Lecture 1 Introduction

○ examples of equations: what and why
○ "intrinsic" view, physical origin, probability, geometry

Intrinsic/abstract

$$F\left(x, Du, D^2u, D^3u, \cdots\right) = 0$$

Recall algebraic equations such as linear (algebra) one and quadratic one: $x^2+y^2 = z^2$, $x^2 + y^2 = 1, x^2 - y^2 = 1, y = x^2$. Now just replace the variables with derivatives, we have partial differential equations, PDE in short.

1st order $b \cdot Du = 0$, $|Du| = 1$

2nd order $u_{11} = 0$, $u_{12} = 0$

u first derivatives Du and double derivatives $D^2u \sim \begin{bmatrix} \lambda_1 & & \\ & & \\ & & \lambda_n \end{bmatrix}$

coordinate free ones

Laplace $\triangle u = \sigma_1 = \lambda_1 + \cdots + \lambda_n = c$

$\sigma_k = \lambda_1 \cdots \lambda_k + \cdots = c$

M-A $\det D^2u = \sigma_n = \lambda_1 \cdots \lambda_n = c$

$\lambda_1 - \lambda_2$ or $\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_3\lambda_1$ hardly make sense.

Adding time, $u_{tt} = \triangle u$, $u_t = \triangle u$

3rd order?

4th order $\triangle^2 u = 0$

.....

combinations of the above.

Concrete

Transport equation $u_t = -\operatorname{div}(u\,\mathrm{V}) \overset{V\ \text{const}}{=} -V \cdot Du$

$u(x,t)$ moisture density

$V(x,t)$ wind velocity field

$figure$

moisture changing rate over domain $\Omega : \frac{d}{dt}\int_\Omega u dx = \int_\Omega u_t dx.$

Via its boundary with exterior unit normal $\gamma : -\int_{\partial\Omega} u\, V \cdot \gamma dA = -\int_\Omega \operatorname{div}(u\ V)\, dx$

As Ω is arbitrary, we have $u_t = -\operatorname{div}(u\,\mathrm{V})$.

Heat conduct $u_t = \triangle u$

$u(x,t)$ temperature/heat

heat changing rate over domain $\Omega : \frac{d}{dt}\int_\Omega u dx = \int_\Omega u_t dx.$

Via boundary, as heat flows from high temp to low along $-Du$ direction: $\int_{\partial\Omega} Du \cdot \gamma dA = \int_\Omega \operatorname{div}(Du)\, dx$

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Again, as Ω is arbitrary, we have $u_t = \operatorname{div}(Du) = \triangle u$.

Probability

Brownian motion

Let us test it by function $f(x)$

$u(x,t) = E[f(B_t(x))]$ expectation/average of f at Brownian motion position B_t after time t, starting from x. Say we in 1-d case

$$\begin{aligned} u\left(x, t+\varepsilon^2\right) &= E\left[f\left(B_{t+\varepsilon^2}(x)\right)\right] = \frac{1}{2}E\left[f\left(B_t(x-\varepsilon)\right)\right] + \frac{1}{2}E\left[f\left(B_t(x+\varepsilon)\right)\right] \\ &= \frac{u(x-\varepsilon,t) + u(x+\varepsilon,t)}{2} \end{aligned}$$

it follows that

$$\frac{u(x,t+\varepsilon^2) - u(x,t)}{\varepsilon^2} = \frac{u(x-\varepsilon,t) + u(x+\varepsilon,t) - 2u(x,t)}{2\varepsilon^2}.$$

Let ε go to 0, we have $u_t = \frac{1}{2}u_{xx}$. Similarly $u_t = \frac{1}{2^n} \triangle u$ in n-d.

Random walk, when hits boundary, the pay off is $\varphi(x)$.

$$\begin{cases} \frac{1}{2}u_{xx} + \frac{1}{2}u_{yy} = 0 & \text{in } \Omega \\ u = \varphi(x) & \text{on } \partial\Omega \end{cases}$$

$$figure$$

Let $u(x)$ be the expectation of pay off, starting from interior point $x \in \Omega$, with directional probability $p_h = 1/2$ and $p_v = 1/2$, say we are in 2d case.

$$u(x) = \frac{1}{2}\left[\frac{u(x+\varepsilon e_h) + u(x-\varepsilon e_h)}{2}\right] + \frac{1}{2}\left[\frac{u(x+\varepsilon e_v) + u(x-\varepsilon e_v)}{2}\right]$$

$\Rightarrow$

$$0 = p_h u_{hh} + p_v u_{vv} = \frac{1}{2}\left(u_{xx} + u_{yy}\right)$$

Wave equation $u_{tt} = \triangle u$

"Vertical" oscillation of string and drum usually can be modelled by 1-d and 2-d wave equation respectively. Sound wave in the air can be conveniently described by a scalar, density or pressure of the air (not clear about other vector ways).

$u(x,t)$ air/gas density at (x,t)

$p = p(u)$ pressure is in terms of u

$V(x,t)$ local average velocity of the air/gas (average velocity makes more sense than "individual" one for each air/gas particle)

As in the above transport equation, the mass conservation law says $\frac{d}{dt}\int_\Omega u dx = -\int_{\partial\Omega} uV \cdot \gamma dA$ or

$$u_t = -\operatorname{div}(uV).$$

Newton's second law of force is $ma = F$. The force comes from the pressure, along $-Dp$. But as the mass density is changing, ma should be changed to the changing rate of the (average) momentum $(uV)_t$. That is

$$\text{Newton (momentum version):} \quad (uV)_t = F = -Dp.$$

Eliminate uV, we have

$$u_{tt} = \operatorname{div}(Dp).$$

When the air/gas is ideal, the pressure is proportional to the density u and temperature, the sound wave equation is (all constants are 1)

$$u_{tt} = \triangle u = u_{xx} + u_{yy} + u_{zz}.$$

When there is no (time for) heat change (called adiabatic), the pressure is nonlinearly proportional to the density $p(u) = u^\beta$, the sound wave equation is quasilinear

$$u_{tt} = \operatorname{div}\left(Du^\beta\right).$$

Schrodinger's wave equation

$$i\hbar u_t = -\frac{\hbar^2}{2m} \triangle u + Vu \quad \hbar = \frac{h}{2\pi} \text{ Planck's constant}$$

Water wave (along river) Korteweg-de Vries equation

$$u_t + u\, u_x + u_{xxx} = 0$$

Scalar curvature equation of $\left(M, u^{4/(n-2)} g_0\right)$
$n \geq 3 \;\; R\left(u^{4/(n-2)} g_0\right) = u^{-\frac{n+2}{n-2}} \left(- \triangle_{g_0} u + c(n) R_0 u\right)$
$n = 2 \;\; R\left(e^{2u} g_0\right) = e^{-2u} \left(- \triangle_{g_0} u + R_0\right)$

Variational
$E[u] = \int_\Omega F(Du)\, dx$
$\varphi \in C_0^\infty(\Omega)$

$$\begin{aligned} \frac{d}{dt} \int F(Du + tD\varphi)\, dx|_{t=0} &= \int \sum F_{p_i}(Du) \frac{\partial \varphi}{\partial x_i} dx \\ &= \int -\sum \frac{\partial}{\partial x_i} \left[F_{p_i}(Du)\right] \varphi d \end{aligned}$$

$$\sum \frac{\partial}{\partial x_i} \left[F_{p_i}(Du)\right] = 0.$$

eg1. $F(Du) = |Du|^2$ Energy $F_{p_i} = 2Du \dashrightarrow \triangle u = 0.$

eg2. $F(Du) = \sqrt{1+|Du|^2}$ Area $F_{p_i} = \frac{Du}{\sqrt{1+|Du|^2}}$ ⇢ mean curvature $H = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0.$

eg3. $E[u] = \int \sigma_{k-1}(\kappa)\sqrt{1+|Du|^2}dx$, E-L equation $\sigma_k(\kappa) = 0$ (Reilly).

RMK. One obvious thing

1d principle curvature of curve $(x, f(x))$

$\kappa = \frac{f_{xx}}{\left(\sqrt{1+f_x^2}\right)^3} = \left(\frac{f_x}{\sqrt{1+f_x^2}}\right)_x$

also $\int \kappa ds = \int \frac{f_{xx}}{\left(\sqrt{1+f_x^2}\right)^3}\sqrt{1+f_x^2}dx = \int (\arctan f_x)_x\, dx = \arctan f_x|_\partial$

2d

$$H = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{1+|Du|^2}}\left[\left(1+u_y^2\right)u_{xx} - 2u_xu_yu_{xy} + \left(1+u_x^2\right)u_{yy}\right].$$

RMK. Equation for 2d steady, adiabatic,irrotational, isentropic flow u ⇢ $\sqrt{-1}u$

$$\left(1-u_y^2\right)u_{xx}a + 2u_xu_yu_{xy} + \left(1-u_x^2\right)u_{yy} = 0.$$

Q. In nd similar thing should happen to the total Gauss curvature

$$\int \sigma_n(\kappa)\sqrt{1+|Du|^2}dx?$$

More variationals

Eg $\sigma_k : E[u] = -\frac{1}{k+1}\int u\sigma_k(D^2u)\,dx + \int udx$

E-L equation $\sigma_k(D^2u) = 1$. This can be derived using the following divergence structure.

$$\begin{aligned} k\sigma_k = D_\lambda\sigma_k\cdot\lambda &= \frac{\partial\sigma_k(D^2u)}{\partial m_{ij}}D_{ij}u \\ &= \frac{\partial}{\partial_{x_i}}\left[\frac{\partial\sigma_k(D^2u)}{\partial m_{ij}}\partial_{x_j}u\right] + \underbrace{\frac{\partial}{\partial_{x_i}}\left[\frac{\partial\sigma_k(D^2u)}{\partial m_{ij}}\right]}_{0}\partial_{x_j}u. \end{aligned}$$

Eg Slag: $A[DU] = \int\sqrt{\det\left(I+(DU)^T DU\right)}dx$, $U:\Omega\to R^n$.

Insist **min**imizer irrotational, i.e. $U = Du$, then E-L

$$D\sum\arctan\lambda_i = 0 \Leftrightarrow \sum\arctan\lambda_i = c.$$

$A[DU] = \int\sqrt{\det\left(I-(DU)^T DU\right)}dx$, $U:\Omega\to R^n$.

Insist **max**imizer irrotational, i.e. $U = Du$, then E-L

$$D\sum \ln \frac{1+\lambda_i}{1-\lambda_i} = 0 \Leftrightarrow \sum \ln \frac{1+\lambda_i}{1-\lambda_i} = c \leftarrow\dashrightarrow \sum \ln \bar{\lambda}_i = c.$$

$$figure?$$

Explicit solutions

$\circ\ H = 0$

catenoid: $|(x, y)| = \cosh z$

helicoid: $z = \arctan \frac{y}{x}$

Sherk's surface: $z = \ln \frac{\cos y}{\cos x}$

$\circ\ H_k = const.$

unit sphere

$\circ\ \triangle u = 0$

complex analysis in even d: $u = \operatorname{Re} z^k,\ z^{-k},\ e^z,\ z_1^3 e^{z_2}, \cdots$

algebraic n-d $u = \sigma_k (x_1, \cdots, x_2)$

radial

$$\triangle u = \partial_r^2 u + \frac{n-1}{r} \partial_r u + \frac{1}{r^2} \triangle_{S^{n-1}} u$$

$u_{rr} + \frac{n-1}{r} u_r = 0$

$r^{n-1} u_{rr} + (n-1) r^{n-2} u_r = 0$ or $(r^{n-1} u_r)_r = 0$

$u_r = \frac{c}{r^{n-1}}$

$$u = \frac{c}{r^{n-2}}\ ,\ \ln|(x_1, x_2)|\,,\ \text{or}\ |x_1|$$

Fluid mechanics

vector field $\vec{V}(x,t) \overset{\text{steady state}}{=} \vec{V}(x)$

incompressible $div\left(\vec{V}\right) = 0$

irrotational $curl\vec{V} = 0 \Longleftrightarrow D\vec{V} = \left(D\vec{V}\right)^T \Longrightarrow \vec{V} = D\varphi$

$\Rightarrow \triangle \varphi = 0$

Navier-Stokes equation (incompressible)

$$\begin{cases} u_t + u \cdot Du - \triangle u + Dp = 0 \\ \operatorname{div} u = 0 \end{cases}$$

Vector $u(x,t)$ velocity field, $p(x,t)$ pressure

$ma = F$ and $X_t = u(X, t)$

Acceleration, $X_{tt} = u_t + X_t \cdot Du = u_t + u_t \cdot Du$

Force comes from two parts: pressure $= -Dp$, and viscosity $\triangle u$ "ad-hoc" due to sheer stress caused by difference of velocity.

Heuristic derivation

$$\text{viscosity } = \frac{\text{average velocity in } B_r(x) - \text{velocity at } x}{r^2} \approx \triangle u$$

Physical derivation

$$\begin{aligned} \text{sheer force } &= c\delta_A \frac{\delta u}{\delta x_i} = c\delta_A \partial_i u \\ \text{viscous force/per unit volume } &= \sum_i \frac{\delta \text{sheer force in } x_i \text{ direction}}{\delta A \ \delta x_i} = c \sum_i \partial_{ii} u = c \triangle u \end{aligned}$$

Maxwell equation
electric field $\vec{E} = (E^1, E^2, E^3)$
magnetic field $\vec{H} = (H^1, H^2, H^3)$

$$\begin{cases} \varepsilon \vec{E}_t = \text{curl}\, \vec{H} \\ \mu \vec{H}_t = \text{curl}\, \vec{E} \end{cases}$$

RMK. div $\vec{E} = 0$, since $\left(\text{div}\, \vec{E}\right)_t = \text{div}\left(\text{curl}\, \vec{H}\right) = 0$ and div $\vec{E} = 0$ at $t = 0$. Similarly div $\vec{H}(t) = 0$ for div $\vec{H} = 0$ at $t = 0$.

Lame elastic wave

$$\vec{U}_{tt} - \mu \triangle U - (\lambda + \mu) D \left(\text{div}\, \vec{U}\right) = 0$$

Harmonic maps

Consider the energy of vector m-valued function of n-variables, let us look at critical point(s) of the energy functional with pointwise constraint

$$E(w) = \int_\Omega \frac{1}{2} |Dw|^2 \, dx \quad \text{with } w \cdot w = 1,$$

the Euler-Lagrangian equation is

$$-\triangle u = |Du|^2 u.$$

For a critical point, function $u : \Omega \to S^{m-1} \subset R^m$ now, we take a variation $\eta \in C_0^\infty(\Omega; R^m)$, but to the sphere $(u + \varepsilon\eta) / |u + \varepsilon\eta|$

$$\begin{aligned} \frac{d}{d\varepsilon} E\left(u + (u + \varepsilon\eta) / |u + \varepsilon\eta|\right) &= \int_\Omega \left\langle D \frac{u + \varepsilon\eta}{|u + \varepsilon\eta|}, D\left(\frac{\eta}{|u + \varepsilon\eta|} - (u + \varepsilon\eta) \frac{(u + \varepsilon\eta) \cdot \eta}{|u + \varepsilon\eta|^3}\right)\right\rangle dx \\ &\overset{\varepsilon=0}{=} \int_\Omega \langle Du, D(\eta - u \ u \cdot \eta)\rangle \, dx \\ &= \int_\Omega \langle Du, D\eta - Du \ u \cdot \eta - uD(u \cdot \eta)\rangle \, dx \\ &= \int_\Omega \langle Du, D\eta - Du \ u \cdot \eta\rangle \, dx \quad \left(u \cdot u = 1 \text{ implies } \sum u^\alpha Du^\alpha = 0\right) \\ &= \int_\Omega \text{div}\left(\eta^T Du\right) - \langle \triangle u, \eta\rangle - |Du|^2 u \cdot \eta \ dx \end{aligned}$$

thus the equation.

For general constraint such as ellipsoid, hyperboloid, paraboloid, etc, we employ Lagrangian multiplier to get the critical equation. Say now the constraint is $S(u) = 0$. The critical equations are critical points of augment functional

$$E(w) - \int_\Omega \lambda(x) S(w)\, dx.$$

The variation w.r.t. $w + \varepsilon\eta$ and $\lambda + \delta f$ $(f \in C_0^\infty(\Omega; R^1))$ leads to respectively

$$-\langle \triangle u, \eta \rangle = \lambda \langle \nabla_u S, \eta \rangle \quad \text{and } S(u) = 0.$$

In order to pin down $\lambda(x)$, we take η as $\nabla_u S$, or rather $f\ \nabla_u S$ with f being one near any fixed interior point of Ω and zero near its boundary. Then near the fixed interior point, we have

$$\lambda \langle \nabla_u S, \nabla_u S \rangle = -\langle \triangle u, \nabla_u S \rangle = -\operatorname{div} \langle Du, \nabla_u S \rangle + \langle Du, D\nabla_u S \rangle = \langle Du, D\nabla_u S \rangle,$$

where we used $0 = D[S(u)] = \langle Du, \nabla_u S \rangle = \partial_{u^\alpha} S\ Du^\alpha$ and the notation $\langle Du, D\nabla_u S \rangle = D_{x_i} u^\alpha\ D_{x_i} \partial_{u^\alpha} S$. Then

$$\lambda = \frac{\langle Du, D\nabla_u S \rangle}{\langle \nabla_u S, \nabla_u S \rangle}$$

and our equation becomes

$$-\triangle u = \frac{\langle Du, D\nabla_u S \rangle}{\langle \nabla_u S, \nabla_u S \rangle} \nabla_u S.$$

Reality check ....

Einstein equation
Canonical metric $Ric(g) = cg$
g Riemannian

$$Ric(g) = -g^{ij} D_{ij} g + (Dg, g)$$

g pseudo Riemannian (general relativity) such as $dx^2 - dt^2$

$$Ric(g) = \text{“} -\triangle_x g + D_{tt} g''.$$

Ricci flow

$$g_t = -Ric(g) = \overbrace{-g^{ij} D_{ij} g}^{\text{diffusion}} + (Dg, g).$$

RMK. The “heat” equation $g_t = \triangle_g g = 0$ is static.

Observation: The most frequent combination is $\triangle u$. So we study

- $\triangle u,\ u_t = \triangle u,\ u_{tt} = \triangle u$
- “ $\triangle$” $u,\ u_t =$ “ $\triangle$” $u,\ u_{tt} =$ “ $\triangle$” u general linear methods
- nonlinear methods.