I. Self-intro
II. Course info
III. What is calculus? The biggest achievement of mankind in the past 1000 years.

III. Derivatives and Integrals: more than one variable in both input and output
Representing problems to be solved (at ease):

eg. Solve $E = -\frac{E}{|E|}$, $E = (x, y, z)$ orbit of the Earth around the Sun.

eg. $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

- multiple variables
  - algebra of vectors
  - vector functions
  - partial derivatives
  - multiple integrals

- one variable
  - series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \infty$
  - $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} + \cdots \approx \frac{\pi^2}{6}$
  - $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$
  - arctan $x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, eg. $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Sec 12.1 Three dimensional coordinate system
Recall 2d coordinate system. Now add one more dimension.

$P(1, 2, 3)$
distance $|PO| = \sqrt{1^2 + 2^2 + 3^2}$

Plane:
$z = 3$ represents horizontal plane at height 3 or with distance 3 from the ground
$x = 1$ vertical plane parallel to $y-z$ plane w/ distance 1
$y = 2$ ....

Q. How to represent the plane through points $i, j, k$?

Sphere:
eg. sphere centered at $P(1, 2, 3)$ w/ radius $\pi$.

$\pi =$ distance between $(x, y, z)$ and $P = \sqrt{(x - 1)^2 + (y - 2)^2 + (z - 3)^2}$

Equ (standard or distance square form): $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = \pi^2$.

General form: $x^2 + y^2 + z^2 - 2x - 4y - 6z + 14 - \pi^2 = 0$.

Going backward to standard /distance formulation of sphere via completing square procedure.

Sec 12.2 Vectors
A vector is a quantity with magnitude(length) and direction, such as velocity with speed and direction.

Now a point in space (in particular ground plane) is perfectly suited to represent a vector: magnitude = distance to the origin, direction is from the origin to the point.

From now on $u = (u_1, u_2, u_3)$ represents both a point and a vector.

"4-1" Algebraic operations of vectors: $+, -, \times$, no $\div$. $\times$ has three versions: scalar multiplication, inner product $\cdot$, and cross product $\times$
Geometrically:
\[ u + v, \ u - v \]
Multiply buy a scalar, \( 100u \) means a vector \( w / \) the same direction and \( 100 \) times length.

Component recording:
\[ u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \]
\[ u - v = (u_1 - v_1, u_2 - v_2, u_3 - v_3) \]
\[ 100u = (100u_1, 100u_2, 100u_3) \]

Base or \( i, j, k \) recording:
\[ i = (1, 0, 0), \ j = (0, 1, 0), \ k = (0, 0, 1) \]
\[ u = u_1i + u_2j + u_3k, \ v = ... \]
eg. \( 2u + 3v = (2u_1 + 3v_1)i + (2u_2 + 3v_2)j + (2u_3 + 3v_3)k \)

Unit vector: a corresponding vector \( w / \) the same direction and length/magnitude

\[ \text{eg. i,j,k vectors} \]
\[ \text{eg. } u = (1, 2, 3), \text{ its corresponding unit vector is } \frac{1}{|u|} (1, 2, 3) = (1, 2, 3) \frac{1}{\sqrt{14}} = (\frac{1,2,3}{\sqrt{14}}). \]

\[ \text{eg7. A 100-lb weight hangs from 2 wires, the left angle is 45° and the right one 30°. Find the tensions (forces) } T_1 \text{ and } T_2 \text{ in both wires.} \]
Sol. \[ T_1 = L = l(-\cos 45°, \sin 45°) = \frac{l}{\sqrt{2}} (-1, 1), \ T_2 = r(\cos 30°, \sin 30°). \]
\[ (0, 100) = T_1 + T_2 = \left( \frac{-l}{\sqrt{2}} + \frac{\sqrt{3}r}{2}, \frac{l}{\sqrt{2}} + \frac{r}{2} \right). \]
\[ |T_2| = r = 100/ \left( \frac{1}{2} + \frac{\sqrt{3}}{2} \right), \ |T_1| = l = 100/ \left( \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{8}} \right). \]

Sec12.3 The dot product

• Cosine law to dot product
Recall/derive cosine law: \( |a| - |b| \cos \theta \)^2 + \( |b| \sin \theta \)^2 = \( |a - b| \)^2, that is \( |a|^2 + |b|^2 - 2 |a| |b| \cos \theta = |a|^2 + |b|^2 - 2 (a_1b_1 + a_2b_2 + a_3b_3) \), thus \( |a| |b| \cos \theta = a_1b_1 + a_2b_2 + a_3b_3. \) So

Define the inner product of \( a \) and \( b \) as \( a \cdot b = a_1b_1 + a_2b_2 + a_3b_3. \)

Then
\[ \cos \theta = \frac{a \cdot b}{|a| |b|}. \]

• Algebraic identity
\[ |a|^2 |b|^2 - |a|^2 |b|^2 \cos^2 \theta = |a|^2 |b|^2 \sin^2 \theta \]
then
\[ |a|^2 |b|^2 \sin^2 \theta = (a_1^2 + a_2^2 + a_3^2) (b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \]
\[ = a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_3^2b_3^2 + a_1^2b_3^2 + a_2^2b_1^2 + b_3^2a_1^2 \]
\[ - (a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + 2a_1b_1a_2b_2 + 2a_2b_2a_3b_3 + 2a_3b_3a_1b_1) \]
\[ = (a_1b_2 - a_2b_1)^2 + (a_2b_3 - b_2a_3)^2 + (a_3b_1 - a_1b_3)^2 \]
That is
\[ |a|^2 |b|^2 = (a \cdot b)^2 + |(a_2b_3 - b_2a_3, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)|^2. \]

Observe \(|a| \cdot |b| \sin \theta = \text{area of the parallelogram spanned by } a \& b.\)

Q. What is vector \((a_2b_3 - b_2a_3, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)\)?

Simple formula for dot product
\[
\begin{align*}
  a \cdot a &= |a|^2, \\
  a \cdot b &= a \cdot b + a \cdot c, \quad (9a) \cdot b = 9 (a \cdot b) = a \cdot (9b)
\end{align*}
\]

eg3 Find the angle between \(a = (2, 2, -1)\) and \(b = (5, -3, 2)\).

RMK. \(a \cdot b > 0 \iff \angle (a, b) \text{ acute}; a \cdot b < 0 \iff \angle (a, b) \text{ obtuse}; a \cdot b = 0 \iff \angle (a, b) = \pi/2.\)

eg4 Show that \(2i - 2j - k\) is perpendicular to \(5i - 4j + 2k.\)

eg5 Find direction angles \(\alpha, \beta, \gamma\) of the vector \(a = (1, 2, 3).\)

• Projections
Scalar projection of \(b\) onto \(a\):\(\text{comp}_a b = |b| \cos \theta = \frac{|a||b| \cos \theta}{|a|} = \frac{b \cdot a}{|a|} a.\)

vector projection of \(b\) onto \(a\):\(\text{vect}_a b = \text{comp}_a b = \frac{a \cdot b}{|a|^2} a.\)

eg. \(a\) is perpendicular to \(b - \text{vect}_a b.\)

eg7. A wagon is pulled a distance of 100m along a horizontal path by a constant force of 70N. The handle of the wagon is held at an angle of 30° above the horizontal. Find the work done by the force.

\(W = \text{force} \cdot \text{distance} = |F| \cos 30° |D| = F \cdot D = 70 \cos 30°, \sin 30°, 0 \cdot 100 (1, 0, 0) = 700 \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right).\)

Sec12.4 The cross product

• Intro of x product
Q. What is vector \((a_2b_3 - b_2a_3, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)\) in the identity \(|a|^2 |b|^2 = (a \cdot b)^2 + |(a_2b_3 - b_2a_3, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)|^2?\)

A. 1st its length is the area of parallelogram spanned by \(a \& b, |a| \cdot |b| \sin \theta,\)

2nd its direction is?? case \(a = i, b = j,\) then it is \((0, 0, 1) = j.\) General case, perpendicular to both \(a \& b,\) since

\[
a \cdot (a_2b_3 - b_2a_3, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) = a_1 (a_2b_3 - b_2a_3) + a_2 (a_3b_1 - a_1b_3) + a_3 (a_1b_2 - a_2b_1) = 0 \quad \text{sort through } b_1, b_2, b_3.
\]

Similarly \(b \cdot (a_2b_3 - b_2a_3, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) = 0.\)

Finally whether the direction is away or toward plane \(a-b\) is determined by right-hand-rule as for \(i \times j = k,\) and \(j \times i = -k.\)

Def. The cross product of \(a \& b\) \(a \times b = (a_2b_3 - b_2a_3, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).\)

Matrix notation
\[
a \times b = \begin{vmatrix} i & j & k \\
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\
  b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\
  b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\
  b_1 & b_2 \end{vmatrix} k.
\]

eg1. \((1, 1, 1) \times (1, 0, 0) = ?\)

eg2. \(a \times a = 0\)
**Geometric implication: area and volume**

eg3. Find a vector perpendicular to the plane through \( P(1, 4, 6) \), \( Q(-2, 5, -1) \), and \( R(1, -1, 1) \).

eg4. Find the area of the triangle with vertices \( PQR \).

\[
\text{Area} = \frac{1}{2} |PQ| |PR| \sin \theta = \frac{1}{2} |PQ \times PR| = \ldots
\]

Formula the volume of the parallelepiped spanned by \( a, b, c \) is

\[
|a \cdot (b \times c)| = \begin{vmatrix} a \\ b \\ c \end{vmatrix}, \text{ take absolute value,}
\]

by symmetry \( |(a \times b) \cdot c| = |(a \times c) \cdot b| \).

Proof. volume = base \cdot height = |a \times b| |c| \cos \angle (a \times b, c) = (a \times b) \cdot c \text{ upto a sign.}

By symmetry other orders.

**Formulas for cross product**

\[
a \times b = -b \times a \\
(\lambda a) \times b = \lambda (a \times b) = a \times (\lambda b) \\
a \times (b + c) = a \times b + a \times c \\
(b + c) \times a = b \times a + c \times a \\
a \cdot (b \times c) = (a \times b) \cdot c \\
a \times (b \times c) = a \cdot c b - a \cdot b c
\]

Proof of **.

0th may assume \( a, b, c \) are unit vectors.

1st \( b \perp c : a = a \cdot b b + a \cdot c c + a \cdot (b \times c) b \times c \)

2nd otherwise: \( c = c - c \cdot b b + c \cdot b b \) and \( b \times c = b \times (c - c \cdot b b) \), reduces to 1st case.

eg6. A bolt is tightened by applying a 40-N force to a 0.25-m wrench as shown. The angle between force and wrench is 75\(^0\). Find the magnitude of the torque about the center of the bolt.

\[
|\tau| = |r \times F| = |r| |F| \sin 75^0
\]

Sec12.5 Equations of lines and planes

**Lines: direction and a point**

* parametric form: \( \gamma = \gamma_0 + tv \) or \( x = x_0 + t\alpha, \ y = y_0 + t\beta, \ z = z_0 + t\gamma \)

* symmetric form: eliminate \( t \) parameter

\[
\frac{x - x_0}{\alpha} = \frac{y - y_0}{\beta} = \frac{z - z_0}{\gamma}
\]

when say, \( \alpha = 0, \beta \neq 0, \gamma \neq 0, \ x = x_0, \ \frac{y - y_0}{\beta} = \frac{z - z_0}{\gamma} \)

when \( \alpha = \beta = 0, \) then \( \gamma \) cannot be 0, \( x = x_0, \ y = y_0, \) \( z \) is free. That is the line the intersection of two vertical planes.
Show that $L_1$ and $L_2$ are not parallel, don’t intersect. In this case, called skew.

**Proof.** • $V_1 = (1, 3, -1) \parallel L_1$, $V_2 = (2, 1, 4) \parallel L_2$. We know $V_1 \parallel V_2 \iff V_1 = \alpha V_2$.

Now $\frac{1}{2} \neq \frac{3}{1} \neq -\frac{1}{4}$. Thus $L_1 \parallel L_2$.

• When intersect, system 
\[
\begin{cases}
1 + t = 2s \\
-2 + 3t = 3 + s \\
4 - t = -3 + 4s
\end{cases}
\]
and $t = 5/3$, which don’t solve (2). Thus $L_1$ and $L_2$ are skew.

Q. What is distance between $L_1$ and $L_2$?

**Planes**

Geometrically any point on the plane $r = (x, y, z)$, actually any vector on the plane $\perp$ normal vector $n$. Then

$$(x, y, z) - (x_0, y_0, z_0) \perp n$$

that is $n \cdot (r - r_0) = 0$.

eg5. Find an equation for the plane passes through $A(1, 0, 0)$, $B(0, 1, 0)$, and $C(0, 0, 1)$.

Sol. Normal vector: $N = AC \times AB = (-1, 0, 1) \times (-1, 1, 0) = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} = -i - j (1) + k (-1) = -(1, 1, 1)$.

RMK. $AB \times AC = (1, 1, 1)$.

Equa: $[(x, y, z) - (1, 0, 0)] \cdot (1, 1, 1) = 0$. Then $x + y + z - A \cdot (1, 1, 1) = 0$, that is

$$x + y + z - 1 = 0.$$ 

For plane $\pi: \alpha x + \beta y + \gamma z + \delta = 0$, contains $(x_0, y_0, z_0)$, then $\delta = -\alpha x_0 - \beta y_0 + \gamma z_0$. Thus the equation takes another “standard” form: $\alpha (x - x_0) + \beta (y - y_0) + \gamma (z - z_0) = 0$. We can read off the normal vector is $(\alpha, \beta, \gamma)$.

eg7. a) Find the angle between $\pi_1: x + y + z = 1$ and $\pi_2: x - 2y + 3z = 1$.

b) Find the symmetric equation for the line of intersection of $\pi_1$ and $\pi_2$.

Sol. a) $n_1 = (1, 1, 1)$, $n_2 = (1, -2, 3)$. Then $\cos \theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{2}{\sqrt{3}\sqrt{14}} = 2/\sqrt{42} > 0$.

We look for the acute angle anyway, $\theta = \arccos 2/\sqrt{42} \approx 70^\circ$.

b) Point-direction formula

Point $\begin{cases}
x + y + z = 1 \\
x - 2y + 3z = 1
\end{cases}$ leads to a family of solutions. We only need one. Let $z = 0$, then $\begin{cases}x + y = 1 \\
x - 2y = 1
\end{cases}$ gives us $y = 0$ and $x = 1$. Thus $(1, 0, 0)$ is on the intersecting line.

Direction: $L \perp n_1$, $L \perp n_2$. Then $L \parallel n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = (5, -2, -3)$.

Symmetric form: $(x - 1, y, z) \parallel (5, -2, -3)$. That is $\frac{x - 1}{5} = \frac{y}{-2} = \frac{z}{-3}$.

5
Remember the simple Pythagorean formula for distance from one point to another.
Next we deal with the distances from a point to a line and a plane.

eg8-. Find the distance from $P(X, Y, Z)$ to the line $\gamma(t) = \gamma(0) + tV = (x_0, y_0, z_0) + t(\alpha, \beta, \gamma)$.

Sol. Simple case first: when the line is $z$ axis, then the dist$(P, z$-axis$) = \sqrt{X^2 + Y^2}$.

General case:

$$\text{dist} = \left| \gamma(0) \right| \sin \left( V, \gamma(0) \right) = \frac{1}{|V|} |V| \left| \gamma(0) \right| \sin \left( V, \gamma(0) \right)$$

$$= \frac{|V \times \gamma(0)|}{|V|} = \frac{|(\alpha, \beta, \gamma) \times (X - x_0, Y - y_0, Z - z_0)|}{|(\alpha, \beta, \gamma)|}.$$

eg8. Find the distance from $P(X, Y, Z)$ to the plane $\alpha x + \beta y + \gamma z + \delta = 0$.

Sol. Simple case first: when the plane is the ground $z = 0$, then dist$(P, \text{ground plane}) = |Z|$.

General case: dist = absolute value of $\frac{|PP_0| \cos \theta}{|PP_0|} = \text{abs of } \frac{PP_0}{|n|} =$

$$\left| \frac{(x_0 - X, y_0 - Y, z_0 - Z) \cdot (\alpha, \beta, \gamma)}{|(\alpha, \beta, \gamma)|} \right| = \frac{\alpha x_0 + \beta y_0 + \gamma z_0 - \alpha X - \beta Y - \gamma Z}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}} = \frac{\alpha X + \beta Y + \gamma Z + \delta}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}.$$

eg10. Find the distance between the two skew lines in eg3.

Claim: dist = abs value of $P_1P_2 \cdot (L_1 \times L_2) / |L_1 \times L_2|$.

In fact, dist =

$$|P_1P_2| \cos \theta = \frac{|n| |P_1P_2\cos \theta = n \cdot P_1P_2|}{|n|} = \frac{n \cdot (P_1P_2 + P_2P_3)}{|n|} = n \cdot P_1P_2 \frac{n}{|n|} , \text{YES, take absolute value.}$$

Q. How to calculate the distance between two parallel lines, eg $L_1 : r = t(1, 2, 3)$ and $L_2 : r = (1, 6, 5) + s(2, 4, 6)$?

Sec12.6 Quadric surfaces, cones, and cylinders

- Quadric surfaces
Recall lines/planes are represented by linear equations:

$$\alpha x + \beta y + \gamma z + \delta = 0 \text{ plane}$$

$$\begin{cases} \frac{x-x_0}{v_1} = \frac{y-y_0}{v_2} = \frac{z-z_0}{v_3} \\ \alpha_1 x + \beta_1 y + \gamma_1 z + \delta_1 = 0 \text{ or} \end{cases} \begin{cases} \alpha_2 x + \beta_2 y + \gamma_2 z + \delta_2 = 0 \text{ line.} \\ \end{cases}$$

Now quadratics
Recall 2-d:

$x^2 + y^2 = 1, \quad \frac{x^2}{4} + \frac{y^2}{9} = 1 \text{ circle/ellipse}$
\[ x^2 - y^2 = 1, \ xy = 1 \text{ hyperbola} \]

\* \[ y = x^2 \text{ parabola} \]

In 3-d

Just as in 2-d, via translation (completing squares), one can “ignore” linear terms; via rotation (change of variables), one can “eliminate” cross quadratic terms \( xy, yz, zx \); general quadratics like \( x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6xz + 7x + 8y + 9z + 10 = 0 \) can be “reduced" to one of the following (while its surface shape doesn’t change):

\* **Ellipsoid**: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \)

\* **Paraboloid**: 
  - elliptic type: \( \frac{z}{\gamma} = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} \)
  - hyperboloid type: \( \frac{z}{\gamma} = \frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} \)

RMK. When general equation has no \( z^2 \), then linear \( z \) term cannot be reduced.

\* **Hyperboloid**: 
  - one sheet: \( \frac{z^2}{\gamma^2} = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - 1 \)
  - two sheet: \( \frac{z^2}{\gamma^2} = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + 1 \).

Now asymptotically, hyperboloid (1/2 sheet) are cones \( \frac{z^2}{\gamma^2} = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} \).

- A Cone is formed by semi-lines/rays from the vertex (here origin).

RMK. There is “no” asymptotes for paraboloids.

Lastly
- A Cylinder is formed by parallel lines (unlike the intersecting rays from one common point).

  eg. \( y = x^2 \) in 3d
  eg. \( y = x^3 \) in 3d
  eg. \( x^2 + y^2 = 1 \) in 3d
  eg. \( x + y + z = 1 \).

Sec10.1 Curved defined by parametric equations
eg7. Cycloid ...

Sec13.1 Vector functions and space curves
Recall we’ve already encountered vector functions of one variable/parameter \( t \) of a line

\[ r \left( t \right) = \left( 1 + t, 2 + 6t, 3 + 5t \right). \]

In general, components as general functions of one variable/parameter \( t \): \( r \left( t \right) = \left( f \left( t \right), g \left( t \right), h \left( t \right) \right) \).

Now do calculus such as limit, derivatives, and integrals component by component to \( r \left( t \right) = \left( f \left( t \right), g \left( t \right), h \left( t \right) \right) \).

eg. Find \( \lim_{t \to 0} \left( 1 + t^3, te^{-t}, \frac{\sin t}{t} \right) \). (Three egs in Calculus I)

The trace or orbit of \( r \left( t \right) = \left( f \left( t \right), g \left( t \right), h \left( t \right) \right) \) is really a space curve.

eg4. Sketch the curve \( r \left( t \right) = \cos t \ i + \sin t \ j + tk \).

On the cylinder, winding up.
eg6. Find a vector function representation of the curve of the intersection of the cylinder \( x^2 + y^2 = 1 \) & plane \( y + z = 1 \).

(Read Sec10.6 Conic sections)
In fact the curve E is an ellipse.

Projection way. On the plane, let X axis be through (0,0,1) and parallel to x-axis, let Y axis be through (0,0,1) or the intersection between the plane and y-z plane. Now as X is parallel to x axis, \( x = X \); as the angle between Y axis and y axis is 45°, \( y = Y/\sqrt{2} \). Then \( X^2 + \frac{Y^2}{2} = x^2 + y^2 = 1 \). Therefore the intersection curve E is an ellipse.

Ball touching way.
* Inscribe two balls w/ radius 1 in the cylinder. The top ball touches/tangent to the plane at F_{BT} from below, and touches/tangent to the cylinder along a top horizontal circle C_T. The bottom ball touches/tangent to the plane at F_{BT} from below, and touches/tangent to the cylinder along a bottom horizontal circle C_B.

* Any point on the intersection has its distance to \( C_B \) & \( C_T \) from above, and touches/tangent to the cylinder along a top horizontal circle C_T.

Therefore, the intersection curve E is an ellipse.

Sec10.2 Calculus with plane curves

• Derivatives

\( \gamma (t) = (x(t), y(t)), \) such as \( (\cos t, \sin t), (t, t^3). \)

\( \gamma'(t) = (x'(t), y'(t)) = \left( \frac{dx}{dt}, \frac{dy}{dt} \right) \) velocity vector

\( m = y_x = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \) (picture), \( \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(y_x)}{dx} \neq \frac{\frac{d}{dt}y}{\frac{d}{dt}x} \)

RMK. Preview space curve \( \gamma'(t) = (x'(t), y'(t), z'(t)) \) ....

• Integrals: area & length

eg3/5 Given one arch of the cycloid \( \begin{cases} x = \theta - \sin \theta \\ y = 1 - \cos \theta \end{cases} \) 0 ≤ \( \theta \leq 2\pi \). a) Find the area between x-axis and the arch. b) Find the length of the arch.

Sol. a) \( A = \int_{x_{\theta}}^{x_{\theta+\frac{\pi}{2}}} y \, dx = \int_{\theta}^{\theta+\frac{\pi}{2}} (1 - \cos \theta)(1 - \cos \theta) \, d\theta = \int_{0}^{2\pi} (1 - 2 \cos \theta + \cos^2 \theta) \, d\theta = \int_{0}^{2\pi} (1 - 2 \cos \theta + \frac{\cos^2 \theta + 1}{2}) \, d\theta = 3\pi \).

b) Length, infinitesimally \( L = \int_{\theta}^{\theta+\frac{\pi}{2}} |\gamma'(t)| \, dt \) 

\( L = \int_{0}^{2\pi} \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta} \, d\theta = \int_{0}^{2\pi} \sqrt{2 - 2 \cos \theta} \, d\theta = \int_{0}^{2\pi} \sqrt{2} \, d\theta = \int_{0}^{2\pi} 2 \sin \theta \, d\theta = -8 \ln \frac{1}{2} = 8. \)

RMK/preview: Space curve like \( \gamma(t) = (\cos t, \sin t, t) \) has length \( \int_{t_0}^{t_1} |\gamma'(t)| \, dt = \int_{t_0}^{t_1} \sqrt{x'_{xx} + y'_{xx} + z'_{xx}} \, dt. \)

Sec10.3 Polar coordinates–curved coordinates
On the plane, two numbers represent a point. Recall rectangular coordinates: $x$ signed distance to y-axis, $y$ signed distance to x-axis.

Now $r$ distance to the origin, $\theta$ signed angle between $OP$ and positive x-axis. 

eg. $(2,30^0)$ and $(2,-330^0)$ represent point $x = \sqrt{3}$, $y = 1$. For angles we use radians, thus $(2, \frac{\pi}{6})$ or $(2, -\frac{\pi}{6})$. 

eg. $(1.99, 131^0)$ or $(1.99, -229^0)$ really should $(1.99, \frac{131}{180}\pi), (1.99, -\frac{229}{180}\pi)$

\[
\begin{aligned}
x &= r \cos \theta \\
y &= r \sin \theta
\end{aligned}
\]

\[\theta = \begin{cases} 
\begin{array}{l}
\arctan \frac{y}{x} & I, IV \text{ quadrant/right } / x > 0 \\
\arctan \frac{y}{x} + \pi & II, III \text{ quadrant/left } / x < 0
\end{array}
\end{cases}\]

eg4. $r = 2$ circle w/ radius 2 centered at the origin.

eg5. $\theta = 1$ line with slope $m = \tan 1$, through the origin.

eg6. Sketch the curve $r = 2 \cos \theta$ and find its Cartesian equation.

Sol. 

\[
\begin{aligned}
\theta &= 0, \pi/4, 2\pi/4, 3\pi/4, \pi, 6\pi/4, 7\pi/4, 8\pi/4 \\
r &= 2, \sqrt{2}, 0, -\sqrt{2}, -2, 0, \sqrt{2}, 2
\end{aligned}
\]

Analytically: $r = \sqrt{x^2 + y^2}$, $\cos \theta = x/\sqrt{x^2 + y^2}$, $\sin \theta = y/\sqrt{x^2 + y^2}$, $x^2 + y^2 = 2x$.

r-way, $r^2 = 2r \cos \theta = 2x$, ...

Also justify circle conclusion geometrically: as $|OP| = 2 \cos \theta$, then $P(r, \theta) = (2,0) \bot OP$. Hence the line through $C(1,0)$ parallel to $PA$ cuts triangle $OCP$ into 2 equal parts. Thus $|CP| = |CO| = 1$, that is $P$ is on the circle centered at $C$ w/ radius 1.

eg. Spiral curve $r = \theta$.

Tangent to curves in polar coordinates

$r = f(\theta)$, then $\gamma(\theta) = (f(\theta) \cos \theta, f(\theta) \sin \theta)$, tangent vector $\gamma_\theta = (f_\theta \cos \theta - f \sin \theta, f_\theta \sin \theta + f \cos \theta)$

\[
\begin{aligned}
\frac{dy}{dx} &= \frac{y_\theta}{x_\theta} = \frac{f_\theta \sin \theta + f \cos \theta}{f_\theta \cos \theta - f \sin \theta}
\end{aligned}
\]

RMK. Arclength $L = \int_{\theta_1}^{\theta_2} |\gamma_\theta| \, d\theta$. Area $A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta$

eg. For the spiral $r = \theta$, find the slope of the tangent line at $\theta = \pi/4$. Find the points when tangents are horizontal or vertical.

Sol. Slope $m = \frac{dy}{dx} = \frac{y_\theta}{x_\theta} = \frac{\sin \theta + \theta \cos \theta}{\cos \theta - \theta \sin \theta} |_{\theta = \pi/4} = \frac{1 + \pi/4}{1 - \pi/4}$

Horizontal tangents: $\sin \theta + \theta \cos \theta = 0$ or $\tan \theta = -\theta$, picture

Vertical tangents: $\cos \theta - \theta \sin \theta = 0$ or $\cot \theta = \theta$, picture.

eg. Find polar/rectangular form of the curve of all points, whose distance to the origin is 1/6 of the one to the line $x = 7$.

Polar: $r = \frac{1}{6} (7 - r \cos \theta)$ or

\[
\begin{aligned}
r &= \frac{17}{1 + \frac{1}{6} \cos \theta} = \frac{e \, d}{1 + e \cos \theta}
\end{aligned}
\]

Rectangular: $x^2 + y^2 = \frac{1}{36} (49 - 14x + x^2), (1 - \frac{1}{36}) x^2 + \frac{14}{36} x + y^2 = \frac{49}{36}, \frac{35}{36} (x + \frac{7}{36})^2 + y^2 = \frac{49}{36} + \frac{49}{36 \cdot 35}$. Ellipse.
RMK. In stead of $< 1$ ratio $1/6$: if the ratio $e$ is 1, then parabola; if the ratio $e$ is $>1$, then hyperbola, as one can see from the rectangular equations.

Sec13.2 Derivatives and integrals of vector functions—calculus w/ space curves

- Derivatives, differentiate it component by component $\gamma (t) = (x(t), y(t), z(t))$

eg3. Find parametric equations for the tangent line to helix $\gamma (t) = (\cos t, \sin t, t)$ at $(0, 1, \pi/2)$.

Sol. Point/direction formula

Sum, difference, product rules for vector functions $u(t), v(t)$, scalar function $f(t)$.

\[
\begin{align*}
\frac{d}{dt}[u(t) \pm v(t)] &= \frac{d}{dt}u(t) \pm \frac{d}{dt}v(t) \\
\frac{d}{dt}[f(t)u(t)] &= f'(t)u(t) + f(t)u'(t) = f_t(t)u(t) + f(t)u_t(t) \\
\frac{d}{dt}[u(t) \cdot v(t)] &= [\frac{d}{dt}u(t)] \cdot v(t) + u(t) \cdot \frac{d}{dt}v(t) \\
\frac{d}{dt}[u(t) \times v(t)] &= [\frac{d}{dt}u(t)] \times v(t) + u(t) \times \frac{d}{dt}v(t) \\
\end{align*}
\]

Indeed $(u \cdot v)' = (u_1v_1 + u_2v_2 + u_3v_3)' = u_1'v_1 + u_2'v_2 + u_3'v_3 + u_1v_1' + u_2v_2' + u_3v_3' = u' \cdot v$

\[
\begin{align*}
(u \times v)' &= (u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1)' = (u_1'v_2v_3 - u_1v_2v_3', u_2'v_3v_1 - u_2v_3v_1', u_3'v_1v_2 - u_3v_1v_2') = u_1v_2v_3' - u_2v_3v_1' + u_3v_1v_2' - u_1v_1v_2' - u_2v_2v_3' + u_3v_3v_2'
\end{align*}
\]

eg4. If $|\gamma(t)| = 128$, then $\gamma'(t)$ is perpendicular to $\gamma(t)$.

Proof. $\gamma'(t) \perp \gamma(t) \iff \gamma'(t)$ is orthogonal to $\gamma$

\[
\gamma(t) \cdot \gamma'(t) = 128^2, \text{ then } \gamma'(t) \cdot \gamma(t) + \gamma(t) \cdot \gamma'(t) = 0, \text{ then } 2\gamma(t) \cdot \gamma'(t) = 0, \text{ that is } \gamma(t) \cdot \gamma'(t) = 0.
\]

eg4'. If $|v(t)| = 10$, then acceleration $a(t) = v'(t)$ is orthogonal to the velocity $v(t)$, as $a(t) \cdot v(t) = 0$. For example, the velocity of the helix curve $\gamma(t) = (\cos t, \sin t, t)$ is $v(t) = (-\sin t, \cos t, 1)$. The speed is $|v(t)| = |v(t)| = \sqrt{2}$. The acceleration $a(t) = (-\cos t, -\sin t, 0)$ is orthogonal to $v(t)$.

- Integrals, integrate it component by component

eg5' Given $v(t) = (-\sin t, \cos t, 1)$. Find the displacement from $t = 0$ to $t = \pi/2$ or $\int_0^{\pi/2} v(t) \, dt$.

Review for Midterm I
* Algebraic (vector operation preparation) for vector function calculus

\[+ - \times (\text{scalar}, \cdot \text{/inner } \cdot, \text{cross/outer } \times)\]

geometric origin

\[|u||v| \cos \theta = u \cdot v = u_1v_1 + u_2v_2 + u_3v_3\]

area of parallelogram $\{u, v\} = |u||v| \sin \theta = $magnitude of

$$
\begin{vmatrix}
i & j & k \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3
\end{vmatrix}$$

* parametric equations
* polar coordinates

Applications
* lines/planes, curves/quartic surfaces
* length/area/volume (only for straight ones)

Sec13.3 Arc length and curvature (1st & 2nd derivative properties)

- **Arc length**
  Recall for straight segment, \( \gamma(t) = x(t) = t^2 \), \( L = \int_1^2 |\gamma'(t)| \, dt = \int_1^2 2t \, dt = t^2 |^1_2 = 4 \).
  Just as length for plane curves \( \gamma(t) = (x(t), y(t)) \), \( L = \int_t^2 |\gamma'(t)| \, dt = \int_t^2 \sqrt{x_t'^2 + y_t'^2} \, dt \).
  Now length for space curve \( \gamma(t) = (x(t), y(t), z(t)) \), \( L = \int_t^2 |\gamma'(t)| \, dt = \int_t^2 \sqrt{x_t'^2 + y_t'^2 + z_t'^2} \, dt \).

- **Curvature**
  How much a wavy curve curves? How fast its tangent vector changes directions?
  eg. straight line, small circle, large circle, general curve.
  eg. same circle w/ different parametrization: \( (\cos t, \sin t) \), \( (\cos 2t, \sin 2t) \)
  Need to normalize: unit tangent vector \( T = \frac{\gamma'(t)}{|\gamma'(t)|} \), arc length parametrization \( \gamma(s), |\gamma'(s)| = |\gamma| = 1 \).

  Def. The curvature of a curve \( \gamma(t) \) is \( \kappa = \left| \frac{dT}{ds} \right| = |T_s| \).

  Q1. How to calculate \( \kappa \) in non-arc-length parametrization?
  Q2. Other interpretations of \( \kappa \)?

  **A1 of Q1.**
  Step1. Move to non-arc-length parametrization

  \[
  \kappa = \left| \frac{dT}{ds} \right| = \left| \frac{dT}{dt} \cdot \frac{dt}{ds} \right| = \left| \frac{\gamma_t}{|\gamma|} \right| \left| \frac{\gamma_{tt} + \gamma_t \left( \frac{1}{|\gamma|} \right)_t}{|\gamma|} \right|.
  \]

  Step2. \( T_s \perp T \)
  As \( T \cdot T = 1 \), then \( 2T_s \cdot T = 0 \). So \( T_s \perp T \).
  Then \( |T_s| = |T_s| |T| \sin \frac{\pi}{2} = |T_s \times T| \).

  Step3. \( \kappa = |T_s| = \left| \frac{\gamma_t}{|\gamma|} \right| \left| \gamma_t \left( \frac{1}{|\gamma|} \right)_t + \gamma_t \left( \frac{1}{|\gamma|} \right)_t \right| \times \frac{\gamma_t}{|\gamma|} = \frac{|\gamma \times \gamma_t|}{|\gamma|^3} \).

  **A2 of Q2.**
  Physically, when unit speed, magnitude of acceleration is the curvature of the orbit/trace curve.
  Circle way: second order touching/kissing/osculating circle’s radius is the reciprocal of the curvature.
  Radius \( 1/\kappa \), Center \( \gamma(t) + \frac{1}{\kappa} N \), where \( N = T_s / |T_s| = T_t / |T_t| \).
Quadratic approximate way: say the curve is already a graph over it tangent line, say x-axis (can always achieve this configuration via translation and rotation) \( \gamma(x) = (x, g(x), h(x)) \). Near the tangent point, \( g \)'s and \( h \)'s quadratic approximation take form
\[
\begin{align*}
g(x) &= g(0) + g'(0)x + \frac{1}{2}g''(0)x^2 + \cdots \\
&= g_0 + h_0 x + \frac{1}{2}h_0 x^2 + \cdots,
\end{align*}
\]
similar for \( h \). That is
\[
\gamma(x) \approx \left( x, \frac{1}{2}g'(0)x^2, \frac{1}{2}h''(0)x^2 \right).
\]
Then \( \kappa = \sqrt{g''(0)^2 + h''(0)^2} \).

eg5. Find the curvature of parabola \( y = x^2 \) at \((0,0), (1,1), \) and \((2,4)\).

Graph both parabola and its curvature function.

eg8. Find the osculating circle of \( y = x^2 \) at \((0,0)\).

Circle: \( x^2 + (y - \frac{1}{2})^2 = \frac{1}{4} \). \( y = \frac{1}{2} - \sqrt{\frac{1}{4} - x^2} = x^2 + x^4 + \cdots \), no cubic terms!

eg7. Find an equation of the normal plane and osculating plane of helix \( \gamma(t) = (\cos t, \sin t, t) \) at \((0,1,\pi/2)\).

Use point/normal formula.

Osculating plane, closest plane to the curve near the tangent point, spanned by the velocity \( \gamma_t \) and acceleration \( \gamma_{tt} \), one normal to the osculating plane is \( \gamma_t \times \gamma_{tt} \).

Normal plane, “furthest” to the curve near the tangent point, normal/perpendicular to the tangent/velocity \( \gamma_t \). That is, a normal to the normal plane is \( \gamma_t \).

RMK. The osculating plane is also spanned by (the unit tangent vector) \( T \) and (the unit principle normal vector to the tangent/velocity) \( N = T_s / |T_s| = T_t / |T_t| \), as \( T = \gamma_t / |\gamma_t| \), \( \gamma_{tt} \), and \( N \) are on the same plane (will see in the beginning of Sec13.4). The unit normal vector to the osculating plane is called the binormal, it is \( B = T \times N \), parallel to \( \gamma_t \times \gamma_{tt} \), and another unit normal vector to the tangent/velocity.

RMK. Normal plane spanned by the two normal vectors \( N \) and \( B \) (to the tangent \( \gamma_t \)), consists of all normal vectors to the the tangent \( \gamma_t \).

Summary.

\[
\begin{align*}
T &= \frac{\gamma_t}{|\gamma_t|}, \\
N &= \frac{T_t}{|T_t|}, \\
B &= T \times N = \frac{\gamma_t \times \gamma_{tt}}{|\gamma_t \times \gamma_{tt}|}, \\
\kappa &= \frac{|dT|}{ds} = \frac{|T_t|}{|\gamma_t|} = \frac{|\gamma_t \times \gamma_{tt}|}{|\gamma_t|^3}.
\end{align*}
\]

Sec13.4 Motion in space: velocity and acceleration

Tangential and normal components of acceleration.

The acceleration has two effects on the velocity, one is on the speed change, the other is on the direction change. Those two changes are respectively tangential component (along the velocity direction) and normal component (perpendicular to the velocity and in the velocity-acceleration plane) of acceleration.

For space curve/orbit \( \gamma(t) = (x(t), y(t), z(t)) \), the arclength is \( s = \int |\gamma_t| dt \), then \( \frac{ds}{dt} = |\gamma_t| = v \).
where the second before the last equality is because

\[ T_i = \frac{dT}{ds} = \frac{\kappa N}{ds} = \frac{ds}{dt} \kappa N = v \kappa N; \]

the last equality is because \( \kappa = |\gamma_t \times \gamma_{tt}| / |\gamma_t|^3 \) calculated in Section 13.3 and an indirect, but easier way of computing \( v_t : 2v v_t = \frac{d}{dt} v^2 = \frac{d}{dt} (\gamma_t \cdot \gamma_t) = 2 \gamma_t \cdot \gamma_{tt}, \) then divide both sides by \( 2v = 2 |\gamma_t| \).

RMK. From \( \ddot{a} = \gamma_{tt} = v_t T + \kappa v^2 \hat{N}, \) we see \( \gamma_{tt}, \gamma_t = v \hat{T}, \) and \( \hat{N} \) are on the same (osculating) plane.

eg. When speed \( v \) is constant, then \( v_t = 0, \) and \( \ddot{a} = \gamma_{tt} = \kappa v^2 N, \) such as in \( \gamma(t) = (2 \cos t, 2 \sin t, 0) : \gamma_t = (-2 \sin t, 2 \cot t, 0), v = |\gamma_t| = 2, a = \gamma_{tt} = (-2 \cos t, -2 \sin t, 0) = \frac{1}{2} \dot{v}^2 (- \cos t, - \sin t, 0); \) when \( \kappa = 0, \) then \( \ddot{a} = \gamma_{tt} = v_t T, \) such as in \( \gamma(t) = (0, 0, 16 - 4.9t^4) : \gamma_t = (0, 0, -9.8t), v = 9.8t, v_t = 9.8, \ddot{a} = \gamma_{tt} = 9.8 (0, 0, -1). \)

Next we provide other ways of deriving the tangential and normal component \( v_t \) and \( \kappa v^2 \) in terms of \( \gamma_t \) and \( \gamma_{tt}. \)

Inner&outer product way:

\[ \left( \frac{\gamma_t}{|\gamma_t|} \right) \cdot \gamma_{tt} = T \cdot \left( v_t T + \kappa v^2 N \right) \left. \right|_{T \cdot T = 0}^{T \cdot \hat{N} = 0} v_t. \]

Take magnitude of both sides

\[ \frac{\gamma_t}{|\gamma_t|} \times \gamma_{tt} = T \times \left( v_t T + \kappa v^2 N \right) \left. \right|_{T \times T = 0}^{T \times \hat{N} = 0} \kappa v^2 T \times N, \]

\[ \frac{|\gamma_t \times \gamma_{tt}|}{|\gamma_t|} = \kappa v^2 \quad \text{as} \quad |T \times N| = 1. \]

Projection way: Note \( T = \gamma_t / |\gamma_t|, \gamma_{tt}, \) and \( N \parallel T \) are on the same plane. Then

\[ \ddot{a} = \gamma_{tt} = \left| \gamma_{tt} \right| |T| \cos \angle (\gamma_{tt}, T) T + \left| \gamma_{tt} \right| |N| \cos \angle (\gamma_{tt}, N) N \]

\[ = \frac{|\gamma_{tt}|}{|T|} \cos \angle (\gamma_{tt}, T) T + |\gamma_{tt}| |T| \sin \angle (\gamma_{tt}, T) N \]

\[ = \frac{\gamma_t \cdot \gamma_{tt} T}{|\gamma_t|} + \left| \gamma_t \right| \frac{\gamma_t \times \gamma_{tt} N}{|\gamma_t|}. \]

eg7Orbit \( \gamma(t) = (t, t^2, t^3) \) or HW13.4 #8 Orbit \( (\cos t, \sin t, 9t) \). Find the tangential and normal components of the acceleration vector.
Discovering the orbit of the Earth around the sun: ellipse!

By the Newton’s second and gravity law,

\[ m_E \ddot{E} = - \frac{G m_e m_S}{|E|^2} \frac{E}{|E|}. \]

For simple notation, we assume \( G m_S = 1 \), then the equation becomes

\[ \ddot{E} = - \frac{E}{|E|^3}. \]

Let’s use the more familiar notation \( \gamma(t) \) for the Earth position \( E(t) \) in space w/ the Sun positioned at the origin, then

\[ \gamma(t) = - \frac{\gamma}{|\gamma|}. \]

Step1. Earth orbit along a fixed plane (not a space curve as for flies/bugs)

Observation: \( \gamma \times \gamma_t = \text{constant vector} = C_\perp \).

In fact \( \frac{d}{dt} \gamma \times \gamma_t = \gamma_t \times \gamma_t + \gamma \times \ddot{\gamma} = \gamma \times \left( -\frac{\gamma}{|\gamma|^2} \right) = 0. \)

As \( \gamma \perp C_\perp \), then \( \gamma \) is always on a (“ground”) plane.

Step2. Recovering sun shine direction \( \frac{\gamma}{|\gamma|} \)

Obs. ground vector \( \frac{\gamma}{|\gamma|} \) and \( \gamma_t \times C_\perp \) differ only by a constant (ground) vector.

This is because their time derivatives are the same:

\[ (\gamma_t \times C_\perp)_t = \gamma_{tt} \times C_\perp = \gamma_{tt} \times (\gamma \times \gamma_t) \]

\[ = \gamma_{tt} \cdot \gamma_t \gamma - \gamma_{tt} \cdot \gamma \gamma_t \]

\[ = -\frac{\gamma \cdot \gamma_t}{|\gamma|^3} \gamma + \frac{1}{|\gamma|} \gamma_t \]

\[ = \left( \frac{\gamma}{|\gamma|} \right)_t \]

So \( \gamma_t \times C_\perp = \frac{\gamma}{|\gamma|} + C_g \).

Step3. Ellipse (not parabola nor hyperbola)

Let the angle between ground vector \( C_g \) and \( \gamma \) be \( \theta \). Let’s take the inner product of the last equation with \( \gamma \):

\[ \gamma \cdot (\gamma_t \times C_\perp) = |\gamma| + \gamma \cdot C_g = |\gamma| + |\gamma||C_g| \cos \theta \]

Note \( \gamma \cdot (\gamma_t \times C_\perp) = (\gamma \times \gamma_t) \cdot C_\perp = |C_\perp|^2 \). Thus

\[ |\gamma| = \frac{|C_\perp|^2}{1 + |C_g| \cos \theta} \]

or

\[ r = \frac{|C_\perp|^2}{1 + |C_g| \cos \theta} \]
in polar coordinates with x-axis along $C_g$ direction.

At this point, the plane orbit could be ellipse, parabola, or hyperbola. But as the Earth orbit is closed/bounded, then it must be ELLIPSE!

Sec4.1 Functions of several variables

Not a single thing depends only on one factor. Thus need to study functions of 2, 3, \cdots variables.

Domain/range, graph, level set

eg4 Find the domain & range of fcn $h(x, y) = \sqrt{9 - x^2 - y^2}$.

$D = \{(x, y) : x^2 + y^2 \leq 9\}$  $R = \{z : 0 \leq z \leq 3\}$

eg6. Graph of $h(x, y) = \sqrt{9 - x^2 - y^2}$. Upper hemisphere.

eg5 Sketch graph of $L(x, y) = 6 - 3x - 2y$. (Q: Why notation L?)

Level set

1-variable functions: level points, eg. $y = \sqrt{9 - x^2}$

2-variable functions: level curves, eg $z = \sqrt{9 - x^2 - y^2}$

3-variable functions: level surfaces, eg $w = \sqrt{9 - x^2 - y^2 - z^2}$, eg. $w = x + 2y + 3z + 4$, $w = -1, w = 0, w = 1$.

Skip Sec14.2 limits & continuity

Sec14.3 Partial derivatives

• The changing rate of a function w.r.t. each variable while holding the remaining variables constant is a partial derivative.

eg1. Let $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(5, 9)$ and $f_y(10, 1)$.

In general,

$f_y(x, y), x \to 10, \frac{df}{dy}(10, y), \text{ afterwads}, 10 \to x$

$f_y = x^23y^2 - 4y^2$

similarly

$f_x(x, y), y \to 9, \frac{df}{dx}(x, 9), \text{ afterwads}, 9 \to y$

$f_x = 3x^2 + 2xy^3$

Various notations for partial derivatives of $z = f(x, y)$

$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_x f$

$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_y f$

• Geometric interpretation of partial derivatives

$f_x(a, b)$ slope of curve/1-d graph $z = f(x, b)$ of intersection of vertical plane $y = b$ and 2-d graph $z = f(x, y)$ at $x = a$.

$f_y(a, b)$ slope of curve/1-d graph $z = f(a, y)$ of intersection of vertical plane $x = a$ and 2-d graph $z = f(x, y)$ at $y = b$.

The “full derivatives” $(f_x(a, b), f_y(a, b))$ describe the tangent plane to the surface $z = f(x, y)$ at $(a, b, f(a, b))$. In fact the normal direction of the tangent plane is $(f_x(a, b), f_y(a, b), -1)$.

RMK. Consider graph $\gamma(x, y) = (x, y, f(x, y))$, two tangent directions are $\gamma_x = (1, 0, f_x)$ and $\gamma_y = (0, 1, f_y)$, then the normal of the tangent plane spanned by $\gamma_x$ and...
\(\gamma_y\) is parallel to the vector
\[
\gamma_x \times \gamma_y = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = (-f_x, -f_y, 1).
\]

**eg3.** \(f (x, y) = \sin \left(\frac{x}{1+y}\right)\) find \(f_x\) & \(f_y\).

**eg4.** Implicit differentiation. Find \(z_x\) & \(z_y\) if \(z\) is implicitly defined as a function of \(x\) & \(y\) by \(x^3 + y^3 + z^3 + 6xyz = 1\).

- Higher order partial derivatives
  Note for \(z = f (x, y), f_x (x, y)\) and \(f_y (x, y)\) are still functions of their own, they still have their own partial derivatives, those are called 2nd order partial derivatives of \(f\):
  \[
  \left\{ \begin{array}{l}
  (f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} \\
  (f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \\
  (f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} \\
  (f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}
  \end{array} \right.
  \]

**eg6** Find all second order derivatives of \(f (x, y) = x^3 + x^2y^3 - 2y^2\).

\(f_{xy} = 6xy^3, f_{yx} = 6x^3y^2\). No coincidence.
Theorem (Clairaut) If \(f_{xy}\) and \(f_{yx}\) are both continuous (namely their graphs have no break), then \(f_{xy} = f_{yx}\).

No difference in switching order, unlike exam solution v.s. solution exam.

- Partial differential equations
  Wrap up those partial derivatives into equations, then we have partial differential equations.

**eg9.** Wave/string equation \(w_{tt} = w_{xx}\) such as \(w (x, t) = \sin (x - t) = \sin x \cos t - \cos x \sin t\).

**eg8.** Laplace (or heat equilibrium–after infinitely long time) equation \(h_{xx} + h_{yy} = 0\) such as \(h (x, y) = e^x \sin y\).

**Sec14.4 Tangent planes and linear approximation**

- Tangent planes
  Recall for 1-d graph \((x, f (x))\) at \((a, f (a))\) the tangent vector is \((1, f_x (a)) \Leftrightarrow\) slope \(m = f_x (a),\) the function for tangent line \(y = L (x) = f (a) + f_x (a) (x - a)\).
  Geometrically, the graph looks like the tangent plane near \((a, f (a))\).
  Analytically, the function is approximately the linear function, \(f (x) \approx f (a) + f_x (a) (x - a),\) near \(x = a\).
  Now for 2-d graph \(\gamma (x, y) = (x, y, f (x, y)),\) tangent plane at \((a, b, f (a, b))\) is spanned by tangent vector \(\gamma_x = (1, 0, f_x (a, b))\) and \(\gamma_y = (0, 1, f_y (a, b))\).
  Q. Function for the tangent plane \(z = L (x, y) = \text{??}\)
  Q’ Tangent plane equation?
  Point \((a, b, f (a, b)) /\) normal direction \(n = \text{??}\)
Thus the function for the tangent plane is
\[ z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b). \]

Thus the function for the tangent plane is
\[ z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b). \]

AN equation for the tangent plane is:
\[ L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b). \]

Eg1. Given graph \( z = \sqrt{9 - x^2 - y^2} \) at \((1, 2, 2)\), find another equation for the tangent plane and the function representing the tangent plane.

Sol. The linear function for the tangent plane \( L(x, y) = 2 - \frac{1}{2} (x - 1) - (y - 2). \)

Another equation for the tangent plane is:
\[ 987z = 987 \cdot 2 - 987 \cdot \frac{1}{2} (x - 1) - 987 \cdot (y - 2). \]

- Linear approximation of function \( z = f(x, y) \).

Geometrically, the graph \((x, y, f(x, y))\) looks like the tangent plane near \((a, b, f(a, b))\).

Analytically, the function is approximately the linear function, \( f(x) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \), near \((x, y) = (a, b)\).

Q. When \( f(x, y) \) can be approximated linearly (i.e. by the function for the tangent plane)?

A. When \( f(x, y) \) is nice \( \implies \) differentiable. In particular, when both the partial derivatives \((f_x, f_y)\) are continuous.

Eg2. Show \( f = xe^{xy} \) is differentiable at \((1, 0)\). Find its linearization and use it to approximate \( f(1.1, -0.1) \).

- Differentials

Recall 1-var function \( y = f(x) \), differential is \( dy = f'(x) \, dx \), linear approximation
\[ \Delta y = f(x + \Delta x) - f(x) \approx f'(x) \, \Delta x. \]
Let \( dx = \Delta x \), then
\[ dy \approx f(x + \Delta x) - f(x) = \Delta y. \]

Now 2/3 variable functions \( z = f(x, y) \)

The differential is defined as \( dz = f_x(x, y) \, dx + f_y(x, y) \, dy. \)

Linear approximation is \( \Delta z \approx f(x + \Delta x, y + \Delta y) - f(x, y) \approx f_x(x, y) \, \Delta x + f_y(x, y) \, \Delta y. \) Let \( dx = \Delta x \) and \( dy = \Delta y \), then
\[ \Delta z \approx dz. \]

Eg4. a) Let \( z = f(x, y) = x^2 + 3xy - y^2 \), find its differential \( dz \). b) If \( x \) changes from 2 to 2.05 and \( y \) changes from 3 to 2.96, compare the values \( \Delta z \) and \( dz \).

Lastly, eg. for a 3 variable function \( w(x, y, z) \), its differential \( dw = w_x \, dx + w_y \, dy + w_z \, dz \approx w(x + \Delta x, y + \Delta y, z + \Delta z) - w(x, y, z). \)
Sec 14.7 Maximum and minimum values

Def Global maximum value at \((a, b)\) if \(f(a, b) \geq f(x, y)\) for all \((x, y) \in \text{Domain}\)

Global max or min values are also called extreme values, absolute values in the everyday life sense.

Def. Local max value near \(c\), if \(f(a, b) \geq f(x, y)\) for all \((x, y)\) near \((a, b)\)

Local max \((a, b)\) is also local max \(a\) for function \(f(x, b)\), then \(f_x(a, b) = 0\); similarly
local max \(b\) for function \(f(a, y)\), then \(f_y(a, b) = 0\). Thus

Def. Critical point \((a, b)\) if \((f_x, f_y)_{(a,b)} = (0, 0)\).

eg1. Find critical points, local maximum/minimum, global max/min for \(f(x, y) = (x - 1)^2 + (y - 3)^2 + 4 = x^2 + y^2 - 2x - 6y + 14\).

Graph, then calculation

eg2. Find global extreme values of \(S(x, y) = y^2 - x^2\).

No extreme values.

Compare examples: \(x^2 + y^2\), \(-x^2 - y^2\), and \(x^2 - y^2\); in particular their double derivatives. Also \(z = xy\)

Summarize as

Second Derivative Test: At critical point \((a, b)\) of \(f(x, y)\). Let (the determinant of the 2nd derivative matrix

\[
\begin{vmatrix}
  f_{xx} & f_{xy} \\
  f_{yx} & f_{yy}
\end{vmatrix}
\]

\(Det = f_{xx}f_{yy} - f_{xy}^2\) \((a, b)\).

- If \(Det > 0\) & \(f_{xx} > 0\), then \(f(a, b)\) is a local min.
- If \(Det > 0\) & \(f_{xx} < 0\), then \(f(a, b)\) is a local max.
- If \(Det < 0\), then \(f(a, b)\) is not a local min or max. (saddle point).

RMK. When Det=0, NO conclusion.

eg3. Find local max/min/saddle values/points of \(f(x, y) = x^4 + y^4 - 4xy + 1\).

Applications.

eg5. Find the shortest distance from \((1, 0, -2)\) to the plane \(x + 2y + z = 4\).

Inner product way.

\[d = \frac{|1 - 2 - 4|}{\sqrt{1^2 + 2^2 + 1^2}} = \frac{5}{\sqrt{6}}\]

Calculus way.

Step 1. Set up. \(d = \sqrt{(x - 1)^2 + y^2 + (z + 2)^2}\). Here \(x/y/z\) are not independent, in fact say, \(z = 4 - x - 2y\). Then

\[d(x, y) = \sqrt{(x - 1)^2 + y^2 + (6 - x - 2y)^2}\]

and now \(x\) and \(y\) are independent. We are looking for

\[
\min_{(x, y) \in \mathbb{R}^2} \sqrt{(x - 1)^2 + y^2 + (6 - x - 2y)^2},
\]
equivalently  
\[ \min_{(x,y) \in \mathbb{R}^2} d^2(x, y). \]

Let \( f(x, y) = (x - 1)^2 + y^2 + (6 - x - 2y)^2 \).

Step 2. Intuitively, the global minimum of \( f \) exists, as the function goes off to infinity as \( x \) and \( y \) get unbounded. The global min point should be a critical one.

\[ \begin{align*}
  f_x &= 2(x - 1) - 2(6 - x - 2y) = 0 \\
  f_y &= 2y - 4(6 - x - 2y) = 0
\end{align*} \]

or \( x + y = 7/2 \) and \( 2x + 5y = 12 \). Then \( y = 5/3 \) and \( x = 11/6 \).

As the critical point is unique, it must be the global minimum point, which gives the shortest distance \( d = \sqrt{(x - 1)^2 + y^2 + (6 - x - 2y)^2} \bigg|_{(\frac{11}{6}, \frac{5}{3})} = 5/\sqrt{6}. \)

eg6. A rectangular box w/o a lid is to be made from 12 m\(^2\) of cardboard. Find the maximum volume of the box.

Set-up \( V = xyz \) with \( xy + 2yz + 2zx = 12 \). Reduce one variable \( z = (12 - xy)/2(x + y) \).

\[ \max_{0 < xy \leq 12} \frac{xy(12 - xy)}{2(x + y)}. \]

Obs. \( V(x, y) = 0 \) along the boundary of \( 0 < xy \leq 12 \).

Thus intuitively, global max exists, it should be a critical point.

\( V_x = 0, V_y = 0 \).

As the only critical point is \((2, 2)\), \( V(2, 2) = 4 \) must be the global max.

Find Extreme values on the bounded set.

- critical values inside \( D \)
- extreme values on the boundary (1-d calculus)
- pick the largest/smallest from the above.

eg7. Find extreme values of \( f(x, y) = x^2 - 2xy + 2y \) on Rectangle \( \{(x, y) : 0 \leq x \leq 3, 0 \leq y \leq 2\} \).

Sec 15.1 Double integrals over rectangles

- Definition.

As the area for (continuously) curved region is DEFINED as the limit of the sum of the areas of those approximated slices. The volume of the solid over rectangle with curved top is DEFINED as the limit of the sum of the volumes of the approximated “thin” cubes:

\[ \begin{align*}
  &\text{cut } a \leq x \leq b \text{ into } m \text{ pieces (evenly), } x_0 = a + 0\frac{b-a}{m},
  x_1 = a + 1\frac{b-a}{m},
  x_i = a + i\frac{b-a}{m},
  x_m = a + m\frac{b-a}{m} = b; \\
  &\text{cut } c \leq x \leq d \text{ into } n \text{ pieces (evenly), } y_0 = c + 0\frac{d-c}{m},
  y_1 = c + 1\frac{d-c}{m},
  y_j = c + j\frac{d-c}{m},
  y_m = c + m\frac{d-c}{m} = d; \\
  &V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \Delta x \Delta y;
\]
take limit as both \( m \) and \( n \) go to infinity (for nice functions, say continuous ones, the limit exists and is unique!) the volume is DEFINED as

\[
V = \lim_{m \to \infty} \lim_{n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \triangle x \triangle y.
\]

Notation for the limit

\[
V = \int \int_R f(x, y) \, dx \, dy = \int \int_R f(x, y) \, dy \, dx = \int \int_R f(x, y) \, dA.
\]

RMK. Pick \((x_{ij}^*, y_{ij}^*)\) as the center/mid point of the each box, then the midpoint rule approximation;
Pick \((x_{ij}^*, y_{ij}^*)\) as the left-bottom point of the each box, then the left-bottom rule approximation;

\[
\text{eg0. } \int_{[0,1] \times [0,1]} 8.736 \, dx \, dy = \text{base} \cdot \text{height} = 1 \times 1 \cdot 8.736 = 8.736.
\]

\[
\text{eg2. } \int_{-2 \leq y \leq 2} \int_{-1 \leq x \leq 1} \sqrt{1 - x^2} \, dA
\]

volume of the cylinder along y-direction. (Similar problems in HW).

- Linear properties for double integrals
  
  \[
  * \int \int_R [f(x, y) + g(x, y)] \, dA = \int \int_R f(x, y) \, dA + \int \int_R g(x, y) \, dA \\
  * \int \int_R cf(x, y) \, dA = c \int \int_R f(x, y) \, dA \\
  * \text{if } f \leq g \text{ on } R, \text{ then } \int \int_R f(x, y) \, dA \leq \int \int_R g(x, y) \, dA.
  \]

Sec15.2 Iterated integrals (Double integrals over rectangle, “partial” integrals)
Recall in differential calculus of 2-var fcns, differentiate twice

Now in integral calculus of 2-var. fcns, integrate twice!

\[
\text{eg0. } \int \int_{-2 \leq y \leq 2} \int_{-1 \leq x \leq 1} \sqrt{1 - x^2} \, dA = \frac{1}{2} \pi \cdot 4.
\]

Slice along y-direction: \( \int_{-2}^{2} \text{Area of cross section at } y \, dy = \int_{-1}^{1} \left( \int_{-2}^{2} \sqrt{1 - x^2} \, dx \right) \, dy = \int_{-2}^{2} \frac{1}{2} \pi \, dy = \frac{1}{2} \pi \cdot 4 \)

Slice along x-direction: \( \int_{-1}^{1} \text{Area of cross section at } dx = \int_{-1}^{1} \left( \int_{-2}^{2} \sqrt{1 - x^2} \, dy \right) \, dx = 4 \cdot \int_{-1}^{1} \sqrt{1 - x^2} \, dx = 4 \cdot \frac{1}{2} \pi. \)

Two different ordering, same result, like inner product \( \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \), unlike \( \vec{a} \times \vec{b} \neq \vec{b} \times \vec{a} \).

\[
\text{eg1. a) } \int_{0}^{3} \left( \int_{1}^{2} x^2 y \, dx \right) \, dy, \text{ b) } \int_{1}^{2} \int_{0}^{3} (x^2 y) \, dx \, dy
\]

Again different orders, same result.
In general, if \( f(x, y) \) is nice, say continuous, then
\[
\int \int_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy.
\]

Fubini Theorem: If \( f(x, y) \) is nice, say continuous, then
\[
\int \int_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy.
\]

eg3. \( \int_{\mathbb{R}} \sin(xy) \, dA, \quad R = [1, 2] \times [0, \pi]. \)

eg4. Find the volume of the solid \( S \) that is bounded by the elliptic paraboloid \( x^2 + 2y^2 + z = 16 \), and the planes \( x = 2 \), \( y = 2 \), and the three coordinate planes.

Graph
Vol= \( \int_{\mathbb{R}} z \, dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2) \, dx \, dy \).

eg5. If \( R = [0, \pi/2] \times [0, \pi/2] \),
\[
\int_0^{\pi/2} \int_0^{\pi/2} \sin x \cos y \, dx \, dy = \int_0^{\pi/2} \cos y \left( \int_0^{\pi/2} \sin x \, dx \right) \, dy = \left( \int_0^{\pi/2} \sin x \, dx \right) \int_0^{\pi/2} \cos y \, dy = \left( -\cos x \big|_0^{\pi/2} \right) \left( \sin y \big|_0^{\pi/2} \right) = 1 \cdot 1 = 1.
\]

In general, \( \int_a^b \int_c^d f(x) \, g(y) \, dy \, dx = \int_a^b f(x) \left( \int_c^d g(y) \, dy \right) \, dx = \left( \int_c^d g(y) \, dy \right) \int_a^b f(x) \, dx. \)

Sec15.3 Double integrals over general domains
Q. For all the same integrands in Sec15.2 and more, if the integral domain is not
a rectangle, how to calculate \( \int \int_R f(x, y) \, dA \)?

Answer.
Straight left&right \( a \leq x \leq b \), curved top&bottom \( B(x) \leq y \leq T(x) \)
\( \int \int_R f(x, y) \, dA = \int_a^b \int_{y=B(x)}^{y=T(x)} f(x, y) \, dy \, dx; \)
Straight top&bottom \( c \leq y \leq y, \) curved left&right \( L(y) \leq x \leq R(y) \)
\( \int \int_R f(x, y) \, dA = \int_c^d \int_{x=L(y)}^{x=R(y)} f(x, y) \, dx \, dy. \)

eg1. \( \int \int_D (x + 2y) \, dA, \) \( D \) is bounded by \( y = 2x^2 \) & \( y = x^2 + 1 \).
eg2. Find the volume of the solid that lies under the paraboloid \( z = x^2 + y^2 \) and
above the region \( D \) in \( x-y \) plane bounded by the line \( y = 2x \) & parabola \( y = x^2. \)

First domain.
Then
Straight left&right–curved top&bottom way.
Straight top&bottom–curved left&right way.

eg4. Find the volume of the tetrahedron bounded by the planes \( x + 2y + z = 2, \)
\( x = 2y, x = 0, \) and \( z = 0. \)

Domain on ground \( z = 0 : x + 2y + z = 2, \) \( x = 2y, \) \( x = 0 \)


eg5’. \( \int_0^1 \int_x^1 e^{-y^2} \, dy \, dx \) impossible to integrate at this order (Recall/Indeed \( \int e^{-y^2} \, dy \)
has no explicit expression in terms of elementary functions; soon we’ll calculate
\( \int_{-\infty}^{\infty} e^{-x^2} \, dx. \)

Integral domain: bounded by \( y-axis, \) \( y=1 \) line and \( y = x \) line.

Switch order: \( \int_0^1 \int_0^y e^{-y^2} \, dx \, dy = \int_0^1 \left( e^{-y^2} \right) \left( \int_0^y dx \right) \, dy = \int_0^1 \left( e^{-y^2} \right) \, dy = -\frac{1}{2} e^{-y^2} \big|_0^1 = \frac{1}{2} (1 - e^1). \)
Sec15.4 Double integrals in polar coordinates

Q. How to calculate double integrals \( \iint_D f dA \) in polar coordinates?

Recall area of the polar rectangle \( \left\{ \begin{array}{l} 0 \leq r \leq 3 \\ 0 \leq \theta \leq 1.5 \end{array} \right\} \) is \( \frac{1}{2} \) height base = \( \frac{1}{2} \times 3 \times 3 = 4.5 \).

Now the area of infinitesimally small polar rectangle \( \left\{ \begin{array}{l} [r, r + dr] \\ [\theta, \theta + d\theta] \end{array} \right\} \) is

\[
\frac{1}{2} (r + dr)^2 d\theta - \frac{1}{2} r^2 d\theta = r dr d\theta + \frac{1}{2} (dr)^2 d\theta.
\]

Thus \( dA = r dr d\theta \) and \( f dA = f(r \cos \theta, r \sin \theta) \ r dr d\theta / r dr dr \)

* RMK. Cross product way:

\[
\begin{align*}
\text{Area of D} &= \int_0^1 \int_0^{2\pi} (r^2 \cos^2 \theta + r^2 \sin^2 \theta) r dr d\theta \\
&= \int_0^1 \int_0^{2\pi} r^3 dr d\theta.
\end{align*}
\]

1. Evaluate \( \iint_R (3x + 4y^2) \ dA \), where \( R \) is the region in the upper half plane and bounded between the circles \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \).

   Domain: \( 1 \leq r \leq 2, \ 0 \leq \theta \leq \pi \)
   Integral: \( \int_0^1 \int_0^{2\pi} (3r \cos \theta + 4r^2 \sin^2 \theta) \ r dr d\theta \) or \( \int_0^\pi \int_1^2 (3r \cos \theta + 4r^2 \sin^2 \theta) \ r dr d\theta \)

eg2. Find the volume of the solid bounded by the plane \( z = 0 \) and the paraboloid \( z = 1 - x^2 - y^2 \).

   Domain: \( z = 0 \) and \( 1 \geq x^2 + y^2 \) or \( 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi \)
   Integral: \( \int_0^1 \int_0^{2\pi} (1 - r^2 \cos^2 \theta - r^2 \sin^2 \theta) \ r dr d\theta \) or \( \int_0^\pi \int_0^1 (1 - r^2 \cos^2 \theta - r^2 \sin^2 \theta) \ r dr d\theta \).

“Curved” polar region:

Straight \( \theta \), varying \( r : \theta_0 \leq \theta \leq \theta_1 \), Inner(\( \theta \)) \( \leq r \leq \) Outer(\( \theta \))

Straight \( r \), varying \( \theta : r_0 \leq r \leq r_2 \), Lower(\( r \)) \( \leq \theta \leq \) Upper(\( r \)).

eg3. Use a double integral to find the area enclosed by one loop of the four leaved rose \( r = \cos (2\theta) \).

Area of D = \( \iint_D 1 \ dA \).

Domain: \( 0 \leq \theta \leq 2\pi, \ “0 \leq r \leq \cos (2\theta)” \) or \( 4 \) of \( \frac{-\pi}{4} \leq \theta \leq \frac{\pi}{4}, \ 0 \leq r \leq \cos (2\theta) \).

(Hard to describe as \( 0 \leq r \leq 1, \ ? \leq \theta \leq ? \))

\[
A=\int_{-\pi/4}^{\pi/4} \int_0^{\cos(2\theta)} 1 \ r dr d\theta
\]

eg4. Find the volume of the solid that lies under the paraboloid \( z = x^2 + y^2 \), above the xy-plane, and inside the cylinder \( x^2 + y^2 = 2x \).

Domain: \( x^2 + y^2 = 2x \iff r = 2 \cos \theta \), then \( -\pi/2 \leq \theta \leq \pi/2, \ 0 \leq r \leq 2 \cos \theta \).

Integral: \( \int_{-\pi/2}^{\pi/2} \int_0^{2\cos \theta} (r^2 \cos^2 \theta + r^2 \sin^2 \theta) \ r dr d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos \theta} r^3 dr d\theta \)

Note the rectangular way is little tough:

Domain: \( 0 \leq x \leq 2, \ -\sqrt{2x-x^2} \leq y \leq \sqrt{2x-x^2} \)

\[
\text{Vol}=\int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} (x^2 + y^2) \ dy dx = \int_0^2 \left[ 2x \sqrt{2x-x^2} + \frac{2}{3} (\sqrt{2x-x^2})^3 \right] dx = ...
\]

eg. \( \int_{-\infty}^{\infty} e^{-x^2} dx = ? \)
Step1. \[\int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy\right)}\]

\[\pi \left(-e^{-r^2}\right)|_0^\infty = \pi.\]

Thus \[\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.\]

Sec15.5 Applications of double integrals

- Density and mass
  - mass of infinitesimally small square = density \cdot dA
  - Total mass = \(\int \int_D \rho dA\)
  - \(\text{eg1. Charge is distributed over the triangle } D\text{ so that the charge density at } (x, y)\text{ is } \sigma (x, y) = xy\text{, measured in coulombs per square meter } C/m^2.\text{ Find the total charge.}\)

\[T: y = 1, \ x = 1, \text{ and } y = 1 - x.\]

Total Charge = \(\int \int_T xy dA = \int_0^1 \int_{1-x}^1 xy dy dx\)

- Moments and center of mass
  - x-moment/moment about y-axis of infinitesimally small square = \(x \cdot \rho \cdot dA\)
  - Total x-moment/moment about y-axis = \(\int \int_D x \rho (x, y) dA\)
  - y-moment/moment about x-axis of infinitesimally small square = \(y \cdot \rho \cdot dA\)
  - Total y-moment/moment about x-axis = \(\int \int_D y \rho (x, y) dA\)
  - Center of mass/average of x/y moments

\[\bar{x} = \frac{\int \int_D x \rho (x, y) dA}{\int \int_D \rho (x, y) dA}, \ \bar{y} = \frac{\int \int_D y \rho (x, y) dA}{\int \int_D \rho (x, y) dA}\]

In particular, when density is constant 1, i.e. \(\rho = 1\): \(D\) a \(x \leq x \leq b, B (x) \leq y \leq T (x)\)

x-moment = \(\int \int_D x dA = \int_b^a \int_{B(x)}^{T(x)} x dy dx = \int_a^b x \left[ T (x) - B (x) \right] dx,\)

y-moment = \(\int \int_D y dA = \int_b^a \int_{B(x)}^{T(x)} y dy dx = \int_a^b \frac{1}{2} \left[ T^2 (x) - B^2 (x) \right] dx,\)

Those are formulas in M125.

- \(\text{eg2. Find the mass and center of mass of a triangle lamina with vertices } (0, 0), (1, 0),\) and \((0, 2)\) if the density function is \(\rho (x, y) = 1 + 3x + y.\)

\(\text{Domain: } 0 \leq x \leq 1, \ 0 \leq y \leq 2 - 2x\)

mass \(m = \int_0^1 \int_{2-2x}^{2-2x} (1 + 3x + y) \ dy \ dx = \cdots = 8/3.\)

x-moment = \(\int_0^1 \int_{2-2x}^{2-2x} x (1 + 3x + y) \ dy \ dx = \cdots = 1\)

y-moment = \(\int_0^1 \int_{2-2x}^{2-2x} y (1 + 3x + y) \ dy \ dx = \cdots = 11/6.\)

Then \(\bar{x} = 3/8\) and \(\bar{y} = 11/16.\)

- \(\text{eg3. The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of the mass of the lamina.}\)

\(\text{Domain: } x^2 + y^2 \leq R^2, \ y \geq 0, \text{ or } 0 \leq r \leq R, 0 \leq \theta \leq \pi.\)

Density \(\rho (x, y) = K \sqrt{x^2 + y^2}, \) \(K\) proportional constant.
mass=\int_A K \sqrt{x^2+y^2} \, dA \text{ easier to integrate in polar coordinates } \int_0^R \int_0^\pi K r \, r \, dr \, d\theta = K \pi R^2/3

x\text{-moment}=\int_A xK \sqrt{x^2+y^2} \, dA = 0 \text{ by symmetry. Details: } \int_0^R \int_0^\pi xK \sqrt{x^2+y^2} \, dx \, dy = \int_0^R 0 \, dy = 0.

y\text{-moment}=\int_A yK \sqrt{x^2+y^2} \, dA \text{ easier to integrate in polar coordinates } \int_0^R \int_0^\pi r \sin \theta K r \, r \, dr \, d\theta = \int_0^R 2K r^3 \, dr = KR^4/2.

center of mass \bar{x} = 0, \bar{y} = 3R/2\pi.

Review for Midterm II
- one variable vector functions/curves
  - arclength; tangent, normal, binormal, osculating plane, normal plane, curvature (shape)
- scalar functions of several variables
  - "partial" Derivatives, tangent planes/linear approximation, implicit differentiation, critical points, 2nd derivative test, local/global max/min
  - "partial" Integrals over rectangles, "half" rectangles, under polar coordinates, application mainly in volume.

Taylor polynomials and Taylor series
TN1. Linear approximation revisit
Reveal/approximate "non-explicit" nonlinear function via explicit polynomial functions.
Recall linear approximation:
Geometrically, graph looks like tangent line.
Analytically, \( f(x) \approx f(o) + f'(o)(x-o) \).
Q. How much difference?
Geometric measure, hard & impossible.
Analytically:
\[
f(x) = f(o) + \int_o^x f'(t) \, dt\]

IBP, \( \int_o^b g'(t)h(t) \, dt = gh|_o^b - \int_o^b gh' \)
1st try, \( \int_o^x f'(t) \, dt = \int_o^x t'f'(t) \, dt = tf'(t)|_o^x - \int_o^x t^2f''(t) \, dt = xf'(x) - \int_o^x tf''(t) \, dt, \) NOT at \( f'(0) \).
2nd try \( t \to t-x \),
\( \int_o^x f'(t) \, dt = \int_o^x (t-x)'f'(t) \, dt = (t-x) f'(t)|_o^x - \int_o^x (t-x) f''(t) \, dt = f'(o)(x-0) + \int_o^x (x-t) f''(t) \, dt. \)
Q. IBP one more time? Postpone to TN2.
So far
\[
f(x) = f(o) + f'(o)(x-o) + \int_o^x (x-t) f''(t) \, dt.\]
or \( f(x) - L(x) = \int_o^x (x-t) f''(t) \, dt. \)
Error estimate: If $|f''(t)| \leq M_2$ for $t$ between $o$ and $x$, then $|f(x) - L(x)| \leq \frac{1}{2} |x - o|^2 M_2$.

Case $x > o$. $|(x-t) f''(t)| \leq (x-t) M_2$, integrate, the error $< \int_o^x (x-t) M_2 dt = \frac{1}{2} (x-o)^2 M_2$.

Case $x < o$. $|(x-t) f''(t)| \leq (t-x) M_2$.

$|\int_o^x (x-t) f''(t) dt| = -\int_o^x (x-t) f''(t) dt | \leq \int_o^x (t-x) M_2 dt = \frac{1}{2} (o-x)^2 M_2$.

In general, $o$ could be 0, 1.1, or any other number/base, $b$:

$$|f(x) - f(b) - f'(b)(x-b)| \leq \frac{1}{2} |x-b|^2 \max_{t \text{ between } x \text{ and } b} |f''(t)|.$$ 

eg1. Find a bound of error in the linear approximation for $f(x) = \arctan x$ based at $b = 1$ on the interval $[0.9, 1.1]$.

Sol. • $(\arctan x)' = \frac{1}{1+x^2}$, $(\arctan x)'' = \frac{-2x}{(1+x^2)^2}$.

• $L(x) = \arctan 1 + f'(1)(x-1) = \frac{\pi}{4} + \frac{1}{2} (x-1)$.

• $|\arctan x - L(x)| \leq \frac{1}{2} |x-1|^2 \max_{x \in [0.9, 1.1]} \left| \frac{-2t}{(1+t^2)^2} \right|$.

Now $x \in [0.9, 1.1]$, then $\frac{1}{2} |x-1|^2 \max_{x \in [0.9, 1.1]} \left| \frac{-2t}{(1+t^2)^2} \right| \leq \frac{1}{2} \frac{0.1^2 \cdot 2.2}{(1+0.92)^2} \leq 0.003$.

eg1. continued. Find an interval $J$ near 1 so that the error bound is at most 0.001 on $J$.

Sol. As $0.001 < 0.003$, this new interval $J$ should be inside $[0.9, 1.1]$, then

$$\max_J |f''(t)| \leq \max_{[0.9, 1.1]} |f''(t)| \leq \frac{2.2}{(1+0.92)^2}.$$ 

Then on $J$ the error bound is less than $\frac{1}{2} (x-1)^2 \frac{2.2}{(1+0.92)^2}$. We want this less than 0.001.

Solve the inequality we have $|x-1| \leq \frac{1+0.92}{\sqrt{1.1}} \sqrt{0.001} < 0.0181 \sqrt{10} < 0.0181 \cdot 4 < 0.08$.

Thus one interval could be $[0.92, 1.08]$.

Q. Here we used a rough bound for

$$\max_{[0.9, 1.1]} |f''(t)| = \max_{[0.9, 1.1]} \frac{2t}{(1+t^2)^2} \leq \frac{2.2}{(1+0.92)^2}.$$ 

How to get the precise maximum $\max_{[0.9, 1.1]} \frac{2t}{(1+t^2)^2}$?

TN2. Quadratic approximation

One more IBP in $f(x) = f(o) + f'(o)(x-o) + \int_o^x (x-t) f''(t) dt$

$= L(x) + \left[ -\frac{1}{2} (x-t) f'(t) \right]_{t=o}^{t=x} + \int_o^x \frac{1}{2} (x-t)^2 f'''(t) dt = L(x) + \frac{1}{2} f''(o) (x-o)^2 + \int_o^x \frac{1}{2} (x-t)^2 f'''(t) dt$.

Notation. $T_1(x) = L(x), T_2(x) = f(o) + f'(o)(x-o) + \frac{1}{2} f''(o)(x-o)^2$.

Error bound on quadratic approximation: If $|f'''(t)| \leq M_3$ for $t$ between $o$ and $x$, then $|f(x) - T_2(x)| \leq \frac{1}{3!} |x-o|^3 M_3$.

Proof. Case $x > o$. $|\int_o^x \frac{1}{2} (x-t)^2 f'''(t) dt| \leq \int_o^x \frac{1}{2} (x-t)^2 dt M_3 \leq \frac{1}{23} |x-o|^3 M_3$.

Case $x < o$ is similar, needs $\int_o^x = -\int_o^x$. 

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eg 1. Find an error bound in the quadratic approximation of \( f(x) = \cos x \) near 0 on interval \([-0.2, 0.2]\).

Sol. \( \cos x' = -\sin x \quad x = 0, \cos x'' = -\cos x \quad x = 0 - 1, \cos x''' = \sin x \). Then \( M_3 \leq 1. \)

\( T_2(x) = 1 + 0(x - 0) + \frac{1}{2}(-1)(x - 0)^2 = 1 - \frac{1}{2}x^2. \)

\( |\cos x - T_2(x)| \leq \frac{1}{3!} |x - 0|^3 M_3 \leq \frac{1}{3!} |x|^3 \leq 0.23^3/6 < 0.0013. \)

TN3. Higher order approximation and Taylor error estimate

Another IBP in quadratic approximation

\[
f(x) = T_2(x) + \int_o^x \frac{1}{2}(x-t)^2 f'''(t) dt = T_2(x) + \left[ -\frac{1}{3!} (x-t)^3 f'''(t) \right]_{t=0}^{t=x} + \int_o^x \frac{1}{3!} (x-t)^3 f^{(4)}(t) dt
\]

\[
= T_2(x) + \frac{1}{3!} f'''(0)(x-0)^3 + \int_o^x \frac{1}{3!} (x-t)^3 f^{(4)}(t) dt
\]

\[
= T_3(x) + \int_o^x \frac{1}{3!} (x-t)^3 f^{(4)}(t) dt
\]

Another IBP \( f(x) = T_3(x) + \frac{1}{3!} f^{(4)}(0)(x-o)^4 + \int_o^x \frac{1}{4!} (x-t)^4 f^{(5)}(t) dt = T_4(x) + \int_o^x \frac{1}{4!} (x-t)^4 f^{(5)}(t) dt. \)

... IBP \( n \) times,

\[
f(x) = T_n(x) + \int_o^x \frac{1}{n!} (x-t)^n f^{(n+1)}(t) dt
\]

\[
= \frac{f(o)}{0!} + \frac{f'(o)}{1!} (x-o) + \frac{f''(o)}{2!} (x-o)^2 + \cdots + \frac{f^{(n)}(o)}{n!} (x-o)^n + \int_o^x \frac{1}{n!} (x-t)^n f^{(n+1)}(t) dt.
\]

Convention. \( 0! = 1. \)

Error estimate (Taylor). If \( |f^{(n+1)}(t)| \leq M_{n+1} \) for \( t \) between \( o \) and \( x \), then \( |f(x) - T_n(x)| \leq \frac{1}{(n+1)!} |x-o|^{n+1} M_{n+1}. \)

Proof. Case \( x > o. \) \( \frac{1}{n!} (x-t)^n f^{(n+1)}(t) \leq \frac{1}{n!} (x-t)^n M_{n+1}, \) integrate, \( \int_o^x \frac{1}{n!} (x-t)^n M_{n+1} = \frac{1}{(n+1)!} |x-o|^{n+1} M_{n+1}. \)

Case \( x < o \) similar, needs \( \int_o^x = -\int_x^0. \)

eg 0. Take a polynomial \( f(x) = x^4 + 10x^3 - 7x^2 + 3x + 2. \) Find its \( T_4(x) \) based at 1.

Sol. \( f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!} (x-1)^2 + \frac{f'''(1)}{3!} (x-1)^3 + \frac{f''''(1)}{4!} (x-1)^4 + \int_1^x \frac{1}{4!} (x-t)^4 f^{(5)}(t) dt. \)

\[
= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!} (x-1)^2 + \frac{f'''(1)}{3!} (x-1)^3 + \frac{f''''(1)}{4!} (x-1)^4.
\]

eg 1. For \( g(x) = \frac{1}{1-x}. \) Find its \( T_n \) based at 0.

Sol. \( g = (1-x)^{-1} x = 0 1, \)

\( g' = (1-x)^{-2} x = 0 1, \)

\( g'' = 2 (1-x)^{-3} x = 0 2, \)

\( g''' = 3 \cdot 2 (1-x)^{-4} x = 0 3! \)
\[ g^{(4)} = 4 \cdot 3! \cdot (1 - x)^{-5} \bigg|_{x=0} = 4! \]

\[ g^{(n)} = n \cdot (n - 1)! \cdot (1 - x)^{-n-1} \bigg|_{x=0} = n! \]

Then \( T_n (0) = 1 + x + x^2 + x^3 + x^4 + \ldots + x^n \).

eg2. \( f (x) = e^x \). a. Find \( T_n \) based at 0. b. Find \( n \) so that \( |T_n (x) - e^x| \leq 0.01 \) on \([-2, 2] \).

c. On smaller interval \([-1, 1]\), how close is \( T_n \) (from b.) to \( e^x \).

Sol. a. \( f^{(n)} (x) = e^x \bigg|_{x=0} = 1 \).

\[ T_n (x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} \]

b. \[ |T_n (x) - e^x| \leq \left| \int_0^x \frac{(x-t)^n}{n!} e^t \, dt \right| \leq \left| \int_0^x \frac{(x-t)^n}{n!} \, dt \right| e^2 = \frac{|x|^{n+1}}{(n+1)!} e^2 \leq \frac{2^{n+1}}{(n+1)!} e^2 \]

it less than 0.01, when \( x \in [-2, 2] \).

\( n=1, \frac{2^2}{2} e^2 > 18 \)

\( n=2, \ldots \)

\( n=9, \frac{2^{10}}{10!} e^2 < 0.00208 \). Good enough.

c. \( x \in [-1, 1], |T_9 (x) - e^x| \leq \left| \int_0^x \frac{(x-t)^9}{9!} e^t \, dt \right| \leq \left| \int_0^1 \frac{|x|^{10}}{10!} e^1 \right| < 0.00000075. \)

Review sigma notation \( \sum \).

TN4. Taylor series

Push Taylor polynomial to \( \infty \) order, we have Taylor series

\[ f^{(0)} + f^{(1)}(x-o) + f^{(2)}(x-o)^2 + \ldots + f^{(n)}(x-o)^n + \ldots \]

\[ = T_\infty = \lim_{n \to \infty} T_n = \lim_{n \to \infty} \sum_{k=0}^n \frac{f^{(k)}(o)}{k!} (x-o)^k \]

PROVIDED the limit exists.

Convention 0! = 1.

Q. When does the Taylor series converges, namely, the above limit exists?

C-eg0. \( f (x) = \frac{1}{1-x} \).

Base at 0, \( T_n (x) = 1 + x + x^2 + \ldots + x^n \).

\[ T_\infty (x) = 1 + x + x^2 + \ldots + x^n + \ldots = \lim_{n \to \infty} T_n = \lim_{n \to \infty} \sum_{k=1}^n x^k. \]

When \( x = 1 \), no way could the series converges.

In fact as

\[ T_n (x) = 1 + x + x^2 + \ldots + x^n, \]

\[ xT_n (x) = x + x^2 + \ldots + x^n + x^{n+1}. \]

Then \((1-x)T_n (x) = 1 - x^{n+1} \), and \( T_n (x) = \frac{1-x^{n+1}}{1-x} \)

That is only when \( |x| < 1, T_\infty (x) = \frac{1}{1-x}, \) or

\[ \frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots = T_\infty (x) = \sum_{k=0}^\infty x^k. \]

eg1. For all \( x, e^x = T_\infty (x) = \sum_{k=0}^\infty \frac{x^k}{k!}. \)

Proof. Step1. \( |e^x - T_n (x)| = \left| \int_0^x \frac{(x-t)^n}{n!} e^t \, dt \right| \leq \frac{x^{n+1}}{(n+1)!} e^x \) for \( x > 0 \), similarly

\[ \leq \frac{|x|^{n+1}}{(n+1)!} e^{|x|} \) for \( x < 0. \)

Step2. Claim. for any fixed \( x \) large or small, \( \frac{|x|^n}{n!} \xrightarrow{n \to \infty} 0. \)
Indeed, say $|x| = 1$ google

$$
\frac{|x|^n}{n!} = \left( \frac{|x|}{1} \frac{|x|}{2} \cdots \frac{|x|}{2^{\text{google}+1}} \frac{|x|}{2^{\text{google}+2}} \cdots \frac{|x|}{n!} \right) \xrightarrow{n \to \infty} 0.
$$

Step 3. Thus for each fixed $x$, $|x|^n e^{\|x\|} \to 0$, and $e^x = T_{\infty} (x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. That is

$$
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.
$$

eg2.

$$
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!},
$$

$$
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.
$$

For the $T_{\infty}$ part, exercise.

Now convergent part: based at 0

$$
|T_n (x) - \sin x| \leq \frac{1}{(n+1)!} |x|^{n+1} \max_t \text{between } x \& 0 |\sin (n+1) t|.
$$

$$
\leq \frac{1}{(n+1)!} |x|^{n+1} \xrightarrow{n \to \infty} 0.
$$

Similar for $\cos x$.

counter-eg3. For $f \left( x \right) = \left\{ \begin{array}{ll} e^{\frac{1}{x}} & x \neq 0 \\ 0 & x = 0 \end{array} \right.$

We can calculate $0 = f^{(n-1)} (0) = f^{(n)} (0) = \cdots = f^{(n)} (0)$ Thus based at 0, $T_n (x) = 0$, and then $T_{\infty} (0) = 0 \neq e^{\frac{1}{0}}$.

TN5. Algebraic and differential operations with Taylor series

Finding Taylor series of new functions based on known series of old functions without going through the standard, long procedure.

eg0. Find $T_{\infty}$ for $e^{-x^2}$ based at 0.

eg5.3 Find $T_{\infty}$ for $\frac{1}{2x-3}$ based at 0.

eg5.2 Find $T_{\infty}$ for $2e^x - \frac{3}{1-x}$ for $|x| < 1$ based at 0.

HW10 #5 Find $T_{\infty}$ for $x^3 e^x$ based at 0; #8 Find $T_{\infty}$ for $f \left( x \right) = \left\{ \begin{array}{ll} \sin x & x \neq 0 \\ 1 & x = 0 \end{array} \right.$

based at 0.

eg5.6 Find $T_{\infty}$ for $g \left( x \right) = \frac{1}{(x-3)^2}$ based at 0.

$$
\text{Sol. } \frac{1}{(x-3)^2} = \frac{d}{dx} \left( \frac{1}{3-x} \right) = \frac{d}{dx} \left[ \frac{1}{3} \left( \frac{1}{1-\frac{x}{3}} \right) \right] = \frac{1}{3} \frac{d}{dx} \left[ 1 + \frac{x}{3} + \left( \frac{x}{3} \right)^2 + \cdots \right].
$$

eg5.7 Find $T_{\infty}$ for arctan $x$ based at 0.
Sol.

\[ \arctan x = \int_0^x \frac{1}{1+t^2} \, dt \quad \text{where} \quad |x| < 1 \]

\[ = \int_0^x \left( 1 - t^2 + t^4 - t^6 + \cdots \right) \, dt = \]

\[ = \left[ t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \cdots \right]^x_0 \]

\[ = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \]

\[ = \sum_{k=0}^\infty \frac{(-1)^k}{2k+1} x^{2k+1}. \]

Review (for Final)

Three representing results we’ve learned:

- Solution of \( \dot{E} = -\frac{E}{|E|^3} \), or the orbit \( E = (x, y, z) \) of the Earth around the Sun is an ellipse.
- \( \int_{-\infty}^\infty e^{-x^2} \, dx = \sqrt{\pi} \).
- \( e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots \).

I Algebra (vector operation preparation) for vector function calculus

+ & - 3 versions of \( x \) (scalar, dot/inner, cross/outer)

geometric origin

\[ |u| |v| \cos \theta = u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 \]

area of parallelogram \( \{u, v\} = |u| |v| \sin \theta \) = magnitude of \( u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \).

II calculus of vector valued functions

* partial derivatives
* “partial” integrals

III Taylor series

Applications

* lines/planes, quadratic surfaces
* tangent/normal/binormal, curvature, position/velocity/acceleration
* length/area/volume, center of mass
* linear approximations, extreme values
* quadratic & higher order approximations

Preview for M324: Fundamental theorem of calculus for vector valued functions.