Math 307 Lecture 2
First Order Linear Equations!

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Today!

Last time:
- What is a differential equation?
- First order differential equations
- Separable and homogeneous equations

This time:
- First-Order Linear Equations
- Exact Linear Equations
- Solving Exact Linear Equations

Next time:
- More First-Order Linear Equations
- Method: Integrating Factors
- Method: Variation of Parameters
What is a First-Order Linear Equation?

Recall that a first order ordinary differential equation (ODE) is an equation of the form

$$\frac{dy}{dt} = f(t, y).$$

**Question**
When is a first order differential equation *linear*?

**Definition**
A first order ODE is called *linear* if it can be written in the form

$$y' = p(t)y + q(t)$$

for some functions $p(t), q(t)$
Examples of Linear Equations

- Linear: $y' = 2ty + 3 \sin(t)$
- Not linear: $y' = y^2$
- Linear: $y' = \cos(3t)y + e^t$
- Not linear: $yy' = t^2$
- Linear: $ty' + t^2y = \sin(t)$ (divide by $t$ on both sides)
Why should we care about first-order linear ODE?

- Arises naturally in many situations
  - Newton's law of cooling
  - Compound interest
  - Mixing fluids in a tank
  - Velocity of a falling body with air friction
- We know how to solve them!
  - General solution for linear: involves one arbitrary constant
  - Not true for nonlinear equations (see for example $y' = y^2$)
- Can approximate solutions of nonlinear equations by solutions of linear ones
  - For example, consider the IVP $\frac{dy}{dt} = e^{ty}; y(0) = 0$
  - By Taylor series $e^{ty} \approx 1 + ty$
  - Solutions to IVP $\frac{dy}{dt} = 1 + ty; y(0) = 0$ are approximations
Example

Let $a, b$ be constants. Find a general solution to the ODE

$$\frac{dy}{dt} = ay + b$$

- We already know how to solve this! Why?
- That’s right, it’s separable!

$$\frac{1}{ay + b} \frac{dy}{dt} = 1$$

$$\frac{1}{ay + b} dy = dt$$
Continuing our calculations...

\[\int \frac{1}{ay + b} \, dy = \int \, dt\]

\[\frac{1}{a} \ln |ay + b| = t + C_0\]

\[\ln |ay + b| = at + C_1\]

\[ay + b = e^{at+C_1}\]

\[ay + b = C_2 e^{at}\]

\[y = C_3 e^{at} - \frac{b}{a}\]

We might in the future drop the indexing of the constants, and just let arbitrary constants be arbitrary :)

\[C_1 = aC_0\]

\[C_2 = e^{C_1}\]

\[C_3 = C_2/a\]
In general, solving first-order linear equations won’t be as easy :(

Even so, we will always be able to solve them (up to an integral)

We will give two methods for this next lecture:
- Method of Integrating Factors
- Method of Variation of Parameters

Their names come from more general methods

You should be careful to know how to solve an equation both ways!

TODAY: how to solve exact linear equations
Motivating Example

Example

Find the general solution of the first-order linear ODE

$$\sin(t)y' + \cos(t)y = \sec(t)\tan(t)$$

- How might we solve this?
  - Observe: $$\sin(t)y' + \cos(t)y = (\sin(t)y)'$$
  - This means: $$(\sin(t)y)' = \sec(t)\tan(t)$$
  - Integrating: $$\sin(t)y = \sec(t) + C$$

Final answer:

$$y = \frac{\sec(t) + C}{\sin(t)}$$
That was cool, right?

- One question: What just happened ?!?!
- Answer: the differential equation was exact

What’s that suppose to mean?

- Consider a first-order ODE \( a(t)y' + b(t)y = c(t) \)
- Note that this is linear, since it may be rewritten in the form

\[
y' = (-b(t)/a(t))y + c(t)/a(t).
\]

**Definition**

A first order linear ODE \( a(t)y' + b(t)y = c(t) \) is exact if

\[
a'(t) = b(t).
\]
Let's try to tell if the following equations are exact

- \((t + 1)e^t y + te^t y' = 1\)
  - Answer: yes!
- \((t + 1)y + ty' = e^{-t}\)
  - Answer: no! (Note: compare to previous)
- \(\cos(t)y' = \sin(t)y + \frac{1}{1+t^2}\)
  - Answer: yup!

Solutions to exact equations are easy to find...

Today, we’ll only talk about how to solve exact linear equations
Consider a first order linear equation of the form

\[ a(x)y' + b(x)y = c(x). \]

This equation is exact if \( a'(x) = b(x) \).

In this case,

\[ (a(x)y)' = a'(x)y + a(x)y' = b(x)y + a(x)y' = c(x). \]

So therefore

\[ (a(x)y)' = c(x) \]
If we now integrate both sides with respect to $x$:

$$\int (a(x)y)'\,dx = \int c(x)\,dx.$$ 

we find the solution is

$$a(x)y = \int c(x)\,dx.$$

$$y = \frac{1}{a(x)} \int c(x)\,dx.$$
Another Example

Example

Find the solution to the equation

\[ e^x y' + e^x y = \cos(x). \]

- This equation is exact
- In particular, \((e^x y)' = e^x y' + e^x y\)
- Original equation becomes: \((e^x y)' = \cos(x)\)
- Integrating: \(e^x y = \sin(x) + C\)
- Solution: \(y = e^{-x} \sin(x) + Ce^{-x}\)
What we did today:

- We learned about linear equations
- We learned about exact equations
- We learned how to solve exact linear equations

Plan for next time:

- More on solving first order linear equations