Lecture 2 Harmonic functions

- invariance
- mean value
  - maximum principle,
  - (higher order) derivative estimates,
  - Harnack
- weak formulations
  - mean value
  - weak/Weyl
  - viscosity

Invariance for Harmonic functions, solutions to $\Delta u = 0$

- $u(x + x_0)$
- $u(Rx)$
- $u(tx)$

RMK. Equations don’t know/care which coordinates they are in.

- $u + v, au$, where $\Delta v = 0$
- $\int u(x - y) \varphi(y) \, dy$
- $\frac{u(x + \varepsilon x) - u(x)}{\varepsilon} \to D_k u$, so is $D^k u$
- $\frac{u(R \varepsilon x) - u(x)}{\varepsilon} \to D_k u = x_i u_j - x_j u_i$
- $\frac{u((1+\varepsilon) x) - u(x)}{\varepsilon} \to Du(x) \cdot x = ru_r$, so are $r \partial_r (ru_r) = ru_r + r^2 u_{rr}, r^3 u_{rrr}, \cdots$
- $|x|^{2-n} u \left( \frac{x}{|x|^2} \right)$ Kelvin transformation

Rmk. “Kelvin” transformation for the heat equation $u_t - \Delta u = 0$, $\frac{1}{r^{n/2}} e^{-\frac{|x|^2}{4t}} u \left( \frac{x}{t}, \frac{1}{t} \right)$.

More harmonic functions.

eg1. $D_1 r^{2-n} = (2-n) r^{1-n} \frac{x_1}{r} = (2-n) r^{-n} x_1 = (2-n) \frac{x_1}{r^n}$

$$D_{11} r^{2-n} = (2-n) \left[ -n r^{-n} \frac{x}{r} x_1 + r^{-n} \right] = (2-n) \frac{r^2 - n x_1^2}{r^{n+2}}$$

$$D_{12} r^{2-n} = (2-n) \frac{-n x_1 x_2}{r^{n+2}}$$

Let $P_k(x)$ be any homogeneous polynomial of degree k, $P_k(D) r^{2-n} = \frac{H_k(x)}{r^{n-2-k}}$. For example, $\sigma_k(D) r^{2-n} = \frac{\sigma_k(x)}{r^{n-2-k}}$. Note $H_k \neq P_k$ in general, but $H_k(x) = r^{2-n} \frac{H_k \left( \frac{x}{r^2} \right)}{|x|^{n-2-k}}$ is the Kelvin transform of harmonic function $P_k(D) r^{2-n}$, thus harmonic.

Exercise: $H_k(x)$ are ALL harmonic polynomials of degree k.

eg2. Harmonic function

$$|x - x_0|^{2-n} - |x|^{2-n} \left| \frac{x}{|x|^2} - x_0 \right|^{2-n} \mid x = 1 |x - x_0|^{2-n} - |x - x_0|^{2-n} = 0,$$

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is Green’s function (up to a multiple) for the unit ball.

Mean value equality

Recall the divergence formula (the fundamental theorem of calculus)

$$\int_{\Omega} \text{div} \left( \vec{V} \right) \, dx = \int_{\partial\Omega} \langle \vec{V}, \gamma \rangle \, dA.$$  

\[ \vec{V} = Du, \text{ then } 0 = \int_{\partial\Omega} u \gamma \, dA. \]

\[ \vec{V} = vDu, \text{ then } \int_{\Omega} \langle Dv, Du \rangle + v \triangle u = \int_{\partial\Omega} vu \gamma \, dA. \]

\[ \vec{V} = uDv, \text{ then } \int_{\Omega} \langle Du, Dv \rangle + u \triangle v = \int_{\partial\Omega} uv \gamma \, dA. \]

Mean value case. Now \( \triangle u = 0 \) in \( B_1 \), \( v = |x|^{2-n} \), \( \Omega = B_1 \setminus B_{\epsilon} \)

\( \int_{\Omega} v \triangle u - u \triangle v = \int_{\partial\Omega} vu \gamma - uv \gamma \, dA. \)

0 = \int_{\partial\Omega} vu \gamma - uv \gamma \, dA, or

\[ \int_{\partial(B_1 \setminus B_{\epsilon})} vu \gamma \, dA = \int_{\partial(B_1 \setminus B_{\epsilon})} uv \gamma \, dA = \int_{\partial B_1} u \frac{2-n}{n-1} \, dA - \int_{\partial B_{\epsilon}} u \frac{2-n}{n-1} \, dA. \]  

\( \text{(*)} \)

We get \( \int_{\partial B_1} udA = \int_{\partial B_{\epsilon}} u \frac{1}{\epsilon^{n-1}} \, dA \xrightarrow{\epsilon \to 0} c_n u(0). \) So \( u(0) = \frac{1}{c_n} \int_{\partial B_1} udA. \) Taking \( u \equiv 1 \) leads to \( c_n = |\partial B_1| = n |B_1|. \)

Also

\[ u(0) = \frac{1}{|\partial B_r|} \int_{\partial B_r} udA. \]

Take a weight function \( |\partial B_r|, u(0) |B_1| = \int_0^1 u(0) |\partial B_r| \, dr = \int_0^1 \int_{\partial B_r} udA \, dr = \int_{B_1} u \, dx. \) So \( u(0) = \frac{1}{|B_1|} \int_{B_1(0)} u \, dx. \)

Also

\[ u(0) = \frac{1}{|B_r|} \int_{B_r(0)} u \, dx. \]

RMK. “... all the women are strong, all the men are good-looking, and all the children are above average.” –A Prairie Home Companion with Garrison Keillor.

Also

\[ u(x) = \int_{\mathbb{R}^n} \frac{-1}{(n-2)n |B_1|} \frac{1}{|x-y|^{n-2}} \triangle u(y) \, dy \text{ for } u \in C_0^\infty(\mathbb{R}^n). \]
Green case. Still $\triangle u = 0$ in $B_1$, but

$$v = G(x, x_0) = \frac{-1}{(n - 2)|\partial B_1|} \left( \left| x - x_0 \right|^{2-n} - \left| x \right|^{2-n} \right) \frac{x}{|x|^2} - x_0 \right)^{2-n},$$

$\Omega = B_1 \setminus B_\varepsilon(x_0)$.

Taking limits on two ends of (*), we get

$$u(x_0) = \int_{\partial B_1} \frac{\partial G(x, x_0)}{\partial \gamma_x} u(x) dA.$$ 

Note $u(x_0) = \int_{\partial B_1} \frac{\partial G(x, x_0)}{\partial \gamma_x} \varphi(x) dA$, as sum of harmonic functions $\frac{\partial G(x, x_0)}{\partial \gamma_x}$, is harmonic for $\varphi \in C^0, L^1, \ldots$.

Application 1. Strong maximum principle (No toughing).

$$\triangle u_1 = \triangle u_2 = 0$$

$$u_1 \geq u_2, \quad \text{"=} \quad \text{at } 0$$

then

$$0 = u_1(0) - u_2(0) = \frac{1}{|B_1|} \int_{B_r} (u_1 - u_2) dx \geq 0.$$ 

It follows that $u_1 \equiv u_2$.

Application 2. Smooth effect and derivative test.

Take radial weight $\varphi(y) = \varphi(|y|) \in C_0^\infty (R^n)$ such that $1 = \int \varphi(y) dy = \int_0^\infty \varphi(r) |\partial B_r| dr$.

Then

$$\int_{R^n} u(y) \varphi(x - y) dy = \int_0^\infty \int_{\partial B_r(x)} u(y) \varphi(x - y) dAdr$$

$$= \int_0^\infty u(x) \varphi(r) |\partial B_r| dr = u(x) \int \varphi(y) dy$$

$$= u(x).$$

Consequence $u(x) = \int_{R^n} u(y) \varphi(x - y) dy$ is smooth for continuous initial $u(y)$, and

$$D^k u(0) = \int u(y) \frac{D^k}{\partial y^k} \varphi(x - y) dy = (-1)^k \int u(y) \frac{D^k}{\partial y^k} \varphi(x - y) dy.$$ 

Thus

$$\left| D^k u(0) \right| \leq C(k, n, \varphi) \| u \|_{L^1(B_1)}.$$
Scaled version

\[ |D^k u(0)| \leq \begin{cases} \frac{C(k,n,\varphi)\|u\|_{L^1(B_R)}}{R^{n+k}} \\ \frac{C(k,n,\varphi)\|u\|_{L^\infty(B_R)}}{R^k} \end{cases}. \]

That is the larger the domain, the flatter the harmonic graph.

Application 3. Harnack inequalities.

eg. Consider positive harmonic functions \( r^{2-n}, x_1 r^{-n} \) on \( \{ x_1 > 0 \} \).

\textit{figure}

Now for \( \triangle u = 0, \ u > 0 \) in \( B_1(0) \),

\[ u(x) = \frac{1}{B_1-|x|}(x) \int_{B_1-|x|} u dx \leq \frac{1}{|B_1-|x||} \int_{B_1(0)} u dx = \frac{|B_1|}{|B_1-|x||} u(0) = \frac{1}{(1-|x|)^n} u(0). \]

Rmk. As those two examples suggest, from Poisson representation, we have

\[ c_n (1-|x|) u(0) \leq u(x) \leq \frac{2}{(1-|x|)^{n-1}} u(0). \]

Cor. Suppose \( \triangle u = 0, \ u > 0 \) in \( B_r(x_0) \). Then we have

\[ \sup_{B_{r/4}(x_0)} u \leq 3^n \inf_{B_{r/4}(x_0)} u \]

Consequences \( \cdots \), for example one sided Liouville for entire harmonic functions.

Weak formulation

Mean value way.

Suppose \( u \in L^1 \) satisfy \( u(x) = \int_{B_r(x)} u(y) dy \) for all \( x \) and \( r \).

Exercise. Then \( u \) is continuous, since

\[ u(x) - u(x_0) = \int_{B_1(x)} u(y) dy - \int_{B_1(x_0)} u(y) dy \xrightarrow{x \to x_0} 0. \]
In turn, we have \( u(x) = \int_{\partial B_r(x)} u(y) \, dy \). In fact

\[
\frac{d}{dr} : r^n |B_1| u(x) = \int_{B_r(x)} u(y) \, dy
\]

\[
nr^{n-1} |B_1| u(x) = \int_{\partial B_r(x)} u(y) \, dy
\]

\[
|\partial B_r| u(x) = \int_{\partial B_r(x)} u(y) \, dy
\]

Then

\[
u(x) = \int_{\mathbb{R}^n} \varphi(x-y) u(y) \, dy \in C^\infty
\]

for \( \varphi(x) = \varphi(|x|) \) with \( \int_{\mathbb{R}^n} \varphi(|x|) \, dx = 1 \). Let us check \( \Delta u = 0 \).

\[
\int_{\partial B_\epsilon(0)} udA = \int_{\partial B_\epsilon(0)} u(0) + Du(0) \cdot x + \frac{1}{2} D_{ij} u(0) x_i x_j + \varepsilon^3 dA
\]

\[
|\partial B_\epsilon| u(0) = |\partial B_\epsilon| u(0) + \frac{1}{2} \left( \lambda_1 \varepsilon^2 + \cdots + \lambda_n \varepsilon^2 \right) |\partial B_\epsilon| + O(\varepsilon^3) |\partial B_\epsilon|
\]

\[
\Rightarrow \frac{1}{2n} \Delta u(0) = 0.
\]

Integration by parts way.

For \( u \in C^0/L^1/distribution \int u \Delta \varphi = 0 \) for any \( \varphi \in C^\infty_0 \). How to move to mean value formulation?

Q. How to find \( \varphi \in C^\infty_0 \) such that

\[
\Delta \varphi = \frac{1}{|B_2|} \chi_{B_2} - \frac{1}{|B_1|} \chi_{B_1}?
\]

\( C^{1,1} \) approach (Caffarelli) \( \varphi \sim \frac{|x|^2}{2n |B_2|} \chi_{B_2} - \frac{|x|^2}{2n |B_1|} \chi_{B_1} \)

\( figure \)

Convolution way: \( \varphi * \Gamma \cdots \).

Fun way. This requires \( \varphi_2 = \frac{|x|^2}{2n |B_2|} - A \) to touch \( r^{2-n} \), in fact \( \frac{-1}{r^{n-2}} \) at \( |x| = 2 \). We have a system \( \frac{r^2}{2n |B_2|} - A = \frac{-1}{r^{n-2}} \) and \( \frac{2r^2}{2n |B_2|} = \frac{(n-2)}{r^{n-2}} \) which implies \( ? = n(n-2) |B_1| \)
and \( A = \frac{2(n-1)}{n(n-2)|B_2|} \). Similarly we get \( \varphi_1 = \frac{|x|^2}{2n|B_1|} - A' \) touching \( \frac{1}{r^{n-2}} = \frac{-1}{n(n-2)|B_1|r^{n-2}} \) at \( |x| = 1 \). Thus \( \varphi = \varphi_2 - \varphi_1 \in C_0^{1,1} \) answers the above question.

For \( u \in L^1 \), \( \int u \triangle \varphi = 0 \Rightarrow \int_{B_2} u = \int_{B_1} u \).

Therefore (exercise)

\[
\lim_{r \to 0} \int_{B_r(x)} u \text{ a.e. at Lebesgue point of } L^1 u.
\]

Cor. (Weyl) \( u \in L^1/C^0 \) satisfying \( \int u \triangle \varphi = 0 \) for any \( \varphi \in C_0^\infty \). Then \( u \in C^\infty \) and \( \triangle u = 0 \).

Warning:

\[
\int \frac{1}{|x|^{n-2}} \triangle \varphi = c_n \varphi(0) \neq 0!
\]

\( C^\infty \) approach (Weyl)

Work for \( u \in \text{distribution} \)

\( \psi(x) = \psi(|x|) \in C_0^\infty \) with \( \int \psi = 1 \)

\( \psi_\varepsilon(x) = \frac{1}{\varepsilon^n} \psi \left( \frac{x}{\varepsilon} \right) \)

**figure**

Step 1. \( \varphi_\varepsilon = \Gamma \ast \psi_\varepsilon = \begin{cases} 
\Gamma & \text{for } |x| \geq \varepsilon \\
\text{smooth for } |x| \leq \varepsilon 
\end{cases} \).

Step 2. \( \triangle \Gamma \ast \psi = \psi \).

Step 3. \( \varphi_{\varepsilon_2} - \varphi_{\varepsilon_1} \in C_0^\infty \)

\[
\int_{\mathbb{R}^n} u \triangle (\varphi_{\varepsilon_2} - \varphi_{\varepsilon_1}) = 0 \Rightarrow \int_{\mathbb{R}^n} u \psi_{\varepsilon_2} = \int_{\mathbb{R}^n} u \psi_{\varepsilon_1}
\]

- \( u \ast \psi_\varepsilon \) is independent of \( \varepsilon \)
- \( u \ast \psi_\varepsilon \in C^\infty \) (Review distribution theory, try it.)
- \( u \ast \psi_\varepsilon = u \) as a distribution (Exercise).

Pointwise (viscosity) way.

**Definition:** \( u \in C^0 \) is a viscosity solution to \( \triangle u = 0 \), if for any quadratic \( P \geq u \) near an interior point \( x_0 \) and "=\( \leq \) at \( x_0 \), then \( \Delta P \geq 0 \).

**Rmk.** We can replace those quadratics by equivalent \( C^2/C^\infty \) testing functions. Certainly \( C^2 \) harmonic functions satisfy this definition. We do have \( C^0 \) but non
$C^2$ solutions to (fully nonlinear) elliptic equations such as Monge-Ampere/Special Lagrangian equations.

We verify $C^0$ harmonic functions in the viscosity sense are in fact smooth by Poisson representation formula. Note explicitly representation for solutions to nonlinear equations are NOT available in general.

Let

$$ h = \int_{\partial B_1} P(x,y) u(y) |_{\partial B_1} dA_y $$

- $h = u$ on $\partial B_1$.
- $\Delta h = 0$ in $B_1$.

Now if $u > h$ somewhere at $x_0 \in B_1$, say $(u - h)(x_0) = \max_{B_1} (u - h) > 0$

\[ \text{figure} \]

$h + \max \geq u$ in $B_1$, “=” at $x_0$.

Also $h + \max' - \varepsilon |x|^2 \geq u$, “=” at $x_0' \in B_1$, yes we can replace.

But $\Delta \left( = -2n\varepsilon < 0 \right)$. This contradiction shows $u \leq h$.

Similarly, if $u < h$ somewhere at $x_0 \in B_1$, say $(u - h)(x_0) = \min_{B_1} (u - h) < 0$

\[ \text{figure} \]

$h + \min \leq u$ in $B_1$, “=” at $x_0$.

Also $h + \min' + \varepsilon |x|^2 \leq u$, “=” at $x_0' \in B_1$, yes we can replace.

But $\Delta \left( = 2n\varepsilon > 0 \right)$. This contradiction shows $u \geq h$.

Thus $u \equiv h$. 

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