\to} c_n u(0)$. So $u(0) = \frac{1}{c_n}\int_{\partial B_1} udA$. Taking $u \equiv 1$ leads to $c_n = |\partial B_1| = n|B_1|$.

Also

$$u(0) = \frac{1}{|\partial B_r|}\int_{\partial B_r} udA.$$

Take a weight function $|\partial B_r|$, $u(0)|B_1| = \int_0^1 u(0)|\partial B_r|dr = \int_0^1\int_{\partial B_r} udAdr = \int_{B_1} udx$. So $u(0) = \frac{1}{|B_1|}\int_{B_1(0)} udx$.

Also

$$u(0) = \frac{1}{|B_r|}\int_{B_r(0)} udx.$$

RMK. "$\cdots$ all the women are strong, all the men are good-looking, and all the children are above average." –A Prairie Home Companion with Garrison Keillor.

Also

$$u(x) = \int_{\mathbb{R}^n}\frac{-1}{(n-2)n|B_1|}\frac{1}{|x-y|^{n-2}}\triangle u(y)\,dy \quad \text{for } u \in C_0^{\infty}(\mathbb{R}^n).$$

Green case. Still $\triangle u = 0$ in B_1, but

$$v = G(x, x_0) = \frac{-1}{(n-2)|\partial B_1|}\left(|x - x_0|^{2-n} - |x|^{2-n}\left|\frac{x}{|x|^2} - x_0\right|^{2-n}\right),$$

$\Omega = B_1 \backslash B_\varepsilon(x_0)$.

$$figure$$

Taking limits on two ends of (*), we get

$$u(x_0) = \int_{\partial B_1} \frac{\partial G(x, x_0)}{\partial \gamma_x} u(x)\, dA.$$

Note $u(x_0) = \int_{\partial B_1} \frac{\partial G(x,x_0)}{\partial \gamma_x} \varphi(x)\, dA$, as sum of harmonic functions $\frac{\partial G(x,x_0)}{\partial \gamma_x}$, is harmonic for $\varphi \in C^0, L^1, \cdots$.

Application 1. Strong maximum principle (No toughing).

$$\triangle u_1 = \triangle u_2 = 0$$
$$u_1 \geq u_2, \text{ “}=\text{'' at } 0$$

then

$$0 = u_1(0) - u_2(0) = \frac{1}{|B_r|}\int_{B_r} (u_1 - u_2)\, dx \geq 0.$$

It follows that $u_1 \equiv u_2$.

Application 2. Smooth effect and derivative test.

Take radial weight $\varphi(y) = \varphi(|y|) \in C_0^\infty(R^n)$ such that $1 = \int \varphi(y)\, dy = \int_0^\infty \varphi(r)\, |\partial B_r|\, dr$. Then

$$\begin{aligned}\int_{R^n} u(y)\,\varphi(x-y)\, dy &= \int_0^\infty \int_{\partial B_r(x)} u(y)\,\varphi(x-y)\, dAdr \\ &= \int_0^\infty u(x)\,\varphi(r)\,|\partial B_r|\, dr = u(x)\int \varphi(y)\, dy \\ &= u(x).\end{aligned}$$

Consequence $u(x) = \int_{R^n} u(y)\,\varphi(x-y)\, dy$ is smooth for continuous initial $u(y)$, and

$$D^k u(0) = \int u(y)\, D_x^k \varphi(x-y)\, dy = (-1)^k \int u(y)\, D_y^k \varphi(x-y)\, dy.$$

Thus

$$\left|D^k u(0)\right| \leq C(k, n, \varphi)\, \|u\|_{L^1(B_1)}.$$

Scaled version

$$\left|D^k u\left(0\right)\right| \leq \begin{cases} \frac{C(k,n,\varphi)\|u\|_{L^1(B_R)}}{R^{n+k}} \\ \frac{C(k,n,\varphi)\|u\|_{L^\infty(B_R)}}{R^k} \end{cases}.$$

That is the larger the domain, the flatter the harmonic graph.

Application 3. Harnack inequalities.

eg. Consider positive harmonic functions r^{2-n}, $x_1 r^{-n}$ on $\{x_1 > 0\}$.

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Now for $\triangle \mathrm{u} = 0,\ u > 0$ in $B_1\left(0\right)$,

$$u\left(x\right) = \frac{1}{B_{1-|x|}\left(x\right)} \int_{B_{1-|x|}(x)} u dx \leq \frac{1}{\left|B_{1-|x|}\right|} \int_{B_1(0)} u dx = \frac{|B_1|}{\left|B_{1-|x|}\right|} u\left(0\right) = \frac{1}{\left(1-|x|\right)^n} u\left(0\right).$$

Rmk. As those two examples suggest, from Poisson representation, we have

$$c_n\left(1-|x|\right) u\left(0\right) \leq u\left(x\right) \leq \frac{2}{\left(1-|x|\right)^{n-1}} u\left(0\right).$$

Cor. Suppose $\triangle \mathrm{u} = 0,\ u > 0$ in $B_r\left(x_0\right)$. Then we have

$$\sup_{B_{r/4}(x_0)} u \leq 3^n \inf_{B_{r/4}(x_0)} u$$

Consequences $\cdots$, for example one sided Liouville for entire harmonic functions.

<u>Weak formulation</u>

Mean value way.

Suppose $u \in L^1$ satisfy $u\left(x\right) = ⨍_{B_r(x)} u\left(y\right) dy$ for all x and r.

Exercise. Then u is continuous, since

$$u\left(x\right) - u\left(x_0\right) = ⨍_{B_1(x)} u\left(y\right) dy - ⨍_{B_1(x_0)} u\left(y\right) dy \overset{x\to x_0}{\to} 0.$$

$figure$

In turn, we have $u(x) = ⨍_{\partial B_r(x)} u(y)\, dy$. In fact

$$\frac{d}{dr}: \; r^n |B_1| u(x) = \int_{B_r(x)} u(y)\, dy$$
$$n r^{n-1} |B_1| u(x) = \int_{\partial B_r(x)} u(y)\, dy$$
$$|\partial B_r| u(x) = \int_{\partial B_r(x)} u(y)\, dy$$

Then

$$u(x) = \int_{R^n} \varphi(x-y)\, u(y)\, dy \in C^\infty$$

for $\varphi(x) = \varphi(|x|)$ with $\int_{R^n} \varphi(|x|)\, dx = 1$. Let us check $\triangle u = 0$.

$$\int_{\partial B_\varepsilon(0)} u dA = \int_{\partial B_\varepsilon(0)} u(0) + Du(0) \cdot x + \frac{1}{2} \underbrace{D_{ij} u(0)\, x_i x_j}_{\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2} + \varepsilon^3 dA$$
$$|\partial B_\varepsilon| u(0) = |\partial B_\varepsilon| u(0) + 0 + \frac{1}{2}\left(\lambda_1 \frac{\varepsilon^2}{n} + \cdots + \lambda_n \frac{\varepsilon^2}{n}\right) |\partial B_\varepsilon| + O\left(\varepsilon^3\right) |\partial B_\varepsilon|$$
$$\Rightarrow \frac{1}{2n} \triangle u(0) = 0.$$

Integration by parts way.

For $u \in C^0/L^1$/distribution $\int u \triangle \varphi = 0$ for any $\varphi \in C_0^\infty$. How to move to mean value formulation?

Q. How to find $\varphi \in C_0^\infty$ such that

$$\triangle \varphi = \frac{1}{|B_2|} \chi_{B_2} - \frac{1}{|B_1|} \chi_{B_1}?$$

$C^{1,1}$ approach (Caffarelli) $\varphi \sim$ "$\frac{|x|^2}{2n|B_2|} \chi_{B_2}$" $-$ "$\frac{|x|^2}{2n|B_1|} \chi_{B_1}$"

figure

Convolution way: $\varphi * \Gamma \cdots$.

Fun way. This requires $\varphi_2 = \frac{|x|^2}{2n|B_2|} - A$ to touch r^{2-n}, in fact $\frac{-1}{? r^{n-2}}$ at $|x| = 2$. We have a system $\frac{2^2}{2n|B_2|} - A = \frac{-1}{? 2^{n-2}}$ and $\frac{2 \cdot 2}{2n|B_2|} = \frac{(n-2)}{? 2^{n-1}}$ which implies $? = n(n-2)|B_1|$

and $A = \frac{2(n-1)}{n(n-2)|B_2|}$. Similarly we get $\varphi_1 = \frac{|x|^2}{2n|B_1|} - A'$ touching $\frac{-1}{?r^{n-2}} = \frac{-1}{n(n-2)|B_1|r^{n-2}}$ at $|x| = 1$. Thus $\varphi = \varphi_2 - \varphi_1 \in C_0^{1,1}$ answers the above question.

$$\text{For } u \in L^1, \int u \triangle \varphi = 0 \Rightarrow \fint_{B_2} u = \fint_{B_1} u.$$

Therefore (exercise)

$$u(x) = \lim_{r \to 0} \fint_{B_r(x)} u \text{ a.e. at Lebesgue point of } L^1 \; u.$$

Cor. (Weyl) $u \in L^1/C^0$ satisfying $\int u \triangle \varphi = 0$ for any $\varphi \in C_0^\infty$. Then $u \in C^\infty$ and $\triangle u = 0$.

Warning:

$$\int \frac{1}{|x|^{n-2}} \triangle \varphi = c_n \varphi(0) \neq 0 \; !$$

C^∞ approach (Weyl)
Work for $u \in$distribution
$\psi(x) = \psi(|x|) \in C_0^\infty$ with $\int \psi = 1$
$\psi_\varepsilon(x) = \frac{1}{\varepsilon^n} \psi\left(\frac{x}{\varepsilon}\right)$

figure

Step 1. $\varphi_\varepsilon = \Gamma * \psi_\varepsilon = \begin{cases} \Gamma & \text{for } |x| \geq \varepsilon \\ \text{smooth} & \text{for } |x| \leq \varepsilon \end{cases}$.
Step 2. $\triangle \Gamma * \psi = \psi$.
Step 3. $\varphi_{\varepsilon_2} - \varphi_{\varepsilon_1} \in C_0^\infty$

$$\int_{\mathbb{R}^n} u \triangle (\varphi_{\varepsilon_2} - \varphi_{\varepsilon_1}) = 0 \Rightarrow \int_{\mathbb{R}^n} u \psi_{\varepsilon_2} = \int_{\mathbb{R}^n} u \psi_{\varepsilon_1}$$

- $u * \psi_\varepsilon$ is independent of ε
- $u * \psi_\varepsilon \in C^\infty$ (Review distribution theory, try it.)
- $u * \psi_\varepsilon = u$ as a distribution (Exercise).

Pointwise (viscosity) way.

Definition: $u \in C^0$ is a viscosity solution to $\triangle u = 0$, if for any quadratic $P \underset{(\leq)}{\geq} u$ near an interior point x_0 and " $=$" at x_0, then $\triangle P \underset{(\leq)}{\geq} 0$.

Rmk. We can replace those quadratics by equivalent C^2/C^∞ testing functions. Certainly C^2 harmonic functions satisfy this definition. We do have C^0 but non

C^2 solutions to (fully nonlinear) elliptic equations such as Monge-Ampere/Special Lagrangian equations.

We verify C^0 harmonic functions in the viscosity sense are in fact smooth by Poisson representation formula. Note explicitly representation for solutions to nonlinear equations are NOT available in general.

Let

$$h = \int_{\partial B_1} P\left(x, y\right) u\left(y\right)|_{\partial B_1} dA_y$$

· $h = u$ on ∂B_1.
· $\triangle h = 0$ in B_1.

Now if $u > h$ somewhere at $x_0 \in \overset{0}{B}_1$, say $\left(u - h\right)\left(x_0\right) = \max_{B_1}\left(u - h\right) > 0$

$figure$

$h + \max \geq u$ in B_1, "$=$" at x_0.
Also $h + \max' - \varepsilon \left|x\right|^2 \geq u$, "$=$" at $x_0' \in \overset{0}{B}_1$, yes we can replace.
But $\triangle$left$= -2n\varepsilon < 0$. This contradiction shows $u \leq h$.
Similarly, if $u < h$ somewhere at $x_0 \in \overset{0}{B}_1$, say $\left(u - h\right)\left(x_0\right) = \min_{B_1}\left(u - h\right) < 0$

$figure$

$h + \min \leq u$ in B_1, "$=$" at x_0.
Also $h + \min' + \varepsilon \left|x\right|^2 < u$, "$=$" at $x_0' \in \overset{0}{B}_1$, yes we can replace.
But $\triangle$left$= 2n\varepsilon > 0$. This contradiction shows $u \geq h$.
Thus $u \equiv h$.