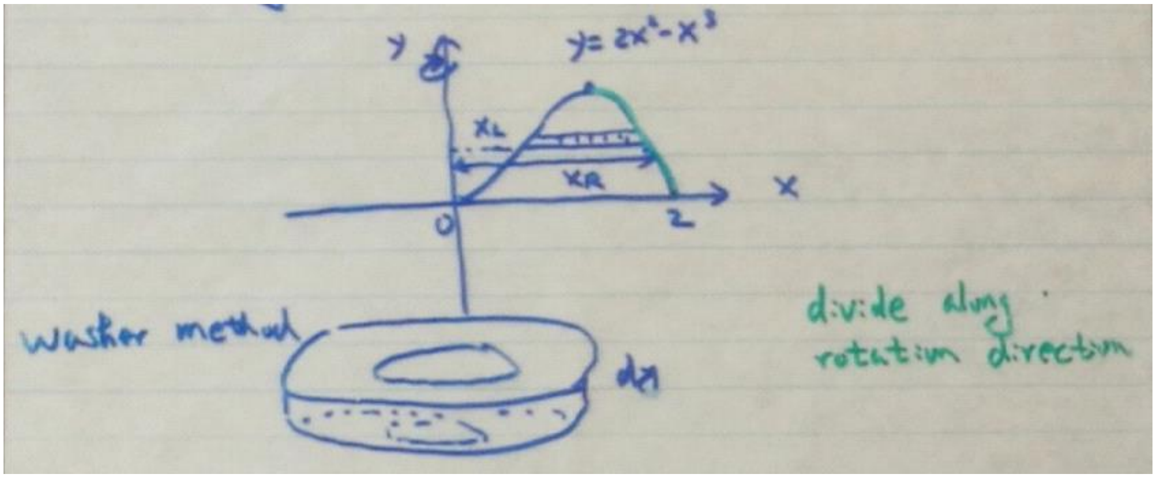
Sec6.3 Volume by cylindrical shell

eg1. Volume of solid by rotating about y-axis the region enclosed by
$y = 2x^2 - x^3$ and $y = 0$.

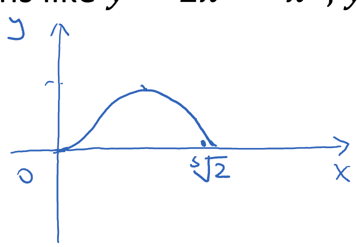


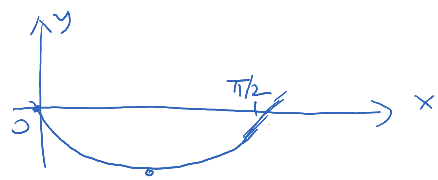
Washer method: slice along the rotating direction

$$vol = \int_0^{y_{max}} \left(\pi x_R^2 - \pi x_L^2\right) dy$$

Q. $y_{max} =?\ x_R =?(y), x_L =?(y)$?

Need to solve cubic equation $y = 2x^2 - x^3$ for x. Hard, or impossible in other situations like $y = 2x^2 - x^7, y = x - \frac{\pi}{2}\sin x$.

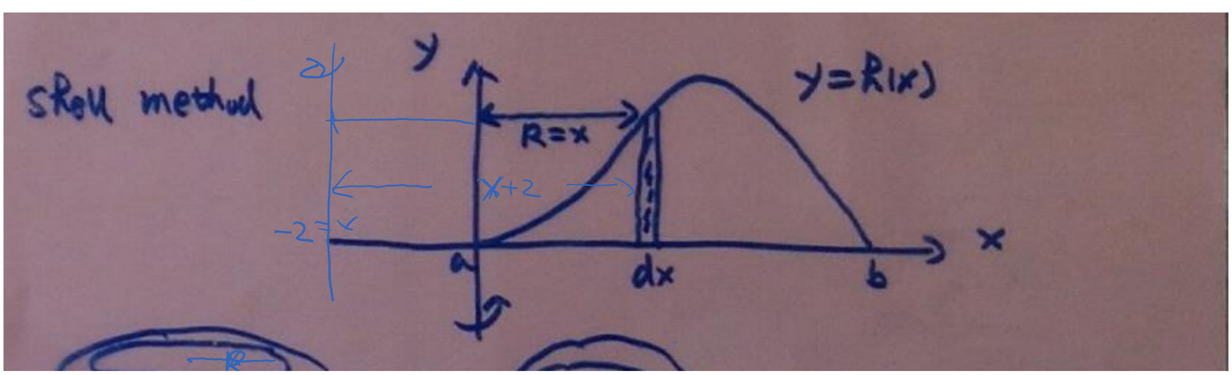




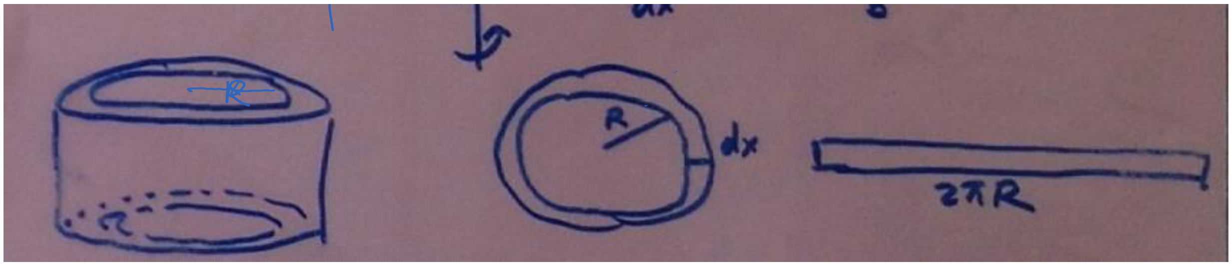
Abandon.

Shell method: slice away from rotating axis.

"tree trunk"



Parallel
slice || rotating
axis



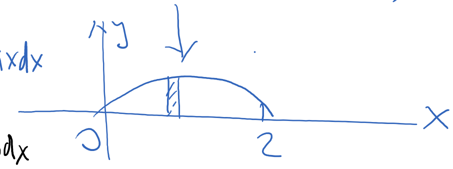
volume of "thin" shell= base · height
= base · y = $2\pi x dx \cdot y = y \cdot 2\pi x dx$

Note. base area $= \pi(x+dx)^2 - \pi x^2 = \pi 2x dx + \pi(dx)^2 \approx 2\pi x dx$.

Another way base area= base length · width= $2\pi x \cdot dx$, or $2\pi(x+dx)\cdot dx \approx 2\pi x dx$.

Q. $y = 2x^2 - x^3 = x^2(2-x)$

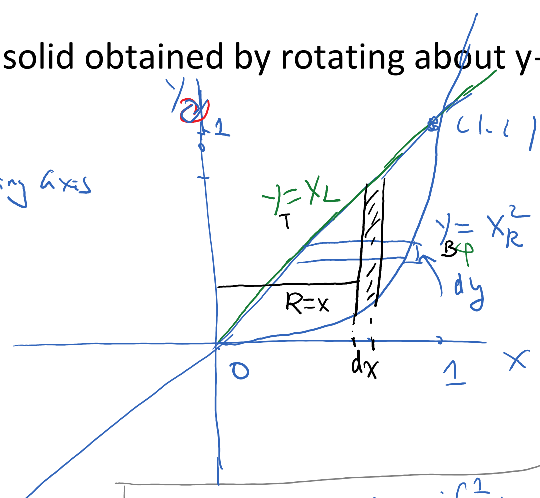
$vol = \int_a^b y(x)\, 2\pi x dx$. $= \int_0^2 (2x^2 - x^3)\, 2\pi x dx$



$= 2\pi \int_0^2 (2x^3 - x^4) dx$

$= 2\pi \left(\frac{1}{2}x^4 - \frac{1}{5}x^5\right)\Big|_0^2 = 2\pi\left(\frac{1}{2}\cdot 16 - \frac{1}{5}32\right) = 2\pi \frac{40-32}{5} = 16\pi/5$

eg2. Find the volume of the solid obtained by rotating about y-axis the region between $y = x$ and $y = x^2$. (w.&s.)



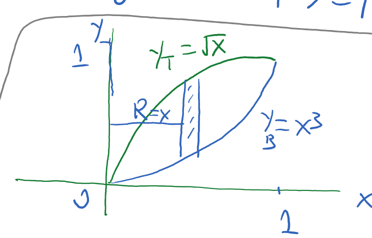
$\begin{cases} y = x \\ y = x^2 \end{cases}$

$\begin{cases} x = 0 \\ y = 0 \end{cases}$ or $\begin{cases} x = 1 \\ y = 1 \end{cases}$

Washer way slice ⊥ rotating axis

vol of short washer
$= (\pi R_{out}^2 - \pi R_{in}^2) dy$
$= (\pi x_R^2 - \pi x_L^2) dy$

$vol = \int_0^1 (\pi y - \pi y^2) dy$
$= \pi \left(\frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big|_0^1 = \pi/6$



$vol = \int_0^1 (y_T - y_B)\, 2\pi R\, dx$
$(\sqrt{x} - x^3)\, 2\pi x$

shell method

vol of shell = height · area of base/cross section $= (y_T - y_B)\cdot 2\pi R\,dx$

$= (x - x^2)\cdot 2\pi x\,dx$

$$\text{Vol} = \int_0^1 (x-x^2)\,2\pi x\,dx = 2\pi\int_0^1 (x^2-x^3)\,dx$$

$$= 2\pi\left(\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right)\Big|_0^1 = 2\pi\cdot\tfrac{1}{12} = \pi/6$$

#14 Find the volume of the solid obtained by rotating about x-axis the region enclosed by $x_L + y = 3$ and $x_R = 4 - (y-1)^2$. (s.)

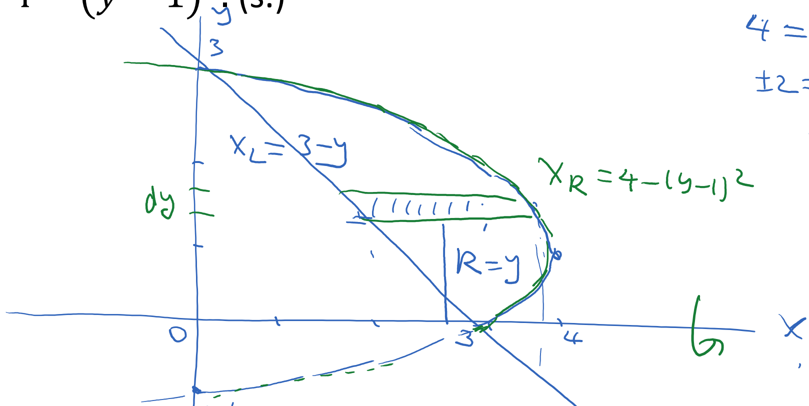
$x_L = 3 - y$

$0 = x = 4-(y-1)^2$

$4 = (y-1)^2$

$\pm 2 = y - 1$

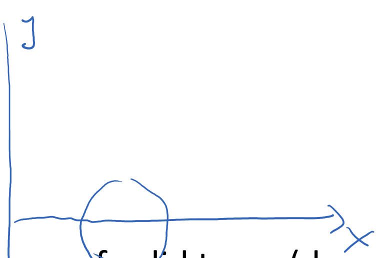
$y = 3$ or -1



vol of shell = height · area of cross section

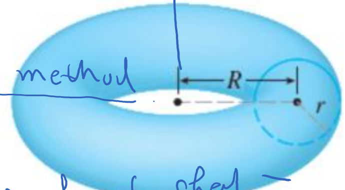
$= (x_R - x_L)\cdot 2\pi R\,dy$

$$\text{vol} = \int_0^3 \left[4-(y-1)^2-(3-y)\right] 2\pi y\,dy$$



HW3C.#9 Volume of solid torus (donut-shaped solid) with radii r and R.

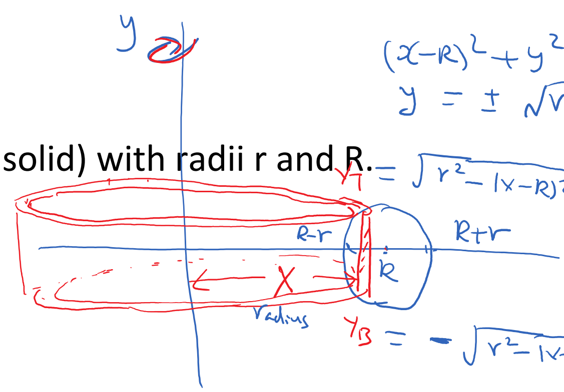
Shell method



$(x-R)^2 + y^2 = r^2$

$y = \pm\sqrt{r^2-(x-R)^2}$

$y_T = \sqrt{r^2-(x-R)^2}$



$y_B = -\sqrt{r^2-(x-R)^2}$

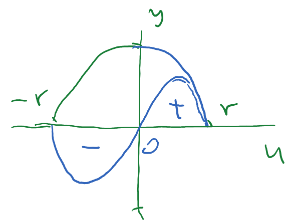
vol of shell = height · base

$= (y_T - y_B)\cdot 2\pi x\,dx$

vol of shell =

$$= (y_T - y_B) \cdot 2\pi x \, dx$$

$$= 2\sqrt{r^2-(x-R)^2}\; 2\pi x \, dx$$



$$\text{vol} = \int_{R-r}^{R+r} 4\pi\, x\sqrt{r^2-(x-R)^2}\,dx$$

$u+R = x$

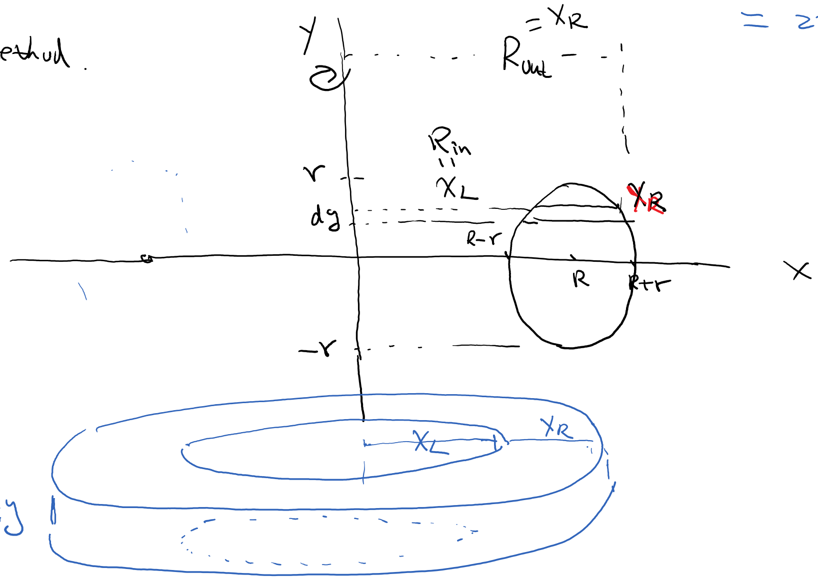
$$\underset{du=dx}{\overset{u=x-R}{=\!=\!=}} \int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,du = 4\pi\Big[\underbrace{\int_{-r}^{r} u\sqrt{r^2-u^2}\,du}_{0} + R\underbrace{\int_{-r}^{r}\sqrt{r^2-u^2}\,du}_{\text{area of } \frac{1}{2} \text{ disk}}\Big]$$

$$= 4\pi \qquad R\,\tfrac{1}{2}\pi r^2$$

$$= 2\pi^2 R r^2$$

$$= 2\pi R \cdot \pi r^2$$

washer method.



$$(x-R)^2 + y^2 = r^2$$

$$x = R \pm \sqrt{r^2-y^2}$$

$$x_R = R + \sqrt{r^2-y^2}$$

$$x_L = R - \sqrt{r^2-y^2}$$

$$(a+b)^2 - (a-b)^2$$

$$= a^2+2ab+b^2 - (a^2-2ab+b^2)$$

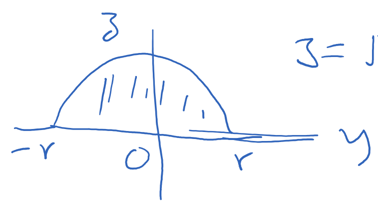
$$= 4ab$$

$$\text{vol of washer} = \text{base} \cdot \text{height} = (\pi x_R^2 - \pi x_L^2)\,dy$$

$$= \pi\left[(R+\sqrt{r^2-y^2})^2 - (R-\sqrt{r^2-y^2})^2\right]dy$$

$$= \pi \cdot 4R\sqrt{r^2-y^2}\,dy$$

$$\text{vol} = \int_{-r}^{r} 4\pi R\sqrt{r^2-y^2}\,dy = 4\pi R\underbrace{\int_{-r}^{r}\sqrt{r^2-y^2}\,dy}_{\frac{1}{2}\text{ disk area}} = 4\pi R\;\tfrac{1}{2}\pi r^2 = 2\pi^2 R r^2$$



$$z = \sqrt{r^2-y^2}$$