Math 307 Lecture 1
Introducing Differential Equations!

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Plan for today:

- What is a differential equation?
- First order differential equations
- Separable and homogeneous equations

Next time:

- Linear equations
- Method: Integrating factors
- Method: Variation of parameters
Outline

1. Introducing Differential Equations!
   - A First Look
   - Real-World Example Bonanza!
   - Scope of this Course

2. First Order Differential Equations
   - What’s a First Order Equation?
   - Slope Fields

3. Separable and Homogeneous Equations
   - Separable Equations
   - Homogeneous Equations
What’s a Differential Equation?

Question
What is a differential equation?

Definition
A *differential equation* is mathematical expression describing a relationship between a function and its derivatives.

Before going further we should think about:
- Examples of differential equations
- Why we care about differential equations
- The scope of our study of diff. equations in MATH 307
Example Diff. Eqn: Motion of a Rigid Pendulum

Newton’s second law:
\[ \tau = I \frac{d^2 \theta}{dt^2} \]

Torque: \( \tau = mgl \sin \theta \)

Moment of inertia:
\( I = ml^2 \)

We get a differential equation:
\[ \frac{d^2 \theta}{dt^2} = \frac{mg}{l} \sin \theta \]

Figure: A physics-type picture you’ve probably seen before
For continuously compounded interest

\[
\frac{dS}{dt} = rS
\]

- $S$ is invested capital
- $r$ is interest rate
- This is a differential equation!
- Solution is $S(t) = S_0 e^{rt}$

(How do we get this?)
Example Diff. Eqn: Falling with air drag

Figure: Diving to the earth from the stratosphere is probably more fun than differential equations. Maybe?

Newton’s second law:
\[ F = ma \]

Using a linear drag model
\[ m \frac{d^2y}{dt^2} = -mg + k \frac{dy}{dt} \]

- \( y \) is your height
- \( g \) is gravitational acceleration
- \( k \) is a drag coefficient
- How can we solve this equation to get \( y \)?

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Math 307 Lecture 1
Figure: A fluid flow is as cool as it is complicated! Below is an example of what are called Von Karman vortices. (2-dim, so not covered in this course)

- Goal: find velocity of the fluid $u = u(x, t)$
- $x$ is position in the fluid
- $t$ is time
- $p$ is pressure
- $\rho$ is density

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{d^2\rho}{dx^2}
\]

- It’s a *partial differential equation* because it has partial derivatives
The What and Wh

Student: Why should we learn about differential equations?
Wizard: Because they naturally come up all over the place!

Student: What kinds of differential equations will we learn about?
Wizard: There’s just too much to learn! We will focus on what are called first and second order equations.

Student: How hard is it to solve a differential equation?
Wizard: Differential equations, even first and second order ones, can be really hard to solve! Our goal: learn to identify ones which are easy to solve and how
Classification of Differential Equations

Figure: A MATH 307 perspective of the "types" of Diff. Eqns

- PDE
- ODE
  - linear
    - 1st order
      - separable
    - 2nd order
      - homogeneous
  - nonlinear
    - 1st order
      - exact
      - non-exact
    - higher order
A first order ordinary differential equation (ODE) is an equation of the form

\[ \frac{dy}{dt} = f(t, y) \]

where \( f \) is a function of the two variables \( t \) and \( y \).

- Our goal is to find **solutions** to first order differential equations
- Algebraically: find function \( y(t) \) satisfying the above equation
- Geometrically: find finding a curve matching a **slope field**
Almost always, an ODE $y' = f(t, y)$ will have lots of different solutions.

However, for nice ODEs, may be exactly one solution satisfying $y(a) = b$.

The additional constraint $y(a) = b$ is called a initial condition.

The differential equation $y' = f(t, y)$ combined with the constraint $y(a) = b$ is called an initial value problem (IVP).
Slope Fields

- Slope field is a geometric representation of a first-order ODE

\[ \frac{dy}{dt} = f(t, y) \]

**PROCESS:**

1. Make a "grid" of points in the \( x, y \)-plane
2. At each grid point \((a, b)\), draw a dash with slope \( f(x, y) \)
3. This process creates a **vector field** representing the ODE

- Let’s look at an example!
Slope Field Example: $\frac{dy}{dx} = x^2 - y^2$
Slope Fields

- Solutions to the ODE fit naturally in this picture!

**PROCESS:**

1. Imagine the slope field as currents in an ocean
2. Put a boat at a point \((a, b)\)
3. Let the boat (quasi-statically) follow the flow

- The path it traces out forms a solution to the ODE
- Example time, how exciting!!
Slope Field Example: Solution satisfying $y(0) = 0$
Of course, where your boat goes depends on where it starts!

Last time it started at \((0, 0)\), and we got a solution to the IVP

\[ y' = x^2 - y^2, \quad y(0) = 0 \]

So if we put our boat at \((0, \pm 0.5)\), we should get a solution to

\[ y' = x^2 - y^2, \quad y(0) = \pm 0.5 \]
Slope Field Example: Solution satisfying $y(0) = 0.5$
Slope Field Example: Solution satisfying $y(0) = -0.5$
Introducing Differential Equations!

First Order Differential Equations

Separable and Homogeneous Equations

Separable Equation

Definition

A first order differential equation

\[ \frac{dy}{dx} = f(x, y) \]

is called *separable* if \( f(x, y) = g(x)h(y) \) for some functions \( g, h \)

Examples:

\[ \frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)} \]

\[ \frac{dy}{dx} = \frac{(y-3) \cos x}{1+2y^2} \]

\[ y' = \frac{e^{-x} - e^x}{3 + 4y} \]

\[ \sin(2x)dx + \cos(3y)dy = 0 \]
Solving a Separable Equation

**Question**

How can we solve a separable equation?

\[
\frac{dy}{dx} = g(x)h(y)
\]

\[
\frac{1}{h(y)}dy = g(x)dx
\]

\[
\int \frac{1}{h(y)}dy = \int g(x)dx
\]

... finish by solving for y
An Example Worked out

**Example**

Find a solution to the differential equation $y' = (1 - 2x)y^2$ satisfying the initial condition $y(0) = -1/6$.

\[
\frac{1}{y^2} y' = (1 - 2x)
\]

\[
\int \frac{1}{y^2} dy = \int (1 - 2x) dx
\]

\[
-\frac{1}{y} = x - x^2 + C
\]

\[
y = \frac{-1}{x - x^2 + C}
\]

$y(0) = -1/6$ implies $C = 6$. Hence $y = \frac{-1}{x - x^2 + 6}$. 

Homogeneous Equations

Definition

A homogeneous equation is a first order differential equation of the form

\[ \frac{dy}{dx} = f(x, y), \]

where \( f(x, y) = g(y/x) \) for some function \( g \).

- Homogeneous equations have "scale invariant" slope fields: if you zoom out, the slope field looks the same!
- Homogeneous equations are separable equations in disguise!

Examples:

\[ \frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} \quad \quad \quad \quad \quad \quad \quad \quad \quad y' = \frac{3y^2 - x^2}{2xy} \]
Steps to solve:

- Do enough algebra to write \( \frac{dy}{dx} = g(y/x) \)
- Define a new variable \( z = y/x \)
- Since \( xz = y \), implicit differentiation says

\[
\frac{dz}{dx} + x \frac{dz}{dx} = \frac{dy}{dx}
\]

- Plugging back into the original DE, we get a separable equation

\[
z + x \frac{dz}{dx} = g(z)
\]

- Solve this separable equation for \( z \) and use \( y = xz \) to WIN
Question

Find a solution to the differential equation \( y' = \frac{x^2+xy+y^2}{x^2} \) satisfying the initial condition \( y(1) = 0 \).

- Notice that \( y' = 1 + \frac{y}{x} + \frac{y^2}{x^2} = g(y/x) \) for \( g(z) = 1 + z + z^2 \)
- If we set \( z = y/x \), then we find \( z + x \frac{dz}{dx} = 1 + z + z^2 \)

\[
\frac{dz}{dx} = \frac{1 + z^2}{x}
\]

\[
\frac{1}{1 + z^2} dz = \frac{1}{x} dx
\]

\[
\int \frac{1}{1 + z^2} dz = \int \frac{1}{x} dx
\]
arctan(z) = ln|x| + C

z = tan(ln|x| + C)

y = xz = x tan(ln|x| + C)

- Since \( y(1) = 0 \), we must have
  \[ 0 = 1 \tan(ln|1| + C) = \tan(C). \]
- This tells us \( C = 0 \). Hence the solution we want is
  \[ y(x) = x \tan(ln|x|) \]
Introducing Differential Equations!

First Order Differential Equations

Separable and Homogeneous Equations

Separable Equations

Homogeneous Equations

Summary!

What we did today:

- We learned what a differential equation is and why we should care
- We caught a glimpse of first order differential equations
- We learned how to solve separable and homogeneous equations

Plan for next time:

- Linear equations
- Method: Integrating factors
- Method: Variation of parameters