M309 Linear Analysis Name:____________________
Midterm
Feb. 11/13/15, 2019 Student Id:____________________

Section: __________

1. (25pts) Find the fundamental matrix e^{Jt} with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ by diagonalizing $J = (V_1\ V_2)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}(V_1\ V_2)^{-1}$. (Recall we have obtained e^{Jt} by Taylor expansion.)

Sol. $\cdot$ E-Vs $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$, and then $\lambda_1 = i, \lambda_2 = -i$

$\lambda_1 = i, \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} V_1 = 0$, and then $V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$; $\lambda_2 = -i = \bar{\lambda}_1, V_2 = \bar{V}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ (10pts)

$\cdot$ Inverse $(V_1\ V_2)^{-1} = \frac{1}{-2i}\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ (5pts)

$\cdot\ e^{Jt} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{i} & \\ & e^{-i} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\frac{1}{2} = \begin{pmatrix} e^{i} & e^{-i} \\ ie^{i} & -ie^{-i} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\frac{1}{2}$

$= \begin{pmatrix} e^{i} + e^{-i} & -i\left(e^{i} - e^{-i}\right) \\ i\left(e^{i} - e^{-i}\right) & -i^2\left(e^{i} + e^{-i}\right) \end{pmatrix}\frac{1}{2} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ (10pts)

2. (25pts) Consider the system $x' = \begin{pmatrix} -7 & -2 \\ 2 & -2 \end{pmatrix} x$.

a) Draw a phase plane/direction field and a few trajectories. Is the origin an attractor/stable note, repeller/unstable note, saddle point, center, or spiral point?

b) Solve the system with initial position $x(0) = \begin{pmatrix} 18 \\ -17 \end{pmatrix}$. Describe its asymptotic behavior as time t goes to ∞.

c) Find the general solutions and describe their asymptotic behavior as time t goes to ∞.

Sol a) $\cdot$ E-Vs $\begin{vmatrix} -7-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = \lambda^2 + 9\lambda + 18 = (\lambda+3)(\lambda+6) = 0$, and then $\lambda_1 = -3$, $\lambda_2 = -6$

$\lambda_1 = -3, \begin{pmatrix} -4 & -2 \\ 2 & 1 \end{pmatrix} V_1 = 0$, and then $V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda_2 = -6, \begin{pmatrix} -1 & -2 \\ 2 & 4 \end{pmatrix} V_2 = 0$, and then $V_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. (6pts)

$\cdot$ Phase plane/portrait ..... The origin is an attractor/stable note. (4pts)

b) $e^{Mt} = (V_1\ V_2)\begin{pmatrix} e^{-3t} & \\ & e^{-6t} \end{pmatrix}\underbrace{(V_1\ V_2)^{-1}x(0)} = \left(e^{-3t}V_1\ e^{-6t}V_2\right)\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$,

where b satisfies $(V_1\ V_2)\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 18 \\ -17 \end{pmatrix}$, and then $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 16/3 \\ 19/3 \end{pmatrix}$.

Then $x(t) = \frac{16}{3}e^{-3t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{19}{3}e^{-6t}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. (8pts)

And $x(t)$ goes to $0_{2\times 1}$ as t goes to ∞. (2pts)

c) General solutions are $c_1 e^{-3t}\begin{pmatrix}1\\-2\end{pmatrix} + c_2 e^{-6t}\begin{pmatrix}2\\-1\end{pmatrix}$. (3pts)
They go to $0_{2\times 1}$ as t goes to ∞. (2pts)

3. (25pts) Find the general solutions to the system $x' = \begin{pmatrix}-5 & 1 & 1\\ 1 & -5 & 1\\ 1 & 1 & -5\end{pmatrix} x$, and describe their asymptotic behavior as time t goes to ∞.

Sol $\cdot$ E-Vs $\begin{vmatrix}-5-\lambda & 1 & 1\\ 1 & -5-1 & 1\\ 1 & 1 & -5-\lambda\end{vmatrix} = \begin{vmatrix}0 & 6+\lambda & 1-(5+\lambda)^2\\ 0 & -6-\lambda & 6+\lambda\\ 1 & 1 & -5-\lambda\end{vmatrix} = (6+\lambda)\begin{vmatrix}1 & 1-(5+\lambda)^2\\ -1 & 6+\lambda\end{vmatrix}$
$= (6+\lambda)\left[6+\lambda+1-(5+\lambda)^2\right] = (6+\lambda)\left[6+\lambda+(6+\lambda)(1-5-\lambda)\right] = (6+\lambda)^2(-3-\lambda) = 0$,
and then $\lambda_1 = \lambda_2 = -6$, $\lambda_3 = -3$. (10pts)

$\lambda_1 = \lambda_2 = -6$, $\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix} V_1 = 0$, and then $V_1 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$, as this matrix is rank 1, there is "another" (linearly independent) eigenvector $V_2 = \begin{pmatrix}1\\0\\-1\end{pmatrix}$, (6pts)

$\lambda_3 = -3$, $\begin{pmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{pmatrix} V_3 = 0$, and then $V_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}$. (3pts)

General solutions are

$$e^{Mt}\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\\ \tilde{c}_3\end{pmatrix} = (V_1\ V_2\ V_3)\begin{pmatrix}e^{-6t} & & \\ & e^{-6t} & \\ & & e^{-3t}\end{pmatrix}(V_1\ V_2\ V_3)^{-1}\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\\ \tilde{c}_3\end{pmatrix} = (e^{-6t}V_1\ e^{-6t}V_2\ e^{-3t}V_3)\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix}$$

$= c_1 e^{-6t}V_1 + c_2 e^{-6t}V_2 + c_3 e^{-3t}V_3$, (4pts)

where we call/denote $(V_1\ V_2\ V_3)^{-1}\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\\ \tilde{c}_3\end{pmatrix} = \begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix}$, and $M = \begin{pmatrix}-4 & 1 & 1\\ 1 & -4 & 1\\ 1 & 1 & -4\end{pmatrix}$.

The general solutions all go to $0_{3\times 1}$ as time t goes to ∞. (2pts)

4. (25pts) Solve the nonhomogeneous system

$$\begin{cases} x' = \begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix} x + \begin{pmatrix}1\\ t\end{pmatrix}\\ x(0) = \begin{pmatrix}-1\\ 1\end{pmatrix}\end{cases}.$$

Sol $\cdot$ E-Vs $\begin{vmatrix}1-\lambda & 1\\ -1 & -1-\lambda\end{vmatrix} = \lambda^2 - 1 + 1 = \lambda^2 = 0$, and then $\lambda_1 = \lambda_2 = 0$. (4pts)

We then "notice" $M^2 = \cdots = M^n = 0_{2\times 2}$, where $M = \begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}$.

$\cdot$ Fundamental matrix

$e^{Mt} = I + Mt + \underset{\longrightarrow 0}{\underline{\frac{M^2t^2}{2!} + \cdots + \frac{M^nt^n}{n!} + \cdots}} = I + Mt$ (10pts)

$e^{M(t-s)} = I + M(t-s)$ (1pt)

· The solution is then

$$x(t) = e^{Mt}x(0) + \int_0^t e^{M(t-s)}F(s)\,ds = (I + Mt)\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \int_0^t (I + M(t-s))\begin{pmatrix} 1 \\ s \end{pmatrix} ds \text{ (5pts)}$$

$$= \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 \\ s \end{pmatrix} + M\begin{pmatrix} (t-s) \\ (t-s)s \end{pmatrix} ds$$

$$= \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \left[\begin{pmatrix} s \\ \frac{1}{2}s^2 \end{pmatrix} + M\begin{pmatrix} ts - \frac{1}{2}s^2 \\ \frac{1}{2}ts^2 - \frac{1}{3}s^3 \end{pmatrix}\right]\Bigg|_{s=0}^{s=t}$$

$$= \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} t \\ \frac{1}{2}t^2 \end{pmatrix} + M\begin{pmatrix} \frac{1}{2}t^2 \\ \frac{1}{6}t^3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} t + \frac{1}{2}t^2 + \frac{1}{6}t^3 \\ -\frac{1}{6}t^3 \end{pmatrix}. \quad \text{(5pts)}$$

Verification:

$$x(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$x'(t) = \begin{pmatrix} 1 + t + \frac{1}{2}t^2 \\ -\frac{1}{2}t^2 \end{pmatrix},\ Mx(t) + F(t) = \begin{pmatrix} t + \frac{1}{2}t^2 \\ -t - \frac{1}{2}t^2 \end{pmatrix} + \begin{pmatrix} 1 \\ t \end{pmatrix} = \begin{pmatrix} 1 + t + \frac{1}{2}t^2 \\ -\frac{1}{2}t^2 \end{pmatrix}, \text{ and YES.}$$