ON THE $L^p$ BOUNDEDNESS OF FOURIER INTEGRAL OPERATORS

Yuanlong Chen

Abstract. In this report, we will mainly discuss the $L^p$ boundedness of the Fourier integral operators.

1. INTRODUCTION.
In this report, we will show the $L^p$ estimate for Fourier integral operators with $m$ order symbol, so that $|1/2 - 1/p| \leq -m/(n-1)$. By the $L^2$ theorem and the complex interpolation theorem, in order to derive the $L^p$ result, it suffices to show that an operator of order $-(n-1)/2$ maps Hardy space $H^1$ into $L^1$. This result is based on the "second dyadic decomposition" techniques in frequency space.

2. SINGULAR INTEGRALS.
2.1. Covering lemmas and maximal function.

Lemma 2.1. (Vitali) Let $E$ be a measurable subset of $\mathbb{R}^n$ that is the union of a finite collection of balls $\{B_j\}$. Then there exists a disjoint subcollection $B_1, \ldots, B_m$ of the $\{B_j\}$ such that $\sum_{k=1}^m \mu(B_k) \geq c \mu(E)$, here $c$ is a positive constant.

Theorem 2.2. (Hardy-Littlewood-Wiener) Let $f$ be a function defined on $\mathbb{R}^n$.

(a) If $f \in L^p$, $1 \leq p \leq \infty$, then $M(f)$ is finite almost everywhere.
(b) If $f \in L^1$, then for every $\alpha > 0$, $\mu(\{x : (Mf) > \alpha\}) \leq \frac{\alpha}{\pi} \int_{\mathbb{R}^n} |f(y)|dy$.
(c) If $f \in L^p$, $1 < p \leq \infty$, then $M(f) \in L^p$ and $\|M(f)\|_p \leq A_p \|f\|_p$, where $A_p$ depends only on $c$ and $p$.

Proof. For part (c), we note that $M$ is $L^\infty$ bounded, then use part (b) and interpolation theorem, we can derive the $L^p$ boundedness for $1 < p < \infty$.

Given a ball $B$ with center $x$ and radius $\delta$, let $B^*$ and $B^{**}$ denote the balls with the same center and radius $c^* \delta$ and $c^{**} \delta$ respectively, where $c^*$ and $c^{**}$ are two positive constants with $1 < c^* < c^{**}$.

Lemma 2.3. (Generalized Whitney covering lemma) Given a closed nonempty set $F$, there exists a collection of balls $B_1, \ldots, B_k, \ldots$ such that

(a) The $B_k$ are pairwise disjoint.
(b) $\bigcup_k B_k^* = O = F^c$.
(c) $B_k^{**} \cap F \neq \emptyset$, for each $k$.

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2.2. Generalization of the Calderón-Zygmund decomposition.

**Theorem 2.4.** Suppose $f \in L^1$ and a positive number $\alpha$, with $\alpha > \frac{1}{p-1}$, then there exists a decomposition of $f, g + b$, with $b = \sum b_k$ and a sequence of balls $\{B_k\}$ such that

(i) $|g(x)| \leq c\alpha$, for a.e. $x$.

(ii) Each $b_k$ is supported in $B_k$, $\int |b_k(x)| \, d\mu(x) \leq c\alpha \mu(B_k)$, and $\int b_k(x) \, d\mu(x) = 0$.

(iii) $\sum_k \mu(B_k) \leq \frac{\alpha}{c} \int |f(x)| \, d\mu(x)$.

**Proof.** Apply generalized Whitney covering lemma to the set $E_\alpha = \{x : \hat{M}f(x) > \alpha\}$, where $\hat{M}$ is the uncentered maximal function. Then construct $g$ and $b_k$ respectively.

2.3. Singular integrals.

We want to show here, for an integral operator of particular type, the boundedness in $L^q$ guarantees the boundedness in $L^p$ for $1 < p < q$. More precisely, the integral operator formally can be expressed in the following form

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, d\mu(y)$$

(1)

where the kernel $K$ is singular near $x = y$ (we can treat the kernel $K$ as distribution in some sense).

We impose two assumptions to the operators. The first one is

$$\|T(f)\|_q \leq A \|f\|_q, \text{ for all } f \in L^q$$

(2)

Moreover, we assume there is a measurable function $K$ associated to $T$, such that for the same constant $A$ and some constant $c > 1$, we have

$$\int_{B_{r(y, \delta)}} |K(x, y) - K(x, \bar{y})| \, d\mu(x) \leq A, \text{ whenever } \bar{y} \in B(y, \delta),$$

(3)

for all $y \in \mathbb{R}^n, \delta > 0$.

**Theorem 2.5.** (Calderón-Zygmund) Under the assumptions (2) and (3), the operator $T$ is bounded in $L^p$ norm on $L^p \cap L^q$, when $1 < p < q$. More precisely,

$$\|T(f)\|_p \leq A_p \|f\|_p$$

(4)

for $f \in L^p \cap L^q$ with $1 < p < q$, where the bound $A_p$ depends only on the constant $A$ appearing in (2) and (3), but not on the kernel $K$, or on $f$.

**Proof.** The key step is to prove that such operator is of weak-type $(1,1)$, then use interpolation theorem to derive the $L^p$ boundedness.

we want to show

$$\mu\{x : |T(f)| > \alpha\} \leq \frac{A'}{\alpha} \int |f| \, d\mu$$

(5)

for $f \in L^1 \cap L^q$ and $\alpha > 0$.

By Theorem (2.4), we can write $f$ as $g + b$, where $g$ is the "good" part of $f$ which is bounded and $b = \sum b_k$ with $b_k$ is supported in $B_k$ and satisfies the cancelation property (behaviors like the atom of $H^1$). Hence, it suffices to prove

$$\mu\{x : |Tg(x)| > (c'/2)\alpha\} + \mu\{x : |Tb(x)| > (c'/2)\alpha\} \leq \frac{A'}{\alpha} \int |f(x)| \, d\mu(x)$$

(6)
For the "good" part, using the fact $g$ is bounded, we can conclude
$$
\|g\|_q^2 \leq c\alpha^{-1} \|f\|_1, 
$$
(7)

Then Chebycheff’s inequality tells
$$
\mu\{x : |Tg(x)| > (c'/2)\alpha\} \leq [(c'/2)\alpha]^{-q} \|Tg\|_q^q \leq A' \|g\|_q^q \leq \frac{A'}{\alpha} \|f\|_1
$$
(8)
for some appropriate constant $c'$.

For the second part, note each $b_k$ is supported in $B_k^*$ and is in $L^q$. If we set $B_k^{**}$ is the ball with the same center as $B_k^*$ but whose radius is expanded by the factor $c$, then the assumption (3) tells
$$
\int_{(B_k^{**})^c} |K(x,y) - K(x,\hat{y}^k)|d\mu(x) \leq A, \text{ if } y \in B_k
$$
(9)
where $\hat{y}^k$ denotes the common center of $B_k^*$ and $B_k^{**}$.

On the other hand, we know when $x \notin B_k^*$ the expression (1) holds for $T(b_k)(x)$, so we have
$$
\int_{(\cup B_k^{**})^c} |Tb(x)|d\mu(x) \leq \sum_k \int_{(B_k^{**})^c} |Tb_k(x)|d\mu(x),
$$
(10)

together with the cancelation property of $b_k$, we can show the last term is dominated by
$$
\sum_k \int_{(B_k^{**})^c} \left( |K(x,y) - K(x,\hat{y}^k)| \int_{B_k^*} |b_k(y)|d\mu(y) \right) d\mu(x)
$$
(11)
Notice that (9) is valid and $\int_{B_k^*} |b_k(y)|d\mu(y) \leq c\alpha \mu(B_k^*)$ and $\sum_k \mu(B_k^*) \leq (c/\alpha) \int |f|d\mu$, we then have
$$
\int_{(\cup B_k^{**})^c} |Tb(x)|d\mu(x) \leq A' \int |f(x)|d\mu(x)
$$
(12)
which shows that
$$
\mu\{x : |Tb(x)| > c'\alpha/2\} \cap (\cup B_k^{**})^c \leq \frac{A'}{\alpha} \int |f|d\mu
$$
(13)
However,
$$
\mu(\cup B_k^{**}) \leq \sum_k \mu(B_k^{**}) \leq c \sum_k \mu(B_k^*) \leq \frac{c}{\alpha} \int |f|d\mu
$$
(14)
These last two estimates show that
$$
\mu\{x : |Tb(x)| > (c'/2)\alpha\} \leq \frac{A'}{\alpha} \int |f|d\mu.
$$

Two remarks here:

**Remark 1.** Since $L^p \cap L^q$ is dense linear subspace of $L^p$ (when $p < \infty$), we can then extend $T$ to $L^p$.

**Remark 2.** If $T$ is of form $\hat{T}f(\xi) = m(\xi)\hat{f}(\xi)$ (the "multiplier") and $m$ satisfies the condition $\left| \left( \frac{2}{\pi} \right)^{\mu} m(\xi) \right| \leq A'_\alpha |\xi|^{-\alpha}$ for all $\alpha$, the corresponding kernel $T$ satisfies condition (3). Roughly speaking, this result means the kernel associated to the pseudo-differential operator with zero order satisfies condition (3).
3. HARDY SPACE AND BMO.

3.1. Some equivalent definitions of $H^p$ spaces.

**Definition 3.1.** For any $\Phi \in \mathcal{S}$ and any tempered distribution $f$, define $M_{\Phi} f(x) = \sup_{r>0} |(f * \Phi_r)(x)|$.

**Definition 3.2.** Let $\mathcal{F} = \| \cdot \|_{\alpha, \beta_i}$ be any finite collection of seminorms on $\mathcal{S}$, set $\mathcal{S}_\mathcal{F} = \Phi \in \mathcal{S} : \| \Phi \|_{\alpha, \beta_i} \leq 1$ for all $\| \cdot \|_{\alpha, \beta_i} \in \mathcal{F}$, define $\mathcal{M}_\mathcal{F} f(x) = \sup_{\Phi \in \mathcal{S}_\mathcal{F}} M_{\Phi} f(x)$.

**Definition 3.3.** Let $f$ be a tempered distribution, set $u(x, t) = (f * P_t)(x)$ be the Poisson integral of $f$, define $u^*(x) = \sup_{|x-y| \leq t} |u(y, t)|$.

**Theorem 3.4.** *(Fefferman-Stein)* Let $f$ be a distribution and let $0 < p \leq \infty$. Then the following conditions are equivalent:

(i) There is a $\Phi \in \mathcal{S}$ with $\int \Phi dx \neq 0$ so that $M_{\Phi} f \in L^p(\mathbb{R}^n)$.

(ii) There is a collection $\mathcal{F}$ so that $\mathcal{M}_\mathcal{F} f \in L^p(\mathbb{R}^n)$.

(iii) The distribution $f$ is bounded and $u^* \in L^p(\mathbb{R}^n)$.

**Definition 3.5.** If any of the properties above are satisfied, we say that $f$ belongs to $H^p(\mathbb{R}^n)$.

One remark here:

**Remark 3.** If $1 < p \leq \infty$, we have $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. For the case $p = 1$, we have $H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$.


**Definition 3.6.** An $H^p$ atom, $p \leq 1$, is a function such that

(i) $a$ is supported in a ball $B$,

(ii) $|a| \leq |B|^{-1/p}$ almost everywhere, and

(iii) $\int x^\beta a(x) dx = 0$ for all $\beta$ with $|\beta| \leq n(p^{-1} - 1)$.

**Theorem 3.7.** *(Atomic decomposition for $H^p$, Coifman-Latter)* If $\{a_k\}$ is a collection of $H^p(p < 1)$ atoms and $\{\lambda_k\}$ is a sequence of complex number with $\sum |\lambda_k|^p < \infty$, then the series $f = \sum_k \lambda_k a_k$ converges in the sense of distribution and $f \in H^p$ with $\|f\|_{H^p} \leq c(\sum |\lambda_k|^p)^{1/p}$. Conversely, every $f \in H^p$ can be written as a sum of $H^p$ atoms, $f = \sum_k \lambda_k a_k$ converges in $H^p$ norms, moreover we have $\sum_k |\lambda_k|^p \leq c\|f\|_{H^p}^p$.

3.3. The duality of $H^1$ and BMO.

**Definition 3.8.** A locally integrable function $f$ will be said to belong to BMO if the inequality $\frac{1}{|B|} \int_B |f(x) - f_B| dx \leq A$ holds for all balls $B$, where $f_B = |B|^{-1} \int_B f dx$ denotes the mean value of $f$ over the ball $B$.

**Theorem 3.9.** *(The duality of $H^1$ and BMO, Fefferman)*

(a) Suppose $f \in \text{BMO}$. Then the linear functional $l$ given by $l(g) = \int_{\mathbb{R}^n} f(x) g(x) dx$, where $g \in H^1$, initially defined on the dense subspace of $H^1$, has a unique bounded extension to $H^1$ and satisfies $\|l\| \leq c\|f\|_{\text{BMO}}$.

(b) Conversely, every continuous linear functional $l$ on $H^1$ can be realized as above, with $f \in \text{BMO}$, and with $\|f\|_{\text{BMO}} \leq c\|l\|$. 
4. PSEUDO-DIFFERENTIAL OPERATORS.

4.1. An $L^2$ theorem.

**Definition 4.1.** A pseudo-differential operator with $m$ order symbol is a mapping $f \mapsto T_a(f)$ given by $(T_a f)(x) = \int_{\mathbb{R}^n} a(x, \xi) \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$, where $\hat{f}$ is the Fourier transform of $f$ and $a$ is an $m$ order symbol (denoted as $a \in S^m$) means $a$ is a $C^\infty$ function of $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and satisfies the differential inequality $|\partial^\alpha_x \partial^\beta_\xi a(x, \xi)| \leq A_{\alpha, \beta} (1 + |\xi|)^{m-|\alpha|}$ for all multi-indices $\alpha, \beta$.

**Theorem 4.2.** Suppose $a \in S^0$, then the operator $T_a$ initially defined on $S$, extend to a bounded operator from $L^2$ to itself.

**Proof.** First we prove this result under the restriction that $a(x, \xi)$ has compact support in $x$, which makes it possible to take Fourier transform in $x$ variable. We write

$$a(x, \xi) = \int \hat{a}(\lambda, \xi) e^{2\pi i \lambda \cdot x} d\lambda \quad (15)$$

where $\hat{a}(\lambda, \xi)$ is the Fourier transform of $a(x, \xi)$ in $x$ variable. For each multi-index $\alpha$, using integration by parts, we know

$$(2\pi i \lambda)^\alpha \hat{a}(\lambda, \xi) = \int [\partial^\alpha_x a(x, \xi)] e^{-2\pi i x \cdot \lambda} dx \quad (16)$$

then we get $|\lambda|^\alpha |\hat{a}(\lambda, \xi)| \leq c_\alpha$, uniformly in $\xi$, as a result, we have $|\sup_\xi \hat{a}(\lambda, \xi)| \leq A_N (1 + |\lambda|)^{-N}$ for arbitrary $N > 0$. Now we can write

$$(T_a f)(x) = \int (T^\lambda f)(x) d\lambda \quad (17)$$

where $(T^\lambda f)(x) = e^{2\pi i x \cdot \lambda} (T_\hat{a}(\lambda, \xi) f)(x)$. Since for each $\lambda$, $T_\hat{a}(\lambda, \xi)$ is a multiplier operator (since $\hat{a}(\lambda, \xi)$ is independent of $x$), by Plancherel’s theorem, we have that

$$\|T_\hat{a}(\lambda, \xi) f\|_{L^2} \leq \sup_\xi |\hat{a}(\lambda, \xi)| \cdot \|f\|_{L^2} = \sum_\xi |\hat{a}(\lambda, \xi)| \cdot \|f\|_{L^2} \quad (18)$$

Therefore, we have $\|T_\lambda\| \leq A_N (1 + |\lambda|)^{-N}$, since $T_a = \int T^\lambda d\lambda$ this shows

$$\|T_a\| \leq A_N \int (1 + |\lambda|)^{-N} d\lambda < \infty \quad (19)$$

For the general case, we shall use the singular integral realization of $T_a$ to prove this theorem. We write

$$(T_a f)(x) = \int_{\mathbb{R}^n} k(x, z) f(x - z) dz \quad (20)$$

where $k(x, \cdot)$ is understood as the distribution which is the inverse Fourier transform of $a(x, \cdot)$. Now give an estimate for the kernel $k$,

$$|k(x, z)| \leq A_N |z|^{-N}, \text{ for all } |z| \geq 1 \text{ and all } N > 0 \quad (21)$$

uniform in $x$. To see this, we note $k(x, \cdot)$ is the distribution which is the inverse Fourier transform of function $a(x, \cdot)$, thus $(-2\pi iz)^\alpha k(x, \cdot)$ equals the inverse Fourier transform of $\partial_x^\alpha a(x, \xi)$, since however $a \in S^0$, we know $\partial_x^\alpha a(x, \xi) \in L^1$ whenever $\alpha \geq n + 1$. This shows $k(x, \cdot)$ agrees with a function $k(x, z)$ away from the origin, and $|z|^N |k(x, z)| \leq A_N$ whenever $N > n$. Thus (21) is valid.
Then we show the pseudo-locality future of the operator $T_a$, that is, for any $x_0 \in \mathbb{R}^n$,
\[
\int_{|x-x_0| \leq 1} |(T_a f)(x)|^2 dx \leq A_N \int_{\mathbb{R}^n} \frac{|f(x)|^2 dx}{(1 + |x-x_0|)^N}, \quad \text{all } N \geq 0
\]  
(22)

Note once this estimate is established, we integrate both sides with respect to $x_0$, choosing $N > n$, then interchanging the orders of the integration, the theorem is then proved.

We prove (22) by taking $x_0 = 0$, for arbitrary $x_0$, the proof is exactly the same. We split $f$ by setting $f = f_1 + f_2$, with $f_1$ supported in $B(3)$ and $f_2$ supported outside $B(2)$. $f_1, f_2$ both smooth with $|f_1|, |f_2| \leq |f|$. Now fix $\eta \in C_0^\infty$ so that $\eta \equiv 1$ in $B(1)$. Then $\eta T_a f_1 = T_a \eta f_1$, and $\eta(x) u(x, \xi)$ has compact support in $x$, apply the previous result, we get
\[
\int_{B(1)} |T_a f_1|^2 dx \leq \int_{\mathbb{R}^n} |T_a f_1|^2 \leq A \int_{\mathbb{R}^n} |f_1|^2 \leq A \int_{B(3)} |f|^2
\]  
(23)

However, if $x \in B(1)$, since $f_2$ is supported away from $B(2)$, we can then write $T_a f_2$ as
\[
(T_a f_2)(x) = \int_{B^c(2)} k(x, x-z) f_2(z) dz
\]  
(24)

since $|x-z| \geq 1$ here, if we invoke (21), we obtain that
\[
|(T_a f_2)(x)| \leq A \int_{B^c(2)} |f(z)| |z|^{-N} dz \leq A \int |f(z)|(1 + |z|)^{-N} dz
\]  
(25)

Then by Schwarz’s inequality, we get (provided that $N > n$)
\[
\int_{B(1)} |(T_a f_2)(x)|^2 dx \leq A \int |f(x)|^2 (1 + |x|)^{-N} dx
\]  
(26)

Since $T_a$ is linear, we know $T_a(f) = T_a(f_1) + T_a(f_2)$, the proceeding argument show that (22) is true when $x_0 = 0$. 

\section{4.2. First dyadic decomposition and estimate of the kernel.}

Fix a $C_0^\infty$ function in $\xi$ space, $\eta$, so that $\eta(\xi) \equiv 1$ for $|\xi| \leq 1$, and $\eta(\xi) \equiv 0$ for $|\xi| \geq 2$. Define $\delta$ by setting $\delta(\xi) = \eta(\xi) - \eta(2\xi)$, then we get the following two "partition of unity" of the $\xi$-space:
\[
1 = \eta(\xi) + \sum_{j=1}^{\infty} \delta(2^{-j}\xi), \quad \text{all } \xi,
\]  
(27)

and
\[
1 = \sum_{j=-\infty}^{\infty} \delta(2^{-j}\xi). \quad \text{all } \xi \neq 0
\]  
(28)

Now we consider in $x$-space. Define $\Phi$ in $x$-space so that $\hat{\Phi}(\xi) = \eta(\xi)$, also define $\Psi$ by $\hat{\Psi}(\xi) = \delta(\xi) = \eta(\xi) - \eta(2\xi)$. Writing $\Phi_0(x) = t^{-n} \Phi(tx)$, then we have $\Phi_{2^{-j}} = \Phi_{2^{-j+1}} - \Phi_{2^{-j+1}}$. Define the operator $S_j$ by setting $S_j(f) = f * \Phi_{2^{-j}}$ and $\Delta_j$ by $\Delta_j(f) = S_j(f) - S_{j-1}(f) = f * \Psi_{2^{-j}}$, in parallel with (27) and (28), we have
\[
I = S_0 + \sum_{j=1}^{\infty} \Delta_j
\]  
(29)
and
\[ I = \sum_{j=-\infty}^{\infty} \Delta_j \]  

Back to the operator \( T_a \); using (29), we can write it as
\[ T_a = T S_0 + \sum_{j=1}^{\infty} T \Delta_j = \sum_{j=0}^{\infty} T a_j, \]  

where \( a_0(x, \xi) = a(x, \xi) \eta(\xi), \) and \( a_j(x, \xi) = a(x, \xi) \delta(2^{-j} \xi) \) for \( j \geq 1 \). Then each \( T a_j \) can be written in singular integral form as:
\[ (T a_j f)(x) = \int k_j(x, z) f(x - z) dz \]

where \( k_j(x, \cdot) \) is the inverse Fourier transform of \( a_j(x, \cdot) \) given by
\[ k_j(x, z) = \int a_j(x, \xi) e^{2\pi i \xi \cdot z} d\xi. \]

We now use this trick to estimate the kernel of the pseudo-differential operators, precisely, we have the following theorem:

**Theorem 4.3.** Suppose \( a \in S^m \). Then the kernel \( k(x, z) \in \mathcal{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \), and satisfies
\[ |\partial_x^\alpha \partial_z^\beta k(x, z)| \leq A_{\alpha, \beta, N} |z|^{-n - m - |\alpha| - N}, \ z \neq 0 \]  

for all multi-indices \( \alpha \) and \( \beta \), and all \( N \geq 0 \) so that \( n + m + \alpha + N > 0 \).

We first prove a lemma:

**Lemma 4.4.** Suppose \( a \in S^m \). Then \( |\partial_x^\alpha \partial_z^\beta k_j(x, z)| \leq A_{M, \alpha, \beta} |z|^{-M - 2j(n + m + |\alpha|)} \), for all \( \alpha, \beta, \) and \( M \geq 0 \).

**Proof.** Note that
\[ (-2\pi i)^\gamma \partial_x^\alpha \partial_z^\beta [k_j(x, z)] = \int \partial_x^\alpha [(2\pi i)^\gamma \partial_z^\beta a_j(x, \xi)] e^{2\pi i \xi \cdot z} d\xi, \]

since \( |\xi| \approx 2^j \), then the volume of the support is \( \approx 2^{\gamma j} \), also, the facts that \( \delta(2^{-j} \xi) \in S^0 \) uniformly in \( j \) and \( a(x, \xi) \in S^m \) show that the integrand is bounded by a multiplier of \( 2^{j(n + m + |\alpha| - |\gamma|)} \). Combine all these facts, we get \( |z|^{\gamma} \partial_x^\alpha \partial_z^\beta k_j(x, z) | \leq A_{M, \alpha, \beta} \cdot 2^{j(n + m + |\alpha| - |\gamma|)} \), whenever \( |\gamma| = M \), this proves the lemma. \( \square \)

Now we prove the theorem:

**Proof.** We just need to show the estimate is true for the sum \( \sum_{j=0}^{\infty} |\partial_x^\alpha \partial_z^\beta \theta(x, z)| \).

First we consider the case when \( 0 < |z| \leq 1 \). We break the sum into two part with first one where \( 2^j \leq |z|^{-1} \), the second where \( 2^j > |z|^{-1} \). For the first part, we use the estimate derived in the lemma with \( M = 0 \), then it is majorized by a multiplier of \( \sum_{2^j \leq |z|^{-1}} 2^{j(n + m + |\alpha|)} \), which is \( O(|z|^{-n - m - |\alpha| - N}) \) under the restriction that \( |z| \leq 1, N \geq 0 \) and \( n + m + |\alpha| + M > 0 \); for the second part sum, choose \( M > n + m + |\alpha| \), by the lemma, we get the estimate \( O(|z|^{-M}) \sum_{2^j > |z|^{-1}} 2^{j(n + m + |\alpha| - m)} = O(|z|^{-n - m - |\alpha|}) \), the last term is \( O(|z|^{-n - m - |\alpha| - N}) \) whenever \( N \geq 0 \), since \( |z| \leq 1 \). Then we consider the case when \( |z| \geq 1 \). We choose \( M > n + m + |\alpha| + N \), by proceeding lemma, we know the sum is majorized by \( O(|z|^{-M}) \), which is \( O(|z|^{-n - m - |\alpha| - N}) \) for every \( N \) since \( |z| \geq 1 \). \( \square \)
5. FOURIER INTEGRAL OPERATORS.

5.1. Basic definitions and examples.

Definition 5.1. A Fourier integral operator $T$ is given by

$$(Tf)(x) = \int_{\mathbb{R}^n} e^{2\pi i \Phi(x, \xi)} a(x, \xi) \hat{f}(\xi) d\xi$$

(35)

here $\hat{f}$ denotes the Fourier transform of $f$, $a \in S^m$, we also assume that it has compact support in $x$. The phase function $\Phi$ is real-valued, homogeneous of degree 1 in $\xi$ and smooth in $(x, \xi)$ for $\xi \neq 0$ on the support of $a$, furthermore it satisfies the non-degeneracy condition, that for $\xi \neq 0$,

$$\det(\frac{\partial^2 \Phi}{\partial x_i \partial \xi_j}) \neq 0$$

(36)
on the support of $a$.

Some remarks here:

Remark 4. The simplest examples arise when we take $\Phi(x, \xi) = x \cdot \xi$, then Fourier integral operator becomes pseudo-differential operator.

Remark 5. Consider the phase function $\Phi(x, \xi) = x \cdot \xi \pm |\xi|$. This is related to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}^1.$$ 

(37)

with initial conditions $u(x, 0) = f_0(x), \frac{\partial u}{\partial t}(x, 0) = f_1(x)$. We have the classical representation for the solution given by

$$u(x, t) = \int_{\mathbb{R}^n} \cos(2\pi |\xi| t) e^{2\pi i x \cdot \xi} \hat{f}_0(\xi) d\xi + \frac{1}{2\pi} \int_{\mathbb{R}^n} \sin(2\pi |\xi| t) \xi e^{2\pi i x \cdot \xi} \hat{f}_1(\xi) d\xi.$$ 

(38)

thus for each fixed $t$, we can write the solution as $u(x, t) = T_0(f_0) + T_1(f_1)$, where $T_0, T_1$ are sums of Fourier integral operators with phases $x \cdot \xi \pm t|\xi|$ and order 0 and -1 symbols, respectively.

5.2. $L^2$ estimate.

We first prove a lemma:

Lemma 5.2. Suppose $\psi$ is smooth function with compact support, and that $\phi$ is a smooth real-valued function that has no critical point in the support of $\psi$. Then

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda \phi(x)} \psi(x) dx = O(\lambda^{-N}),$$

(39)
as $\lambda \to \infty$, for every $N \geq 0$.

Proof. We first prove the result when dimension is 1. Let $D$ denote the differential operator given by $Df(x) = (i\lambda \phi'(x))^{-1} \cdot \frac{df}{dx}$, and let $D^t$ denote its transpose given by $D^t f(x) = \frac{df}{dx}(\frac{1}{i\lambda \phi'(x)})$. Then $D^N(e^{i\lambda \phi}) = e^{i\lambda \phi}$ for every $N$, and integration by parts shows that $\int e^{i\lambda \phi} \psi dx = \int e^{i\lambda \phi} \cdot (D^t)^N(\psi) dx$, which shows $|I(\lambda)| \leq A_N \lambda^{-N}$.

For general case, we note that for each $x_0$ in the support of $\psi$, we can choose a unit vector $\xi$ and a small ball $B(x_0)$, such that $\xi \cdot (\nabla \phi)(x) \geq c > 0$ for all $x$ lies in the ball. Using partition of unity subordinated to these balls, we can decompose the integral
as \( I(\lambda) = \sum_k \int e^{i\lambda \Phi(x)} \psi_k(x) dx \), where each \( \psi_k \) is smooth and compactly supported in one of these balls. It then suffices to prove the corresponding estimate for each of the integral. We choose a coordinate system \( x_1, \ldots, x_n \) so that \( x_1 \) lies along \( \xi \). Then \( \int e^{i\lambda \Phi(x)} \psi(x) dx = \int \left( \int e^{i\lambda \Phi(x_1, \ldots, x_n)} \psi(x_1, \ldots, x_n) dx_1 \right) dx_2 \cdots dx_n \), apply the result derived for 1 dimension to the inner integral, we obtain the estimate. \( \square \)

Now we state the \( L^2 \) estimate result:

**Theorem 5.3.** (Hörmander- Maslov) Let \( T \) be the Fourier integral operator defined in (35). Assume that \( a \in S^0 \) with compact support in \( x \). Then, \( T \), initially defined on \( S \), extends to a bounded operator form \( L^2(\mathbb{R}^n) \) to itself.

**Proof.** First note, we can write \( T = SF \), where \( F(f) = \hat{f} \) is the Fourier transform of \( f \), and \( S(f) = \int e^{2\pi i \Phi(x, \xi)} a(x, \xi) f(\xi) d\xi \). By Plancherel's theorem, it suffices to prove the statement for \( S \). We first pay attentions to the situation that the symbol \( a(x, \xi) \) is contained in an sufficiently "narrow cone" in the \( \xi \)-space, more precisely, this means that there exists a small constant \( \mathcal{C} \), so that whenever \( \xi \) and \( \eta \) belong to the cone and \( |\eta| \leq |\xi| \), then, if we write \( \eta = \rho \xi + \xi' \) with \( \eta' \) orthogonal to \( \xi \), we have \( |\eta'| \leq \mathcal{C} |\xi| \). Using proper partition of unity, we can write \( S \) as the finite sum of operators corresponding to such "narrow cone".

We now claim that, whenever \( \xi \) and \( \eta \) belong to the same narrow cone, the following estimate

\[
|\nabla_x [\Phi(x, \xi) - \Phi(x, \eta)]| \geq c|\xi - \eta|; \tag{40}
\]

In the sequel, we will, for brevity, write \( \Phi_x \) for \( \nabla_x [\Phi(x, \xi)] \), and \( \Phi_x, \xi \) for \( [\partial^2 \Phi / \partial x_i \partial \xi_j] \). By homogeneity and non-degeneracy condition of \( \Phi \), we have

\[
|\Phi_x, \xi(x, \xi)| \geq c|u| \text{ and } |\Phi_x(x, \xi)| \geq c|\xi|. \tag{41}
\]

By homogeneity of \( \Phi \), to prove (40) it will only suffice to establish it when \( |\xi| = 1, |\eta| \leq 1 \). Assume first that \( |\xi - \eta| \leq c_1 \), where \( c_1 \) is another small constant. Then we have \( \Phi_x(x, \xi) - \Phi_x(x, \eta) = \Phi_x, \xi(x, \xi) \cdot (\xi - \eta) + O(|\xi - \eta|^2) \), so we derive \( |\Phi_x(x, \xi) - \Phi_x(x, \eta)| \geq c|\xi - \eta| \). Next, if \( |\xi - \eta| \geq c_1 \), then we write \( \eta = \rho \xi + \eta' \), and note that \( \Phi_x(x, \xi) - \Phi_x(x, \eta) = \Phi_x(x, \xi) - \Phi_x(x, \rho \xi) + \Phi_x(x, \rho \xi) - \Phi_x(x, \rho \xi + \eta') \). The first quantity in brackets equals \( (1 - \rho)\Phi_x(x, \xi) \), whose length is \( (1 - \rho)|\Phi_x(x, \xi)| \geq c(1 - \rho) \geq c|\xi - \eta| \), while the second is majorized by \( O(|\eta'|) \leq O(\mathcal{C}) = O(\text{small constant} \cdot |\xi - \eta|) \), if we set \( \mathcal{C} \) small. Hence (40) is true.

We now turn to prove the theorem. Roughly speaking, we shall show here that \( S^*S \) behaviors like a pseudo-differential operator with 0 order symbol. We first assume that \( a(x, \xi) \) has compact support in \( \xi \), but our estimate here will be independent of the size of the support, which will allow us to pass the result to general case. Consider operator the \( S^*S \), it is given by

\[
(S^*Sf)(\xi) = \int_{\mathbb{R}^n} K(\xi, \eta) f(\eta) d\eta \tag{42}
\]

where the kernel \( K \) is given by

\[
K(\xi, \eta) = \int_{\mathbb{R}^n} e^{2\pi i [\Phi(x, \eta) - \Phi(x, \xi)]} a(x, \xi) a(x, \eta) dx. \tag{43}
\]

By (40), \( |\nabla_x [\Phi_x(x, \eta) - \Phi_x(x, \xi)]| \leq c|\xi - \eta| \), thus by Lemma (5.2) above, we have that for every \( N \geq 0 \), \( |K(\xi, \eta)| \leq A_N (1 + |\xi - \eta|)^{-N} \). Hence, this is an integrable kernel, by Schur's lemma, we know \( S^*S \) is bounded from \( L^2 \) to itself.

For general case, \( a(x, \xi) \) is not compactly supported in \( \xi \). We set \( a_\varepsilon(x, \xi) = a(x, \xi) \cdot \gamma(\varepsilon \xi) \), where \( \gamma \in C_0^\infty \) with \( \gamma(0) = 1 \); then we know that the operator \( S_\varepsilon \)
associated to \( a_e \) have a uniform bound for their norms. Since \( S_\nu(f) \to S(f) \) in the topology of \( S \), whenever \( f \in S \). Hence the theorem is proved.

We now apply the proceeding theorem to show \( L^p \to L^2 \) result for Fourier integral operators. In order to do this, we need the following "Hardy-Littlewood-Sobolev" inequality:

**Theorem 5.4.** (**Hardy-Littlewood-Sobolev**) The operator \( f \mapsto f \ast (|x|^{-\gamma}) \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \), whenever \( 1 < p < q < \infty \) and \( 1/q = 1/p - 1 + \gamma/n \).

We now state the theorem for Fourier integral operator:

**Theorem 5.5.** Let \( T \) be a Fourier integral operator whose symbol is of order \( m \), with \( -n/2 < m < 0 \). Then \( T \), initially defined on \( S \), extends as a bounded operator from \( L^p(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \), if \( 1/p = 1/2 - m/n \).

**Proof.** We can write \( T = T_0 T_1 \), where \( T_0 \) is a Fourier integral operator with 0 order symbol \( a_0(x, \xi) = a(x, \xi) \cdot (1 + |\xi|^2)^{-m/2} \), and \( T_1 \) is the pseudo-differential operator with \( m \) order symbol \( 1 + |\xi|^2)^{m/2} \). By the theorem we just prove, \( T_0 \) is a bounded map from \( L^2 \) to itself, so we only need to show that \( T_1 : L^p \to L^2 \) is bounded. Using the singular integral realization of pseudo-differential operator, we can write \( T_1 f(x) = \int k(x-y) f(y) dy \), Theorem (4.3) then gives the estimate for kernel, that is, \(|k(z)| \leq A |z|^{-n-m} \). Apply the Hardy-Littlewood-Sobolev inequality, with \( \gamma = n + m, q = 2 \), we then derive the result.

**5.3. \( L^p \) estimates.**

For a Fourier integral operator of order 0, in general, is not bounded from \( L^p \) to itself, if \( p \neq 2 \). For \( L^p \) boundedness property, some restrictions to the symbol should be imposed, as stated in the following theorem:

**Theorem 5.6.** (**Seeger-Sogge-Stein**) Let \( T \) be a Fourier integral operator whose symbol \( a \) is of order \( m \), with \((1-n)/2 < m \leq 0 \). Then \( T \), initially defined on \( S \), extends to a bounded operator from \( L^p(\mathbb{R}^n) \) to itself, whenever

\[
\left| \frac{1}{2} - \frac{1}{p} \right| \leq \frac{-m}{n-1}.
\]  

(44)

The key part of the proof is to show that such operator is bounded from \( H^1(\mathbb{R}^n) \) to \( L^1(\mathbb{R}^n) \). Before starting this, we first show an analogous result for the pseudo-differential operator to see how it works there:

**Theorem 5.7.** (**Fefferman-Stein**) Suppose \( T \) is a pseudo-differential operator with 0 order symbol \( m(\xi) \) (the "multiplier"). Then it is bounded from \( H^1(\mathbb{R}^n) \) to \( L^1(\mathbb{R}^n) \), that is,

\[
||T(f)||_{L^1(\mathbb{R}^n)} \leq A ||f||_{H^1(\mathbb{R}^n)}.
\]  

(45)

**Proof.** First observe that we can write \( \hat{Tf}(\xi) = m(\xi) \hat{f}(\xi) \), for \( f \in L^2 \), \( m \in S^0 \) implies that it is \( L^\infty \), by Plancherel's theorem, we know \( ||Tf||_{L^2} \leq A ||f||_{L^2} \). Now let \( K \) denote the tempered distribution such that \( \hat{K} = m \), then we can write \( T \) as \( (Tf)(x) = \int K(x-y)f(y)dy \), for \( f \in L^2 \) with compact support and \( x \) outside the support of \( f \). By Remark(2), we know the kernel satisfies the following estimate:

\[
\int_{|x| > 2|y|} |K(x-y) - K(x)| dx \leq A, \text{ whenever } y \neq 0.
\]  

(46)
In view of the atomic decomposition for $H^1$ space, it will suffice the inequality in the theorem for an arbitrary $H^1$ atom $a$, since $T$ is translation-invariant, we may also assume that $a$ is support in $B$, which is a ball with radius $r$ and centered at origin. If $B^*$ is the ball concentric with $B$ having twice its radius, by $L^2$ boundedness of $T$, we have

$$\int_{B^*} |Ta|^2 dx \leq A^2 \int |a|^2 dx \leq A^2 |B|^{-2} = A^2 |B|^{-1},$$

(47)

So by Schwarz’s inequality

$$\int_{B^*} |Ta| dx \leq A |B^*| \cdot |B|^{-1} = 2^n A.$$  

(48)

For $x \notin B^*$, we use the singular integral representation for $T$, that is

$$Ta(x) = \int K(x - y)a(y)dy = \int [K(x - y) - K(x)]a(y)dy,$$  

(49)

by the cancelation property for atom $a$. Now if we invoke the estimate (46), together with Fubini theorem, we get

$$\int_{(B^*)^c} |Ta| dx \leq \int |a(y)| dy \leq A.$$  

(50)

This completes the proof.

In the proceeding argument, in order to prove

$$\int_{\mathbb{R}^n} |Ta(x)| dx \leq A,$$  

(51)

we divide the integration to two parts: one is $B^*$, for which we use the $L^2$ boundedness of $T$ to derive the estimate; the other part is the complement of $B^*$, where we use the crucial estimate for the kernel $K$, that is,

$$\int_{(B^*)^c} |K(x,y) - K(x,y')| dx \leq A,$$  

(52)

where $y, y' \in B$. Now we use the same idea to prove Theorem (5.6).

**Proof.** We first construct the corresponding $B^*$ associated to $B$, whose radius, here, is assumed to be at most $1$. First, for each positive integer $j$, we consider a equally spaced set of points with grid length $2^{-j/2}$ on the unit sphere $S^{n-1}$; that is, we fix a collection $\{\xi_j^\nu\}$ of unit vectors, $|\xi_j^\nu| = 1$, such that

(i) $|\xi_j^\nu - \xi_j^{\nu'}| \geq 2^{-j/2}$, if $\nu \neq \nu'$

(ii) If $\xi \in S^{n-1}$, then there exists a $\xi_j^\nu$, so that $|\xi - \xi_j^\nu| \leq 2^{-j/2}$

Notice that there are at most $c2^{(n-1)}$ elements in the collection $\{\xi_j^\nu\}$.

Now let $B = B(\bar{y}, \delta)$ be the ball about $\bar{y}$ with radius $\delta \leq 1$. We define the ”rectangles” $R_j^\nu = R_j^\nu(B)$ in $x$-space as follows. First, we let the rectangles $\hat{R}_j^\nu$ in $y$-space be given by

$$\hat{R}_j^\nu = \{y : |y - \bar{y}| \leq c2^{-j/2}, |\pi_j^\nu(y - \bar{y})| \leq c2^{-j}\}.$$
where $\pi^\nu_j$ is the orthogonal projection in the direction $\xi^\nu_j$ and $\bar{c}$ is a large constant (independent of $j$), to be fixed later. Next, the mapping

$$x \mapsto y = \Phi_\xi(x, \xi)$$

has, for each $\xi$, a nonvanishing Jacobian. So if $2^{-j} \leq 1$, we take $R^\nu_j$ to be the inverse under $\Phi_\xi$, with $\xi = \xi^\nu_j$, of the rectangle $\tilde{R}^\nu_j$, that is,

$$R^\nu_j = \{x : |\bar{y} - \Phi_\xi(x, \xi^\nu_j)| \leq \bar{c}2^{-j/2}, |\pi^\nu_j(\bar{y} - \Phi_\xi(x, \xi^\nu_j))| \leq \bar{c}2^{-j}\}.$$  \hspace{1cm} (53)

Now let $B^* = \bigcup_{2^{-j} \leq \delta} \bigcup \nu R^\nu_j$, then

$$|B^*| \leq \sum_{2^{-j} \leq \delta} \sum_{\nu} |R^\nu_j| \leq c \sum_{2^{-j} \leq \delta} \sum_{\nu} |\tilde{R}^\nu_j|$$

$$\leq c \sum_{2^{-j} \leq \delta} 2^{-j(n+1)/2} \cdot 2^{j(n-1)/2} = c \sum_{2^{-j} \leq \delta} 2^{-j} \leq c\delta.$$

We now turn to the estimate for $T(a)$ where $a$ is an atom supported in the ball $B$. If the radius of $B$ is greater than 1, we then have the estimate:

$$\int |T(a)|dx \leq c ||T(a)||_{L^2} \leq c'||a||_{L^2}.$$  

The first inequality comes from the fact that the symbol of $T$ is compactly supported in $x$, while the second inequality is because the $L^2$ boundedness of $T$ (since it has 0 order symbol). Now since $a$ is an atom, $|a(x)| \leq |B|^{-1}$, we then have $||a||_{L^2} \leq |B|^{-1/2} \leq c$, hence (51) holds in this case.

Therefore, it suffices to prove the theorem for that radius of $B$, $\delta \leq 1$. Then

$$\int_{B^*} |Ta|dx \leq ||Ta||_{L^2} \cdot |B^*|^{1/2} \leq c\delta^{1/2}||Ta||_{L^2}.$$  

However, by Theorem(5.5), $||Ta||_{L^2} \leq A||a||_{L^p}$, provided that

$$\frac{1}{p} = \frac{1}{2} + \frac{n - 1}{2n},$$  \hspace{1cm} (54)

Notice that $|a(x)| \leq |B|^{-1}$, $a$ is supported in $B$, we then get

$$\int_{B^*} |Ta|dx \leq A\delta^{1/2} \cdot \delta^{-1+1/p} = A.$$  

The last equality is because (54) is equivalent to $1/2 + n(-1 + 1/p) = 0$. Therefore we have that

$$\int_{B^*} |Ta|dx \leq A.$$  \hspace{1cm} (55)

To estimate the integral over $(B^*)^c$, we need to establish the estimate for kernel analogous to (52), for which we need the second dyadic decomposition, as we will describe as follows.

We recall the unit vectors $\{\xi^\nu_j\}$ used above. Let $\Gamma^\nu_j$ denote the corresponding cone in $\xi$-space whose central direction is $\xi^\nu_j$, that is,

$$\Gamma^\nu_j = \{\xi : |\xi/|\xi| - \xi^\nu_j| \leq 2 \cdot 2^{-j/2}\}.$$
We can then construct an associated partition of unity: it is given by functions $\chi_j^\nu$, each homogeneous of degree 0 in $\xi$ and supported in $\Gamma_j^\nu$, with
\[
\sum_{\nu} \chi_j^\nu(\xi) = 1 \quad \text{for all } \xi \neq 0 \text{ and all } j, \tag{56}
\]
and
\[
|\partial^\alpha_j \chi_j^\nu(\xi)| \leq A_0 2^{\lfloor |\alpha|/2 \rfloor} |\xi|^{-|\alpha|}. \tag{57}
\]

From (56), we can get the following dyadic decomposition:
\[
1 = \hat{\Phi}_0(\xi) + \sum_{j=1}^{\infty} \sum_{\nu} \chi_j^\nu(\xi) \cdot \hat{\Phi}_j(\xi) \tag{58}
\]

With this decomposition, we define operators $T_j^\nu$ by
\[
T_j^\nu f(x) = \int_{\mathbb{R}^n} e^{2\pi i \Phi(x,\xi)} a_j^\nu(x,\xi) \hat{f}(\xi) d\xi,
\]
where $a_j^\nu(x,\xi) = \chi_j^\nu(\xi) \cdot \hat{\Phi}_j(\xi) \cdot a(x,\xi)$. We then have
\[
T_j = \sum_{\nu} T_j^\nu. \tag{59}
\]

Let $K_j$ denote the kernel of the operator $T_j$. The crucial estimates we shall make here are:
\[
\int_{\mathbb{R}^n} |K_j(x,y)| dx \leq A, \quad \forall y \in \mathbb{R}^n, \tag{60}
\]
\[
\int_{\mathbb{R}^n} |K_j(x,y) - K_j(x,y')| dx \leq A |y - y'| \cdot 2^j, \quad \forall y, y' \in \mathbb{R}^n, \tag{61}
\]
\[
\int_{(B^*)^c} |K_j(x,y)| dx \leq \frac{A \cdot 2^{-j}}{\delta}, \quad \text{if } y \in B \text{ and } 2^{-j} \leq \delta, \tag{62}
\]
where the bound $A$ is independent of $j, y, y'$ and $\delta$ is the radius of $B$.

Because of (59), it suffices to make similar estimates for the kernels of the $T_j^\nu$, denoted by $K_j^\nu$, the only difference is just a factor $2^{-j(n-1)/2}$ for the right-hand side of the inequalities above, since there are essentially $2^{j(n-1)/2}$ terms in the second dyadic decomposition for each $j$.

We now give the majorization of the kernels. Observe that the kernels $K_j^\nu$ is given by
\[
K_j^\nu(x,y) = \int e^{2\pi i \Phi(x,\xi) - y \cdot \xi} a_j^\nu(x,\xi) d\xi.
\]

For brevity, we set $\tilde{\xi} = \xi^\nu$, and choose the axes in the $\xi$-space so that $\xi_1$ is in the direction of $\tilde{\xi}$ and $\xi' = (\xi_2, \ldots, \xi_n)$ is perpendicular to $\tilde{\xi}$.

Now $\Phi(x,\xi) - y \cdot \xi = [\Phi(x,\tilde{\xi}) - y \cdot \tilde{\xi}] + [\Phi(x,\xi) - \Phi(x,\tilde{\xi}) \cdot \xi]$. We set
\[
h(\xi) = \Phi(x,\xi) - \Phi(x,\tilde{\xi}) \cdot \xi.
\]

We claim the following two estimates for $h(\xi)$:
\[
\left| \left( \frac{\partial}{\partial \xi_1} \right)^N h(\xi) \right| \leq A_N \cdot 2^{-Nj}, \tag{63}
\]
\[
|\nabla^{N/2} h(\xi)| \leq A_N \cdot 2^{-Nj/2}. \tag{64}
\]
where $N \geq 1$ and $\xi$ is restricted to the support of $a_j^\nu(x, \xi)$; the latter requires that, in our coordinates,

\[
2^j \leq |\xi| \leq 2^{j+1}, \quad \text{and} \quad |\xi'| \leq c2^{j/2}.
\]

Since $\Phi(x, \xi)$ is homogeneous of degree 1 in $\xi$, Euler’s identity says,

\[
\Phi_\xi(x, \xi) \cdot \xi = \Phi(x, \xi);
\]

hence $h(\xi_1, 0, \ldots, 0) = 0$. Moreover, it is clear that

\[
(\nabla_\xi h)(\xi_1, 0, \ldots, 0) = 0.
\]

As a result

\[
\left( \frac{\partial}{\partial \xi_1} \right)^N h(\xi_1, 0, \ldots, 0) = \nabla_\xi \left( \frac{\partial}{\partial \xi_1} \right)^N h(\xi_1, 0, \ldots, 0) = 0.
\]

Thus

\[
\left( \frac{\partial}{\partial \xi_1} \right)^N h(\xi_1, \xi') = O(|\xi'|^2 \cdot |\xi|^{-N-1}),
\]

since $h$ is homogeneous of degree 1. However, when $|\xi'| \leq c2^{j/2}$ and $|\xi| \approx 2^j$, we have that

\[
|\xi'|^2 \cdot |\xi|^{-N-1} = O(2^j \cdot 2^{-j(N+1)}) = O(2^{-Nj}),
\]

which proves (63).

Again, $(\nabla_\xi h)(\xi_1, 0, \ldots, 0) = 0$, so

\[
\nabla_\xi h(\xi_1, \xi') = O(|\xi'| \cdot |\xi|^{-1}),
\]

which proves (64) when $N = 1$. For $N \geq 2$, we have that

\[
|((\nabla_\xi)^N h(\xi)| \leq A|\xi|^{1-N},
\]

by homogeneity, and $2^{1-N} \leq 2^{-jN/2}$, so (64) is proved for all $N \geq 1$.

We now give an improvement for (57) when we differentiate in the $\xi_1$-direction. In fact we have

\[
\left| \left( \frac{\partial}{\partial \xi_1} \right)^N \chi_j^\nu(\xi) \right| \leq AN|\xi|^{-N}, \quad N \geq 1.
\]

This is because in the cone of support of $\chi_j^\nu$ we can write

\[
\frac{\partial}{\partial \xi_1} = \frac{\partial}{\partial r} + O(2^{-j/2}) \cdot \nabla_\xi
\]

where $\partial/\partial r$ is the radial derivative; since $\chi$ is homogeneous of degree 0, $(\partial/\partial r)^N \xi = 0$.

We now turn to prove the estimate of the kernel. We write $K_j^\nu$ as

\[
K_j^\nu(x, y) = \int_{\mathbb{R}^n} e^{2\pi i [\Phi_\xi(x, \xi) - y \cdot \xi]} b_j^\nu(x, \xi) d\xi
\]

where

\[
b_j^\nu(x, \xi) = e^{2\pi ih(\xi)} \chi_j^\nu(\xi) \cdot \hat{\phi}_j(\xi) \cdot a(x, \xi)
\]

we next introduce the operator $L$ defined by

\[
L = 1 - 2^{2j} \frac{\partial^2}{\partial \xi_1^2} - 2^j (\nabla_\xi)^2
\]

Because of (57), (65), (63), (64), and the fact $a \in S^{-(n-1)/2}$, we get that

\[
|L^N (b_j^\nu(x, \xi))| \leq AN \cdot 2^{-j(n-1)/2}.
\]
However, \( L^N(e^{2\pi i\Phi(x, \xi)-y} \cdot \xi) \) equals
\[
\{1 + 4\pi^2 2^j |(\Phi_\xi(x, \xi) - y)_1|^2 + 4\pi^2 2^j |(\Phi_\xi(x, \xi) - y)'|^2\}^N \cdot e^{2\pi i\Phi(x, \xi)-y} \cdot \xi.
\]

We introduce this in (66), pass the differentiation onto \( b_j \), and note that the support of \( b_j \) has at most \( c2j \cdot 2^{j(n-1)/2} \). Thus we obtain that
\[
|K_j^\nu(x, y)| \leq c2^j \{1 + 2^j |(\Phi_\xi(x, \xi_j^\nu) - y)_1| + 2^{j/2} |(\Phi_\xi(x, \xi_j^\nu) - y)'|\}^{-2N}.
\]

(67)

Here we have replaced \( \bar{\xi} \) by its original name \( \xi_j^\nu \); also \((\cdot)_1\) indicates the component in the direction \( \xi_j^\nu \), and \((\cdot)'\) denotes the orthogonal component.

In carrying out the estimate for \( \int |K_j^\nu(x, y)|dx \), use the majorization above, as well as the change of variables \( x \mapsto \Phi_\xi(x, \xi_j^\nu) \), whose Jacobian is bounded from below. The result is then
\[
\int |K_j^\nu(x, y)|dx \leq A2^j \int (1 + |2^j(x - y)_1| + |2^{j/2}(x - y)'|)^{-2N}dx
\]
\[
\leq A2^{-j(n-1)/2},
\]
if we choose \( N \) so that \( 2N > n \). Therefore
\[
\int |K_j^\nu(x, y)|dx \leq A2^{-j(n-1)/2}.
\]

(68)

A similar estimate holds for \( \nabla_y K_j^\nu(x, y) \), once we observe that the differentiation in \( y \) introduces the factors bounded by \( 2^j \). As a result,
\[
\int |\nabla_y K_j^\nu(x, y)|dx \leq A2^j \cdot 2^{-j(n-1)/2},
\]
and so
\[
\int |K_j^\nu(x, y) - K_j^\nu(x, y')|dx \leq A|y - y'| \cdot 2^j \cdot 2^{-j(n-1)/2}
\]

(69)

Now we shall estimate
\[
\int_{(B^*)_c} |K_j^\nu(x, y)|dx
\]
where \( B \) is the ball of radius \( \delta \), centered at \( \bar{y} \), and \( 2^{-j} \leq \delta \). Suppose \( k \) is the integer so that \( 2^{-k} < \delta \leq 2^{-k+1} \). Then there is a unit vector \( \xi_k^\nu \), so that \( |\xi_j^\nu - \xi_k^\nu| \leq 2^{-k/2} \). Since \( B^* = \bigcup_{-2^{j-1} \leq \nu} R_j^\nu \), we have the inclusion \( (B^*)_c \subset (R_j^\nu)_c \). However, by (53) for \( x \in (R_j^\nu)_c \),
\[
2^k |\pi_k^\nu(\Phi_\xi(x, \xi_k^\nu) - \bar{y})| + 2^{k/2} |\Phi_\xi(x, \xi_k^\nu) - \bar{y}| \geq \bar{c}.
\]

If \( y \in B \), then \( |y - \bar{y}| \leq 2^{-k+1} \), and since \( \bar{c} \) is assumed sufficiently large, we get
\[
2^j |(\Phi_\xi(x, \xi_j^\nu) - y)_1| + 2^{j/2} |(\Phi_\xi(x, \xi_j^\nu) - y)'| \geq c2^{(j-k)/2}
\]
c when \( j \geq k \). Inserting this in the bound (67) and arguing as before, we obtain
\[
\int_{(B^*)_c} |K_j^\nu(x, y)|dx \leq A2^j 2^{-(j-k)} \int (1 + |2^j(x, y)_1| + |2^{j/2}(x - y)'|)^{2-2N}dx
\]
If we choose \( N \) so large that \( 2N - 2 > n \), the result is then
\[
\int_{(B^*)_c} |K_j^\nu(x, y)|dx \leq A \cdot \frac{2^{-j}}{\bar{c}} \cdot 2^{-j(n-1)/2}, \quad y \in B.
\]

(70)

Summing the inequalities above in \( \nu \), we then derive the corresponding estimates for the kernel \( K_j \).
Now we turn to give an estimate for $T(a)$, for an atom $a$. We here write

$$T(a) = \sum_{j=0}^{\infty} T_j(a) = \sum_{1} + \sum_{2},$$

where $\sum_{1}$ is taken over those $j$ with $2^j \leq \delta^{-1}$, and $\sum_{2}$ is taken over those $j$ with $2^j > \delta^{-1}$.

For the second sum, we use (62), which yields,

$$\sum_{2} \int_{(B^*)^c} |T_j(a)| dx \leq A \left( \int |a(y)| dy \right) \cdot \left( \sum_{2^j > \delta^{-1}} 2^{-j} \right) \cdot \delta^{-1} \leq A$$

For the first sum, we use (61), and write

$$T_j(a) = \int K_j(x, y) a(y) dy = \int_B [K_j(x, y) - K_j(x, \bar{y})] a(y) dy,$$

where we use the fact that $\int a = 0$. Thus

$$\int |(T_j a)(x)| dx \leq A \cdot 2^{j} \cdot \|a\|_1 \cdot \delta,$$

and

$$\sum_{1} \int_{\mathbb{R}^n} |(T_j a)(x)| dx \leq \sum_{2^j \leq \delta^{-1}} 2^{j} \cdot \delta \leq A.$$ 

Put all estimate together, we have

$$\int_{(B^*)^c} |T(a)| dx \leq A,$$

which, combined with (55), we know that $T$ maps $H^1$ to $L^1$ boundedly.

We now complete the proof by using the complex interpolation. Define the analytic family of operators $T_s$ in the strip $0 < \Re(s) \leq 1$ by

$$T_s = e^{(s-\theta)^2} \int_{\mathbb{R}^n} e^{2\pi i \Phi(x, \xi)} a(x, \xi) \cdot (1 + |\xi|^2)^{\gamma(s)/2} \hat{T}(\xi) d\xi,$$

where $\gamma(x) = -m - \frac{a(n-1)}{2}$ and $\theta = \frac{-2m}{n-1}$; note that $\gamma(\theta) = 0$.

When $\Re(s) = 0$, the symbol $a(x, \xi) \cdot (1 + |\xi|^2)^{\gamma(s)/2}$ has order 0, and thus each $T_s$ is bounded from $L^2$ to itself. Moreover, the argument used in the proof of $L^2$ estimate for Fourier integral operators with 0 order symbol shows that bounds for only finitely many derivatives of the symbol are involved. The bounds for the corresponding derivatives of $a(x, \xi) \cdot (1 + |\xi|^2)^{\gamma(s)/2}$, with $s = it$, have at most polynomial growth in $t$. Given the rapid decrease, as $|t| \to \infty$, of the factor $e^{(s-\theta)^2}$, we obtain therefore

$$\|T_{it}(f)\|_{L^2} \leq A \|f\|_{L^2}, -\infty < t < \infty. \quad (71)$$

When $\Re(s) = 1$, the symbol $a(x, \xi) \cdot (1 + |\xi|^2)^{\gamma(s)/2}$ has order $-(n-1)/2$. Then we can show that

$$\|T_{1+it}(f)\|_{BMO} \leq A \|f\|_{L^\infty}, f \in L^2 \cap L^\infty, t \in \mathbb{R} \quad (72)$$

Apply the complex interpolation theorem, we get

$$\|T_{\theta}(f)\|_{L^p} \leq A \|f\|_{L^p}, \quad (73)$$
with $\theta = 1 - 2/p$. However, $T_{1 - 2/p} = T$, and the relation between $\theta$ and $p$ gives $1/2 - 1/p = -m/(n-1)$. Using the dual operator $T^*$ instead of $T$ gives the similar result when $1/p - 1/2 = -m/(n-1)$, so the theorem is proved when $|1/2 - 1/p| = -m/(n-1)$. Our theorem is therefore proved.

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E-mail address: ylchen88@math.washington.edu