Analysis 101:

Functions of a Single Variable

ABSTRACT

These notes are a chapter in Real Analysis, While primarily standard, the reader will find a discussion of certain topics that are ordinarily not covered in the standard accounts.

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FUNCTIONS OF A SINGLE VARIABLE

Ref: *Advanced Analysis on the Real Line*, R. Kannan and Carole King Krueger, Springer-Verlag, 1996

§0. *RADON MEASURES*

Let X be a locally compact Hausdorff space.

1: NOTATION $C(X)$ is the set of real valued continuous functions on X and $BC(X)$ is the set of bounded real valued continuous functions on X.

2: DEFINITION Given $f \in C(X)$, its support, denoted $spt(f)$, is the smallest closed subset of X outside of which f vanishes, i.e., the closure of $\{x : f(x) \neq 0\}$, and f is said to be compactly supported provided $spt(f)$ is compact.

3: NOTATION $C_c(X)$ is the subset of $C(X)$ whose elements are compactly supported.

4: DEFINITION A function $f \in C(X)$ is said to vanish at infinity if $\forall\, \varepsilon > 0$, the set

$$\{x : |f(x)| \geq \varepsilon\}$$

is compact.

5: NOTATION $C_0(X)$ is the subset of $C(X)$ whose elements vanish at infinity.

6: N.B. $C_c(X) \subset C_0(X) \subset BC(X)$.

7: LEMMA $C_0(X)$ is the closure of $C_c(X)$ in the uniform metric:

$$d(f,g) = ||f - g||_\infty .$$

8: DEFINITION A linear functional $I : C_c(X) \to R$ is positive if

$$f \geq 0 \Rightarrow I(f) \geq 0.$$

9: LEMMA If I is a positive linear functional on $C_c(X)$, then for each compact set $K \subset X$ there is a constant $C_K \geq 0$ such that

$$|I(f)| \leq C_K ||f||_\infty$$

for all $f \in C_c(X)$ such that $\operatorname{spt}(f) \subset K$.

10: DEFINITION A Radon measure on X is a Borel measure μ that is finite on compact sets, outer regular on Borel sets, and inner regular on open sets.

11: EXAMPLE Take $X = R^n$ -- then the restriction of Lebesgue measure λ to the Borel sets in X is a Radon measure.

Every Radon measure μ on X gives rise to a positive linear functional on $C_c(X)$, viz. the assignment

$$f \rightarrow \int_X f \, d\mu.$$

And all such arise in this fashion:

12: RIESZ REPRESENTATION THEOREM If I is a positive linear functional on $C_c(X)$, then there exists a unique Radon measure μ on X such that

$$I(f) = \int_X f \, d\mu$$

for all $f \in C_c(X)$.

13: EXAMPLE Take $X = R$ and define I by the rule

$$I(f) = \int_R f \, dx \quad \text{(Riemann integral)}.$$

Then the Radon measure in this setup per the RRT is the restriction of Lebesgue measure λ on the line to the Borel sets.

14: RAPPEL $C_c(X)$ is a complete topological vector space when equipped with the inductive topology, i.e., the topology of uniform convergence on compact sets.

15: DEFINITION A distribution of order 0 is a continuous linear functional $T:C_c(X) \to R$.

16: LEMMA A linear functional $T:C_c(X) \to R$ is a distribution of order 0 iff for each compact set $K \subset X$ there is a constant $C_K > 0$ such that

$$|T(f)| \leq C_K ||f||_\infty$$

for all $f \in C_c(X)$ such that $\operatorname{spt}(f) \subset K$.

Therefore a positive linear function $I:C_c(X) \to R$ is a distribution of order 0, hence is continuous in the inductive topology.

Denote the set of distributions of order 0 by the symbol $\mathcal{D}^0$.

17: LEMMA $\mathcal{D}^0$ is a vector lattice.

If $T \in \mathcal{D}^0$, then its Jordan decomposition is given by

$$T = T^+ - T^-,$$

where

$$\begin{cases} T^+(f) = \sup\limits_{0 \leq g \leq f} T(g) \\ T^-(f) = -\inf\limits_{0 \leq g \leq f} T(g). \end{cases}$$

Here $T^+, T^- \in \mathcal{D}^0$ are positive linear functionals and

$$T = T^+ - T^-.$$

Therefore

$$\left[\begin{array}{l} T^+ \longleftrightarrow \mu^+ \\ \\ T^- \longleftrightarrow \mu^- \end{array}\right. \quad \text{(Radon)},$$

so $\forall\ f \in C_c(X)$,

$$T(f) = \int_X f\, d\mu^+ - \int_X f\, d\mu^-$$

and

$$|T|(f) = \int_X f\, d(\mu^+ + \mu^-).$$

18: N.B. Both μ^+ and μ^- might have infinite measure, thus in general their difference is not defined.

19: REMARK As we have seen, the positive linear functionals on $C_c(X)$ can be identified with the Radon measures. Bearing in mind that $C_0(X)$ is the uniform closure of $C_c(X)$, the positive linear functionals on $C_0(X)$ can be identified with the finite Radon measures.

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Let X be a compact Hausdorff space.

20: N.B. It is clear that in this situation $C_c(X) = C(X)$.

Equip $C(X)$ with the uniform norm:

$$||f||_\infty = \sup_X |f|.$$

Then the pair $(C(X), ||\cdot||_\infty)$ is a Banach space. Let $C(X)^*$ be the dual space of $C(X)$, i.e., the linear space of all continuous linear functionals Λ on $C(X)$ -- then the prescription

$$||\Lambda||^* = \inf\{M \geq 0: |\Lambda(f)| \leq M\ ||f||_\infty\ (f \in C(X))\}$$

is a norm on $C(X)^*$ under which the pair $(C(X)^*, ||\cdot||^*)$ is a Banach space.

<u>21:</u> <u>N.B.</u> $\forall\ f \in C(X)$, $\forall\ \Lambda \in C(X)^*$,

$$|\Lambda(f)| \leq ||\Lambda||^*\ ||f||_\infty.$$

<u>22:</u> RAPPEL A <u>signed Radon measure</u> is a signed Borel measure μ whose positive variation μ^+ is Radon and whose negative variation μ^- is Radon.

[Note: As usual, $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ and its total variation, denoted $|\mu|$, is by definition $|\mu| = \mu^+ + \mu^-$. In addition, μ is finite if $|\mu|$ is finite, i.e., if $|\mu|(X) < +\infty$.]

<u>23:</u> RIESZ REPRESENTATION THEOREM Given a $\Lambda \in C(X)^*$, there exists a unique finite signed Radon measure μ such that $\forall\ f \in C(X)$,

$$\Lambda(f) = \int_X f\, d\mu.$$

And

$$||\Lambda||^* = |\mu|(X).$$

<u>24:</u> NOTATION $M(X)$ is the set of finite signed Radon measures on X.

<u>25:</u> LEMMA $M(X)$ is a vector space of R.

<u>26:</u> NOTATION Given $\mu \in M(X)$, put

$$||\mu||_{M(X)} = |\mu|(X).$$

27: LEMMA $||\cdot||_{M(X)}$ is a norm on $M(X)$ under which the pair $(M(X), ||\cdot||_{M(X)})$ is a Banach space.

28: THEOREM Define an arrow

$$\Lambda : M(X) \to C(X)^*$$

by the rule

$$\Lambda(\mu)(f) = \int_X f \, d\mu.$$

Then Λ is an isometric isomorphism.

[E.g.:

$$|\Lambda(\mu)(f)| = |\int_X f \, d\mu|$$

$$\leq \int_X |f| \, d|\mu| \leq ||f||_\infty |\mu|(X)$$

$$= ||f||_\infty ||\mu||_{M(X)}.$$

Therefore

$$\Lambda(\mu) \in C(X)^*]$$

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If X is not compact, then the story for $C_0(X)$ is the same as that for $C(X)$ when X is compact. Without stopping to spell it all out, once again the bounded linear functionals are in a one-to-one correspondence with the finite signed Radon measures and

$$||\Lambda||^* = |\mu|(X).$$

§1. *VARIATION OF A FUNCTION*

Let $[a,b] \subset R$ be a closed interval $(a < b, -\infty < a < b < +\infty)$.

1: DEFINITION A partition of $[a,b]$ is a finite set $P = \{x_0,\ldots,x_n\} \subset [a,b]$, where

$$a = x_0 < x_1 < \cdots < x_n = b.$$

2: NOTATION The set of all partitions of $[a,b]$ is denoted by $\mathcal{P}[a,b]$.

3: EXAMPLE

$$\{a,b\} \in \mathcal{P}[a,b].$$

Let (X,d) be a metric space and let $f:[a,b] \to X$ be a function.

4: DEFINITION Given a partition $P \in \mathcal{P}[a,b]$, put

$$\overset{b}{\underset{a}{V}}(f;P) = \sum_{i=1}^{n} d(f(x_i),f(x_{i-1})),$$

the variation of f in P.

5: NOTATION Put

$$T_f[a,b] = \sup_{P\in\mathcal{P}[a,b]} \overset{b}{\underset{a}{V}}(f;P),$$

the total variation of f in $[a,b]$.

6: N.B. Here, (X,d) is implicit... .

One can then develop the basics at this level of generality but we shall instead specialize immediately and take

$$X = R, \ d(x,y) = |x - y|,$$

thus now $f:[a,b] \to R$. Later on, we shall deal with the situation when the domain $[a,b]$ is replaced by the open interval $]a,b[$ (or in principle, by any nonempty open set $\Omega \subset R$ (recall that such an Ω can be written as an at most countable union of pairwise disjoint open intervals), e.g. $\Omega = R$). As for the range, we shall stick with R for the time being but will eventually consider matters when R is replaced by R^M $(M = 1,2,\ldots)$ (curve theory).

§2. *LIMIT AND OSCILLATION*

Let $f:[a,b] \to R$.

1: DEFINITION Given a closed subinterval $I = [x,y] \subset [a,b]$, put

$$v(f;I) = |f(y) - f(x)|,$$

the variation of f in I.

2: DEFINITION Given a partition $P \in \mathcal{P}[a,b]$, put

$$\overset{b}{\underset{a}{V}}(f;P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

$$= \sum_{i=1}^{n} v(f;I_i) \quad (I_i = [x_{i-1},x_i]),$$

the variation of f in P.

3: NOTATION Put

$$T_f[a,b] = \sup_{P\in\mathcal{P}[a,b]} \overset{b}{\underset{a}{V}}(f;P),$$

the total variation of f in [a,b].

4: DEFINITION A function $f:[a,b] \to R$ is of bounded variation in [a,b] provided

$$T_f[a,b] < +\infty.$$

5: NOTATION BV[a,b] is the set of functions of bounded variation in [a,b].

6: EXAMPLE Take $[a,b] = [0,1]$ and define $f:[0,1] \to R$ by the rule

$$f(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ 1 \text{ if } x \text{ is rational.} \end{cases}$$

Then $f \notin BV[0,1]$.

7: NOTATION Given $P \in \mathcal{P}[a,b]$, put

$$||P|| = \max(x_i - x_{i-1}) \quad (i = 1,\ldots,n).$$

8: THEOREM Let $f \in BV[a,b]$. Assume: f is continuous -- then

$$T_f[a,b] = \lim_{||P|| \to 0} \overset{b}{\underset{a}{V}} (f;P).$$

[Note: The continuity assumption is essential. E.g., take $[a,b] = [-1, +1]$ and consider $f(0) = 1$, $f(x) = 0$ $(x \neq 0)$.]

Let $f:[a,b] \to R$.

9: DEFINITION Given a closed subinterval $I = [x,y] \subset [a,b]$, denote by M and m the supremum and infimum of f in I and put

$$\text{osc}(f;I) = M - m,$$

the oscillation of f in I.

[Note: Since the diameter of $f(I)$ is the supremum of the distances between pairs of points of $f(I)$, it follows that

$$M - m = \text{diam } f(I)$$

or still,

$$\text{osc}(f;I) = \text{diam } f(I).$$

And, of course,

$$v(f;I) \leq \text{diam } f(I).]$$

Let

$$v(f;[a,b]) = \sup_{P \in \mathcal{P}[a,b]} \sum_{i=1}^{n} \text{osc}(f;I_i).$$

<u>10:</u> THEOREM

$$T_f[a,b] = v(f;[a,b]).$$

PROOF It is obvious that

$$T_f[a,b] \leq v(f;[a,b]).$$

To go the other way, fix $\varepsilon > 0$. Choose a partition P of [a,b] such that if $\Delta_i = \text{osc}(f;I_i)$, then

$$\sigma = \sum_{i=1}^{n} \Delta_i$$

is greater than $v(f;[a,b]) - \varepsilon$ or ε^{-1} according to whether $v(f;[a,b]) < +\infty$ or $v(f;[a,b]) = +\infty$. To deal with the first possibility, note that in each interval $I_i = [x_{i-1}, x_i]$ there are two points ξ_i', ξ_i'' with

$$|f(\xi_i'') - f(\xi_i')| > \Delta_i - \frac{\varepsilon}{n}.$$

The points ξ_i', ξ_i'' divide I_i into one or two or three subintervals. Call

$$Q = \{y_0, \ldots, y_m\} \quad (n \leq m \leq 3n)$$

the partition of [a,b] thereby determined -- then the sum (i) $\Sigma|f(y_j) - f(y_{j-1})|$ ($[y_{j-1}, y_j]$ contained in $[x_{i-1}, x_i]$) is $> \Delta_i - \frac{\varepsilon}{n}$. Therefore

$$\sum_{j=1}^{m} |f(y_j) - f(y_{j-1})|$$

$$= \sum_{i=1}^{n} (i)\Sigma|f(y_j) - f(y_{j-1})|$$

$$> \sum_{i=1}^{n} (\Delta_i - \frac{\varepsilon}{n})$$

$$= \sum_{i=1}^{n} \Delta_i - \frac{\varepsilon}{n} \sum_{i=1}^{n} 1$$

$$= \sigma - \varepsilon$$

$$> v(f;[a,b]) - \varepsilon - \varepsilon,$$

from which

$$T_f[a,b] \geq v(f;[a,b]).$$

§3. *FACTS AND EXAMPLES*

1: FACT Suppose that $f \in BV[a,b]$ -- then f is bounded on $[a,b]$.

[Given $a \leq x \leq b$, write

$$\begin{aligned} |f(x)| &= |f(x) - f(a) + f(a)| \\ &\leq |f(x) - f(a)| + |f(a)| \\ &\leq |f(x) - f(a)| + |f(b) - f(x)| + |f(a)| \\ &\leq T_f[a,b] + |f(a)| < +\infty.] \end{aligned}$$

2: FACT A function $f:[a,b] \to \mathbb{R}$ is constant iff $T_f[a,b] = 0$.

[A constant function certainly has the stated property. Conversely, if f is not constant on $[a,b]$, then the claim is that $T_f[a,b] \neq 0$. Thus choose $x_1 \neq x_2 \in [a,b]$ such that $f(x_1) \neq f(x_2)$, say $x_1 < x_2$ -- then

$$T_f[a,b] \geq |f(x_1) - f(a)| + |f(x_2) - f(x_1)| + |f(b) - f(x_2)|$$

=>

$$T_f[a,b] \geq |f(x_2) - f(x_1)| > 0.]$$

3: FACT If $f:[a,b] \to \mathbb{R}$ is increasing, then $f \in BV[a,b]$ and

$$T_f[a,b] = f(b) - f(a).$$

[If $P = \{x_0,\ldots,x_n\}$ is a partition of $[a,b]$, then

$$\begin{aligned} \overset{b}{\underset{a}{V}}(f;P) &= \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\ &= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = f(b) - f(a).] \end{aligned}$$

4: FACT If $f:[a,b] \to R$ satisfies a Lipschitz condition, then $f \in BV[a,b]$.

[To say that f satisfies a Lipschitz condition means that there exists a constant $K > 0$ such that for all $x,y \in [a,b]$,

$$|f(x) - f(y)| \leq K|x - y|.]$$

5: FACT If $f:[a,b] \to R$ is differentiable on [a,b] and if its derivative $f':[a,b] \to R$ is bounded on [a,b], then $f \in BV[a,b]$.

[The mean value theorem implies that f satisfies a Lipschitz condition on [a,b].]

[Note: Therefore polynomials on [a,b] are in BV[a,b].]

6: FACT If $f:[a,b] \to R$ has finitely many relative maxima and minima, say at the points

$$a < \xi_1 < \cdots < \xi_n < b,$$

then

$$T_f[a,b] = |f(a) - f(\xi_1)| + \cdots + |f(\xi_n) - f(b)|$$
$$< +\infty,$$

so $f \in BV[a,b]$.

7: EXAMPLE Take $f(x) = \sin x$ $(0 \leq x \leq 2\pi)$ — then $T_f[0,2\pi] = 4$.

Neither continuity and/or boundedness on [a,b] suffices to force bounded variation.

8: EXAMPLE Take $[a,b] = [0,1]$ and let

$$f(x) = \begin{cases} x \sin(1/x) & (0 < x \leq 1) \\ 0 & (x = 0). \end{cases}$$

Then f(x) is continuous and bounded but $f \notin BV[0,1]$.

[Note: On the other hand,

$$f(x) = \begin{cases} x^2 \sin(1/x) & (0 < x \le 1) \\ 0 & x = 0 \end{cases}$$

is continuous and of bounded variation in [0,1].]

The composition of two functions of bounded variation need not be of bounded variation.

9: EXAMPLE Work on [0,1] and take $f(x) = \sqrt{x}$,

$$g(x) = \begin{cases} x^2 \sin^2(1/x) & (0 < x \le 1) \\ 0 & (x = 0). \end{cases}$$

Then $f:[0,1] \to R$, $g:[0,1] \to [0,1]$ are of bounded variation but $f \circ g:[0,1] \to R$ is not of bounded variation.

10: FACT Suppose that $f:[a,b] \to [a,b]$ — then the composition $f \circ g \in BV[a,b]$ for all $g:[a,b] \to [a,b]$ of bounded variation iff f satisfies a Lipschitz condition.

[In one direction, suppose that

$$|f(x) - f(y)| \le K|x - y| \quad (x,y \in [a,b]).$$

Let $P \in \mathcal{P}[a,b]$:

$$\overset{b}{\underset{a}{V}} (f \circ g;P) = \sum_{i=1}^{n} |(f \circ g)(x_i) - (f \circ g)(x_{i-1})|$$

$$\leq \sum_{i=1}^{n} K|g(x_i) - g(x_{i-1})|$$

$$\leq K \overset{b}{\underset{a}{V}} (g;P) \leq KT_g[a,b] < +\infty.]$$

§4. *PROPERTIES*

1: THEOREM If $f,g \in BV[a,b]$, then $f + g \in BV[a,b]$ and

$$T_{f+g}[a,b] \leq T_f[a,b] + T_g[a,b].$$

2: THEOREM If $f \in BV[a,b]$ and $c \in R$, then $cf \in BV[a,b]$ and

$$T_{cf}[a,b] = |c| T_f[a,b].$$

3: SCHOLIUM $BV[a,b]$ is a linear space.

4: THEOREM If $f,g \in BV[a,b]$, then $fg \in BV[a,b]$ and

$$T_{fg}[a,b] \leq (\sup_{[a,b]} |g|) T_f[a,b] + (\sup_{[a,b]} |f|) T_g[a,b].$$

5: SCHOLIUM $BV[a,b]$ is an algebra.

6: THEOREM Let $f \in BV[a,b]$ and let $a < c < b$ -- then

$$\begin{cases} f \in BV[a,c] \\ f \in BV[c,b] \end{cases}$$

and

$$T_f[a,b] = T_f[a,c] + T_f[c,b].$$

7: CRITERION Suppose given a function $f:[a,b] \to R$ with the property that $[a,b]$ can be divided into a finite number of subintervals on each of which f is monotonic -- then $f \in BV[a,b]$.

8: EXAMPLE A function of bounded variation need not be monotonic in any subinterval of its domain.

[Take $[a,b] = [0,1]$ and let $r_1, r_2, \ldots$ be an ordering of the rational numbers in $]0,1[$. Fix $0 < c < 1$ and define $f:[0,1] \to R$ by

$$f(x) = \begin{cases} c^k & (x = r_k) \\ 0 & \text{otherwise.} \end{cases}$$

Then f is nowhere monotonic but it is of bounded variation in $[0,1]$:

$$T_f[0,1] = \frac{2c}{1 - c}.]$$

9: THEOREM

$$f \in BV[a,b] \Rightarrow |f| \in BV[a,b].$$

Therefore $BV[a,b]$ is closed under the formation of the combinations

$$\begin{cases} \frac{1}{2}(f + |f|) \\ \frac{1}{2}(f - |f|). \end{cases}$$

§5. *REGULATED FUNCTIONS*

Given a function $f:[a,b] \to R$ and a point $c \in]a,b[$,

$$\begin{cases} f(c+) = \text{limit from the right} = \lim_{x \downarrow c} f(x) \\ f(c-) = \text{limit from the left} = \lim_{x \uparrow c} f(x). \end{cases}$$

[Note: Define $f(a+)$ and $f(b-)$ in the obvious way.]

1: DEFINITION f is said to be regulated if

- $f(c+)$ exists for all $a \leq c < b$.
- $f(c-)$ exists for all $a < c \leq b$.

2: NOTATION $REG[a,b]$ is the set of regulated functions in $[a,b]$.

3: THEOREM $REG[a,b]$ is a linear space.

[Sums and scalar multiples of regulated functions are regulated.]

4: N.B. Continuous functions $f:[a,b] \to R$ are regulated, i.e.,

$$C[a,b] \subset REG[a,b].$$

5: THEOREM Let $f \in REG[a,b]$ — then the discontinuity set of f is at most countable.

6: DEFINITION A function $f:[a,b] \to R$ is right continuous if for all $a \leq c < b$,

$$f(c) = f(c+).$$

7: DEFINITION Let $f \in REG[a,b]$ — then the right continuous modification

f_r of f is defined by

$$f_r(x) = f(x+) \quad (a \leq x < b).$$

8: LEMMA Up to an at most countable set, $f_r = f$.

[The set of points at which f is not right continuous is a subset of the set of points at which f is not continuous.]

9: LEMMA f_r is right continuous.

[For

$$f_r(c+) = \lim_{x \downarrow c} f_r(x) = \lim_{x \downarrow c} f(x) = f(c+) = f_r(c).]$$

10: DEFINITION Let $f:[a,b] \to R$.

- If $f(x) = \chi_I(x)$, where $I = [a,b]$, or $]a,b[$, or $[a,b[$, or $]a,b]$, then f is said to be a single step function.

- If f is a finite linear combination of single step functions, then f is said to be a step function.

11: LEMMA A function $f:[a,b] \to R$ is a step function iff there are points

$$a = x_0 < x_1 < \cdots < x_n = b$$

such that f is constant on each open interval $]x_{i-1},x_i[$ $(i = 1,\ldots,n)$.

12: THEOREM Let $f:[a,b] \to R$ — then f is regulated iff f is a uniform limit of a sequence of step functions.

13: N.B. Regulated functions are bounded.

[Take an $f \in REG[a,b]$ and choose a step function g such that $||f - g||_\infty \leq 1$, hence $\forall\, x \in [a,b]$,

$$|f(x)| \leq ||f - g||_\infty + ||g||_\infty \leq 1 + ||g||_\infty.]$$

<u>14:</u> THEOREM Let $f \in BV[a,b]$ — then f is regulated.

PROOF Suppose that $a < c \leq b$ and $f(c-)$ does not exist — then there is a positive number ε and a sequence of real numbers c_k increasing to c such that for all k,

$$f(c_k) - f(c_{k+1}) < -\varepsilon < \varepsilon < f(c_{k+2}) - f(c_{k+1}).$$

It therefore follows that for all n,

$$+\infty > T_f[a,b] \geq \sum_{k=1}^{n} |f(c_k) - f(c_{k+1})| > n\varepsilon,$$

an impossibility. In the same vein, $f(c+)$ must exist for all $a \leq c < b$.

<u>15:</u> SCHOLIUM

$$BV[a,b] \subset REG[a,b].$$

In particular: The discontinuity set of an $f \in BV[a,b]$ is at most countable.

<u>16:</u> THEOREM $REG[a,b]$ is a Banach space in the uniform norm and $BV[a,b]$ is a dense linear subspace of $REG[a,b]$, thus

$$\overline{BV[a,b]} = REG[a,b]$$

per $||\cdot||_\infty$.

§6. *POSITIVE AND NEGATIVE*

1: NOTATION Given a real number x, put

$$\begin{cases} x^+ = \max(x,0) = \frac{1}{2}(|x| + x) \\ x^- = \max(-x,0) = \frac{1}{2}(|x| - x). \end{cases}$$

Given a function $f:[a,b] \to R$, let

$$\begin{cases} T_f^+[a,b] = \sup\limits_{P \in \mathcal{P}[a,b]} \sum\limits_{i=1}^{n} (f(x_i) - f(x_{i-1}))^+ \\ T_f^-[a,b] = \sup\limits_{P \in \mathcal{P}[a,b]} \sum\limits_{i=1}^{n} (f(x_i) - f(x_{i-1}))^-, \end{cases}$$

the

positive / negative — total variation

of f in [a,b].

Obviously

$$\begin{cases} 0 \leq T_f^+[a,b] \leq T_f[a,b] \leq +\infty \\ 0 \leq T_f^-[a,b] \leq T_f[a,b] \leq +\infty, \end{cases}$$

so $T_f^+[a,b]$, $T_f^-[a,b]$, $T_f[a,b]$ are all finite if $f \in BV[a,b]$.

2: N.B. Abbreviate

$$\sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \text{ to } \Sigma,$$

$$\sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^+ \text{ to } \Sigma^+,$$

$$\sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^- \text{ to } \Sigma^-.$$

Then

$$\Sigma^+ + \Sigma^- = \Sigma, \ \Sigma^+ - \Sigma^- = f(b) - f(a)$$

=>

$$2\Sigma^+ = \Sigma + f(b) - f(a), \ 2\Sigma^- = \Sigma - f(b) + f(a).$$

3: THEOREM If $f \in BV[a,b]$, then

$$\begin{cases} T_f^+[a,b] + T_f^-[a,b] = T_f[a,b] \\ T_f^+[a,b] - T_f^-[a,b] = f(b) - f(a). \end{cases}$$

Replace "b" by "x" and assume that $f \in BV[a,b]$.

- $T_f^+[a,x] = 2^{-1}(T_f[a,x] + f(x) - f(a))$

=>

$$\frac{1}{2}(T_f[a,x] + f(x)) = T_f^+[a,x] + 2^{-1}f(a)$$

- $T_f^-[a,x] = 2^{-1}(T_f[a,b] - f(x) + f(a))$

=>

$$\frac{1}{2}(T_f[a,x] - f(x)) = T_f^-[a,x] - 2^{-1}f(a).$$

4: LEMMA The functions

$$\begin{cases} x \to \frac{1}{2}(T_f[a,x] + f(x)) \\ x \to \frac{1}{2}(T_f[a,x] - f(x)) \end{cases}, \ T_f[a,a] = 0$$

are increasing.

PROOF Let $a \leq x < y \leq b$.

- $\frac{1}{2} (T_f[a,y] + f(y)) - \frac{1}{2} (T_f[a,x] + f(x))$

$$= \frac{1}{2} (T_f[a,y] - T_f[a,x] + f(y) - f(x))$$

$$\geq \frac{1}{2} (T_f[x,y] - |f(y) - f(x)|) \geq 0.$$

- $\frac{1}{2} (T_f[a,y] - f(y)) - \frac{1}{2} (T_f[a,x] - f(x))$

$$= \frac{1}{2} (T_f[a,y] - T_f[a,x] - f(y) + f(x))$$

$$\geq \frac{1}{2} (T_f[x,y] - |f(y) - f(x)|) \geq 0.$$

5: DEFINITION The representation

$$f(x) = \frac{1}{2} (T_f[a,x] + f(x)) - \frac{1}{2} (T_f[a,x] - f(x))$$

is the Jordan decomposition of f.

6: REMARK To arrive at a representation of f as the difference of two strictly increasing functions, write

$$f(x) = (\frac{1}{2} (T_f[a,x] + f(x)) + x - (\frac{1}{2} (T_f[a,x] - f(x)) + x).$$

7: THEOREM Suppose that $f \in BV[a,b]$ — then f is Borel measurable.

[For this is the case of an increasing function.]

§7. CONTINUITY

<u>1:</u> THEOREM Let $f \in BV[a,b]$. Suppose that f is continuous at $c \in [a,b]$ -- then $T_f[a,-]$ is continuous at $c \in [a,b]$.

PROOF The function $x \to T_f[a,x]$ is increasing, hence both one sided limits exist at all points $c \in [a,b]$, the claim being that

$$\lim_{x \to c} T_f[a,x] = T_f[a,c].$$

To this end, it will be shown that the right hand limit of $T_f[a,x]$ as $x \to c$ is equal to $T_f[a,c]$, where $a \leq c < b$, the discussion for the left hand limit being analogous. So let $\varepsilon > 0$ and choose $\delta > 0$ such that

$$0 < x - c < \delta \Rightarrow |f(x) - f(c)| < \frac{\varepsilon}{2}.$$

Partition $[c,b]$ by the scheme

$$T_f[c,b] < \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \frac{\varepsilon}{2} \ (x_0 = c,\ x_n = b).$$

If $x_1 - c < \delta$, then

$$T_f[c,b] - \frac{\varepsilon}{2} < |f(x_1) - f(c)| + \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})|$$

$$< \frac{\varepsilon}{2} + T_f[x_1,b]$$

$\Rightarrow$

$$T_f[c,b] - T_f[x_1,b] < \varepsilon.$$

On the other hand, if $x_1 - c \geq \delta$, add a point x to the partition subject to

$x - c < \delta$, thus

$$T_f[c,b] - \frac{\varepsilon}{2} < |f(x_1) - f(x_0)| + \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})|$$

$$\leq |f(x_1) - f(x)| + |f(x) - f(x_0)|$$

$$+ \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})|$$

$$< |f(x_1) - f(x)| + \frac{\varepsilon}{2} + \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})|.$$

Since

$$\{x, x_1, \ldots, x_n\}$$

is a partition of $[x,b]$, it follows that

$$T_f[c,b] - \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + T_f[x,b]$$

=>

$$T_f[c,b] - T_f[x,b] < \varepsilon.$$

Finally

$$T_f[a,b] - T_f[x,b]$$

$$= T_f[c,x] = T_f[a,x] - T_f[a,c]$$

$$< \varepsilon$$

if $x - c < \delta$. Therefore

$$T_f[a,c+] = T_f[a,c],$$

so $T_f[a,x]$ is right continuous at c.

2: SCHOLIUM If $f \in BV[a,b] \cap C[a,b]$, then

$$T_f[a,\text{--}] \in C[a,b].$$

3: REMARK It is also true that

$$\begin{cases} T_f^+[a,-] \in C[a,b] \\ T_f^-[a,-] \in C[a,b]. \end{cases}$$

Proof:

$$\begin{cases} T_f^+[a,x] = 2^{-1}(T_f[a,x] + f(x) - f(a)) \\ T_f^-[a,x] = 2^{-1}(T_f[a,x] - f(x) + f(a)). \end{cases}$$

4: THEOREM If $f \in BV[a,b]$ is continuous, then f can be written as the difference of two increasing continuous functions.

[In view of what has been said above, this is obvious.]

5: LEMMA Let $f \in BV[a,b]$. Assume: $T_f[a,-]$ is continuous at $c \in [a,b]$ — then f is continuous at $c \in [a,b]$.

PROOF For

$$\begin{cases} c < x \Rightarrow |f(x) - f(c)| \leq T_f[c,x] = T_f[a,x] - T_f[a,c] \\ x < c \Rightarrow |f(c) - f(x)| \leq T_f[x,c] = T_f[a,c] - T_f[a,x]. \end{cases}$$

6: RAPPEL Let $f:[a,b] \to R$ be increasing and let $x_1, x_2, \ldots$ be an enumeration of the interior points of discontinuity of f — then the saltus function $s_f:[a,b] \to R$ attached to f is defined by

$$s_f(a) = 0$$

and if $a < x \leq b$, by

$$s_f(x) = (f(a+) - f(a)) + \sum_{x_k<x} (f(x_k+) - f(x_k-))$$

$$+ (f(x) - f(x-)).$$

7: FACT The difference $f - s_f$ is an increasing continuous function.

Assume again that $f \in BV[a,b]$ and put

$$V(x) = T_f[a,x], \quad F(x) = V(x) - f(x) \quad (a \leq x \leq b).$$

8: N.B. $V(x)$ and $F(x)$ are increasing functions of x.

Let

$$\{x_1, x_2, \ldots\} \quad (a < x_k < b)$$

be the set comprised of the discontinuity points of V.

9: REMARK The discontinuity set of V coincides with the discontinuity set of f and the discontinuity set of F is contained in the discontinuity set of f.

Introduce

$$s_V(x) = (V(a+) - V(a)) + \sum_{x_k<x} (V(x_k+) - V(x_k-))$$

$$+ (V(x) - V(x-))$$

and

$$s_F(x) = (F(a+) - F(a)) + \sum_{x_k<x} (F(x_k+) - F(x_k-))$$

$$+ (F(x) - F(x-)),$$

where $a < x \leq b$ and take

$$s_V(a) = 0, \; s_F(a) = 0.$$

10: LEMMA s_V is the saltus function of V and s_F is the saltus function of F.

[Per V, this is true by its very construction. As for F, if x_k is not a discontinuity point, then

$$F(x_k+) - F(x_k-) = 0,$$

thus such a term does not participate.]

11: DEFINITION The saltus function $s_f:[a,b] \to R$ attached to f is the difference

$$s_f = s_V - s_F.$$

Spelled out,

$$s_f(a) = 0$$

and

$$s_f(x) = (f(a+) - f(a)) + \sum_{x_k<x} (f(x_k+) - f(x_k-))$$

$$+ (f(x) - f(x-))$$

subject to $a < x \leq b$.

12: SCHOLIUM The functions

$$x \to \begin{cases} V(x) - s_V(x) \\ F(x) - s_F(x) \end{cases}$$

are increasing and continuous. Therefore

$$f(x) - s_f(x) = V(x) - F(x) - (s_V(x) - s_F(x))$$

$$= (V(x) - s_V(x)) - (F(x) - s_F(x))$$

is a continuous function of bounded variation.

§8. *ABSOLUTE CONTINUITY I*

1: DEFINITION A function $f:[a,b] \to R$ is absolutely continuous if $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

whenever

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq b$$

for which

$$\sum_{k=1}^{n} (b_k - a_k) < \delta.$$

2: NOTATION $AC[a,b]$ is the set of absolutely continuous functions in $[a,b]$.

3: THEOREM An absolutely continuous function is uniformly continuous.

4: THEOREM

$$f \in AC[a,b] \Rightarrow |f| \in AC[a,b].$$

5: THEOREM If $f,g \in AC[a,b]$, then so do their sum, difference, and product.

6: THEOREM

$$AC[a,b] \subset BV[a,b].$$

7: SCHOLIUM If $f \in C[a,b]$ but $f \notin BV[a,b]$, then $f \notin AC[a,b]$.

8: CRITERION If f is continuous in $[a,b]$ and if f' exists and is bounded in $]a,b[$, then f is absolutely continuous in $[a,b]$.

[Define $M > 0$ by $|f'(x)| < M$ for all x in $]a,b[$. Take $\varepsilon > 0$ and consider

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)|,$$

where

$$\sum_{k=1}^{n} (b_k - a_k) < \frac{\varepsilon}{M}.$$

Owing to the Mean Value Theorem, $\exists\ x_k \in]a_k, b_k[$ such that

$$\frac{f(b_k) - f(a_k)}{b_k - a_k} = f'(x_k).$$

Therefore

$$\begin{aligned}\sum_{k=1}^{n} |f(b_k) - f(a_k)| &= \sum_{k=1}^{n} \left| \frac{f(b_k) - f(a_k)}{b_k - a_k} \right| |b_k - a_k| \\ &= \sum_{k=1}^{n} |f'(x_k)|\ |b_k - a_k| \\ &< \sum_{k=1}^{n} M|b_k - a_k| \\ &= M \sum_{k=1}^{n} |b_k - a_k| \\ &< M \frac{\varepsilon}{M} = \varepsilon.]\end{aligned}$$

9: EXAMPLE It can happen that a continuous function with an unbounded derivative is absolutely continuous.

[Consider $f(x) = \sqrt{x}$ $(0 \le x \le 1)$ — then $f \in AC[0,1]$ but

$$f'(x) = \frac{1}{2\sqrt{x}} \quad (0 < x < 1).]$$

10: EXAMPLE Consider

$$f(x) = \begin{cases} x^2 \sin(1/x) & (0 < x \leq 1) \\ 0 & (x = 0). \end{cases}$$

Then $f \in BV[0,1]$. But more is true, viz. $f \in AC[0,1]$. In fact, in $]0,1[$,

$$f'(x) = 2x \sin(1/x) - \cos(1/x)$$

=>

$$|f'(x)| \leq 2|x| \ |\sin(1/x)| + |\cos(1/x)|$$

$$\leq 3.$$

11: THEOREM Let $f \in BV[a,b]$ — then $f \in AC[a,b]$ iff $T_f[a,-] \in AC[a,b]$.

PROOF Suppose first that f is absolutely continuous. Given $\varepsilon > 0$, introduce the pairs

$$\{(a_1,b_1), (a_2,b_2),\ldots, (a_n,b_n)\}$$

subject to

$$\sum_{k=1}^{n} (b_k - c_k) < \delta,$$

thus

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon.$$

For each k, let

$$P_k : a_k = x_{k0} < x_{k1} < \cdots < x_{kn_k} = b_k$$

be a partition of $[a_k,b_k]$ — then

$$\sum_{k=1}^{n} \sum_{i=1}^{n_k} (x_{k_i} - x_{k_{i-1}}) = \sum_{k=1}^{n} (b_k - a_k) < \delta$$

=>

$$\sum_{k=1}^{n} \sum_{i=1}^{n_k} |f(x_{k_i}) - f(x_{k_{i-1}})| < \varepsilon.$$

Vary now the P_k through $\mathcal{P}([a_k,b_k])$ and take the supremum, hence

$$\sum_{k=1}^{n} T_f[a_k,b_k] < \varepsilon$$

or still,

$$\sum_{k=1}^{n} T_f[a,b_k] - T_f[a,a_k] < \varepsilon.$$

So $T_f[a,—] \in AC[a,b]$. In the other direction, simply note that

$$|f(b_k) - f(a_k)| \leq T_f[a,b_k] - T_f[a,a_k].$$

Recall that the Jordan decomposition of f is the representation

$$f(x) = \frac{1}{2}(T_f[a,x] + f(x)) - \frac{1}{2}(T_f[a,x] - f(x)).$$

<u>12:</u> SCHOLIUM If $f \in AC[a,b]$, then f can be represented as the difference of two increasing absolutely continuous functions.

Here is a useful technicality.

<u>13:</u> LEMMA Suppose that $f:[a,b] \to \mathbb{R}$ is absolutely continuous — then $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that for an arbitrary finite or countable system of pairwise

disjoint open intervals $\{(a_k,b_k)\}$ with

$$\sum_k (b_k - a_k) < \delta,$$

the inequality

$$\sum_k \operatorname{osc}(f;[a_k,b_k]) < \varepsilon$$

obtains.

<u>14:</u> DEFINITION A function $f:[a,b] \to R$ is said to have <u>property (N)</u> if f sends sets of Lebesgue measure 0 to sets of Lebesgue measure 0:

$$E \subset [a,b] \ \& \ \lambda(E) = 0 \Rightarrow \lambda(f(E)) = 0.$$

<u>15:</u> THEOREM If $f:[a,b] \to R$ is absolutely continuous, then f has property (N).

PROOF Suppose that $\lambda(E) = 0$ and assume that $a \notin E$, $b \notin E$ (this omission has no bearing on the final outcome). Notationally ε, δ, and $\{(a_k,b_k)\}$ are per #13, thus

$$\sum_k (b_k - a_k) < \delta \Rightarrow \sum_k \operatorname{osc}(f;[a_k,b_k]) < \varepsilon.$$

To fix the data and thereby pin matters down, start by putting

$$m_k = \min_{[a_k,b_k]} f, \ M_k = \max_{[a_k,b_k]} f,$$

hence

$$\operatorname{osc}(f;[a_k,b_k]) = M_k - m_k.$$

Since $\lambda(E) = 0$, there exists an open set $S \subset [a,b]$ such that

$$E \subset S, \ \lambda(S) < \delta.$$

Decompose S into its connected components $]a_k,b_k[$, so

$$\sum_k (b_k - a_k) < \delta.$$

Next

$$f(E) \subset f(S) = \sum_k f(]a_k,b_k[)$$

$$\subset \sum_k f([a_k,b_k])$$

or still

$$\lambda^*(f(E)) \leq \sum_k \lambda^*(f([a_k,b_k])).$$

But

$$f([a_k,b_k]) = [m_k,M_k].$$

Therefore

$$\lambda^*(f(E)) \leq \sum_k (M_k - m_k) < \varepsilon.$$

Since ε is arbitrary, it follows that

$$\lambda(f(E)) = 0.$$

16: THEOREM If $f:[a,b] \to R$ is continuous, then f has property (N) iff for every Lebesgue measurable set $E \subset [a,b]$, $f(E)$ is Lebesgue measurable.

PROOF Assuming that f has property (N), take an E and write

$$E = (\bigcup_{j=1}^{\infty} K_j) \cup S \quad (K_1 \subset K_2 \subset \ldots),$$

where each K_j is compact and S has Lebesgue measure 0. Since f is continuous, $f(K_j)$ is compact, hence

$$\bigcup_{j=1}^{\infty} f(K_j)$$

is Lebesgue measurable. But f has property (N), hence $f(S)$ has Lebesgue measure 0. Therefore

$$f(E) = (\bigcup_{j=1}^{\infty} f(K_j)) \cup f(S)$$

is Lebesgue measurable. In the other direction, suppose that f does not possess property (N), thus that there exists a set $E \subset [a,b]$ of Lebesgue measure 0 such that $f(E)$ is not a set of Lebesgue measure 0.

- If $f(E)$ is Lebesgue measurable, then it contains a nonmeasurable subset.
- If $f(E)$ is not Lebesgue measurable, then it contains (is...) a nonmeasurable set.

So there exists a nonmeasurable set $A \subset f(E)$. Put $S = f^{-1}(A) \cap E$: S is Lebesgue measurable (being a subset of E, a set of Lebesgue measure 0), yet $f(S) = A$ is not Lebesgue measurable.

17: SCHOLIUM An absolutely continuous function sends Lebesgue measurable sets to Lebesgue measurable sets.

18: REMARK Let $E \subset [a,b]$ be Lebesgue measurable -- then its image $f(E)$ under a continuous function $f:[a,b] \to R$ need not be Lebesgue measurable.

19: RAPPEL If $E \subset R$ is a set of Lebesgue measure 0, then its complement E^C is a dense subset of R.

[In fact, $E^C \cap I \neq \emptyset$ for every open interval I.]

20: LEMMA Suppose that $f,g:[a,b] \to R$ are continuous and $f = g$ almost everywhere -- then $f = g$.

[The set

$$E = \{x \in [a,b]:f(x) \neq g(x)\}$$

is a set of Lebesgue measure 0.]

21: APPLICATION Two absolutely continuous functions which are equal almost everywhere are equal.

§9. DINI DERIVATIVES

1. DEFINITION Let $f:[a,b] \to \mathbb{R}$.

- Given $x \in [a,b[$,

$$(D^+f)(x) = \limsup_{h\downarrow 0} \frac{f(x+h) - f(x)}{h}$$

is the upper right derivative of f at x and

$$(D_+f)(x) = \liminf_{h\downarrow 0} \frac{f(x+h) - f(x)}{h}$$

is the lower right derivative of f at x.

- Given $x \in]a,b]$,

$$(D^-f)(x) = \limsup_{h\uparrow 0} \frac{f(x+h) - f(x)}{h}$$

is the upper left derivative of f at x and

$$(D_-f)(x) = \liminf_{h\uparrow 0} \frac{f(x+h) - f(x)}{h}$$

is the lower left derivative of f at x.

2: N.B. Collectively, these are the Dini derivatives.

3: EXAMPLE Suppose that $a < b$ and $c < d$. Let

$$f(x) = \begin{cases} ax\left[\sin\frac{1}{x}\right]^2 + bx\left[\cos\frac{1}{x}\right]^2 & (x > 0) \\ 0 & (x = 0) \\ cx\left[\sin\frac{1}{x}\right]^2 + dx\left[\cos\frac{1}{x}\right]^2 & (x < 0). \end{cases}$$

Then

$$\begin{cases} (D^+f)(0) = b > a = (D_+f)(0) \\ (D^-f)(0) = d > c = (D_-f)(0). \end{cases}$$

If $(D^+f)(x) = (D_+f)(x)$, then the common value is called the <u>right derivative</u> of f at x, denoted $(D_rf)(x)$, and f is said to be <u>right differentiable</u> at x if this common value is finite.

If $(D^-f)(x) = (D_-f)(x)$, then the common value is called the <u>left derivative</u> of f at x, denoted $(D_\ell f)(x)$, and f is said to be <u>left differentiable</u> at x if this common value is finite.

<u>4:</u> EXAMPLE Take $f(x) = |x|$ — then

$$\begin{cases} (D^+f)(0) = 1 \\ (D_+f)(0) = 1 \end{cases} \Rightarrow (D_rf)(0) = 1$$

and

$$\begin{cases} (D^-f)(0) = -1 \\ (D_-f)(0) = -1 \end{cases} \Rightarrow (D_\ell f)(0) = -1.$$

If $(D_rf)(x)$ and $(D_\ell f)(x)$ exist and are equal, then their common value is denoted by $f'(x)$ and is called the <u>derivative</u> of f at x, f being <u>differentiable</u> at x if $f'(x)$ is finite.

[So the relations

$$\pm\infty \neq (D^+f)(x) = (D_+f)(x) = (D^-f)(x) = (D_-f)(x) \neq \pm\infty$$

are tantamount to the differentiability of f at x.]

5: EXAMPLE Take $f(x) = \frac{1}{x}$ $(x \neq 0)$, $f(0) = 0$ — then

$$\begin{cases} (D_r f)(0) = +\infty \\ (D_\ell f)(0) = +\infty. \end{cases}$$

Therefore $f'(0) = +\infty$ but f is not differentiable at 0.

There is much that can be said about Dini derivatives but we shall limit ourselves to a few points that are relevant for the sequel.

6: THEOREM Let $f:[a,b] \to R$ — then for any real number r, each of the following sets is at most countable:

$$\{x: (D_+f)(x) \geq r \text{ and } (D^-f)(x) < r\},$$

$$\{x: (D_-f)(x) \geq r \text{ and } (D^+f)(x) < r\},$$

$$\{x: (D^+f)(x) \leq r \text{ and } (D_-f)(x) > r\},$$

$$\{x: (D^-f)(x) \leq r \text{ and } (D_+f)(x) > r\}.$$

7: APPLICATION Let $f:[a,b] \to R$ — then up to an at most countable set,

$$\begin{cases} (D^+f)(x) \geq (D_-f)(x) \\ (D^-f)(x) \geq (D_+f(x). \end{cases}$$

8: THEOREM Let $f:[a,b] \to R$ be a Lebesgue measurable function — then its

Dini derivatives are Lebesgue measurable functions.

To fix the ideas, let us consider a special case. So suppose that $f:[a,b] \rightarrow R$ is a Lebesgue measurable function and $E \subset [a,b[$ is a Lebesgue measurable subset of $[a,b]$. Assume: $D_r f$ exists on E — then $D_r f$ is a Lebesgue measurable function on E.

To establish this, extend the definition of f to R by setting $f = 0$ in $R - [a,b]$. Define a sequence $g_1, g_2, \ldots$ of Lebesgue measurable functions via the prescription

$$g_n(x) = n(f(x + \frac{1}{n}) - f(x)).$$

Let D_e be the subset of R comprised of those x such that $\lim_{n \rightarrow \infty} g_n(x)$ exists in $[-\infty, +\infty]$ — then D_e is a Lebesgue measurable set and

$$\lim_{n \rightarrow \infty} g_n : D_e \rightarrow [-\infty, +\infty]$$

is a Lebesgue measurable function. Take now an $x \in E$ and write

$$(D_r f)(x) = \lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{n \rightarrow \infty} \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}} = \lim_{n \rightarrow \infty} g_n(x).$$

Consequently $E \subset D_e$ and

$$D_r f = \lim_{n \rightarrow \infty} g_n$$

in E, hence $D_r f$ is a Lebesgue measurable function on E.

<u>9:</u> <u>N.B.</u> Analogous considerations apply to $D_\ell f$ and f'.

§10. *DIFFERENTIATION*

We shall first review some fundamental points.

<u>1:</u> FACT Let $f:[a,b] \to R$ be an increasing function — then f is differentiable in $]a,b[- E$, where E is a set of Lebesgue measure 0 contained in $]a,b[$.

[Note: Bear in mind that "differentiable" means that at $x \in]a,b[- E$, $f'(x)$ exists and is finite. Moreover $f'(x) = +\infty$ is possible only on a set of Lebesgue measure 0.]

<u>2:</u> <u>N.B.</u>

$$f':[a,b] - E \to R_{\geq 0}$$

is a Lebesgue measurable function.

<u>3:</u> REMARK If $E \subset]a,b[$ is a set of Lebesgue measure 0, then it can be shown that there exists a continuous increasing function f which is not differentiable at any point of E.

<u>4:</u> RAPPEL If ϕ is a Lebesgue measurable function and if $\psi = \phi$ almost everywhere, then ψ is a Lebesgue measurable function.

<u>5:</u> FACT Let $f:[a,b] \to R$ be an increasing function — then f' is integrable on $[a,b]$ and

$$\int_a^b f' \leq f(b) - f(a).$$

[Note: This estimate can be sharpened to

$$\int_a^b f' \leq f(b-) - f(a+).]$$

6: EXAMPLE One can construct a function $f:[a,b] \to R$ that is continuous and strictly increasing in $[a,b]$ such that $f' = 0$ almost everywhere, hence

$$0 = \int_a^b f' < f(b) - f(a).$$

7: FACT Given an $f \in L^1[a,b]$, put

$$F(x) = \int_a^x f \quad (a \le x \le b).$$

Then $F \in AC[a,b]$ and $F' = f$ almost everywhere.

8: FACT Suppose that $f:[a,b] \to R$ is absolutely continuous — then

$$f(x) = f(a) + \int_a^x f' \quad (a \le x \le b).$$

9: FUBINI'S LEMMA Let $\{f_n\}$ $(n = 1,2,\ldots)$ be a sequence of increasing functions in $[a,b]$. Assume that the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges pointwise in $[a,b]$ to a function F — then F is differentiable almost everywhere in $[a,b]$ and

$$F'(x) = \sum_{n=1}^{\infty} f_n'(x)$$

off of a set of Lebesgue measure 0.

PROOF Without loss of generality, take $f_k(a) = 0$ for all k and observing that F is increasing, let E be the set of points $x \in]a,b[$ such that the derivatives $F'(x)$, $f_1'(x)$, $f_2'(x)$, ... all exist and are finite — then $[a,b] - E$ has Lebesgue measure 0. Let

$$F_n(x) = \sum_{k=1}^{n} f_k(x).$$

Suppose that $x \in E$ and h is chosen small enough to ensure that $x + h \in [a,b]$ — then

$$\frac{F(x+h) - F(x)}{h} = \sum_{k=1}^{\infty} \frac{f_k(x+h) - f_k(x)}{h}$$

=>

$$\frac{F(x+h) - F(x)}{h} \geq \sum_{k=1}^{n} \frac{f_k(x+h) - f_k(x)}{h}$$

=>

$$F'(x) \geq \sum_{k=1}^{n} f_k'(x) = F_n'(x).$$

The f_k' are nonnegative and the sequence

$$\{F_n'(x)\} \quad (n = 1,2,\ldots)$$

is bounded above by $F'(x)$, hence is convergent. It remains to establish that

$$\lim_{n \to \infty} F_n' = F'$$

almost everywhere in $[a,b]$. Since

$$\lim_{n \to \infty} F_n(b) = F(b),$$

there exists a subsequence $\{F_{n_j}(b)\}$ such that

$$F(a) - F_{n_j}(a) = 0 \leq F(b) - F_{n_j}(b) \leq 2^{-j}.$$

But $F - F_{n_j}$ is an increasing function, thus

$$0 \leq F(x) - F_{n_j}(x) \leq 2^{-j}$$

for all $x \in [a,b]$ and so the series

$$\sum_{j=1}^{\infty} (F' - F_{n_j}')$$

is a pointwise convergent series of increasing functions. Reasoning as above, we conclude that the series

$$\sum_{j=1}^{\infty} (F' - F'_{n_j})$$

is convergent almost everywhere in [a,b] and from this it follows that

$$F'(x) - F'_n(x) \to 0$$

as $n \to \infty$ for almost all $x \in [a,b]$.

10: APPLICATION Suppose that $f: [a,b] \to R$ is increasing and let $s_f: [a,b] \to R$ be the saltus function attached to f — then $s'_f = 0$ almost everywhere.

[In general, s_f is not continuous. Still, a continuous singular function is a continuous function whose derivative exists and is zero almost everywhere. To illustrate, write

$$f = (f - s_f) + s_f = r_f + s_f,$$

where by construction r_f is increasing and continuous. And almost everywhere

$$f' = r'_f + s'_f = r'_f.$$

Introduce F by the rule

$$F(x) = \int_a^x f'$$

and set

$$f_{cs} = r_f - F.$$

Then almost everywhere

$$f'_{cs} = r'_f - F' = f' - f' = 0.$$

Therefore f_{cs} is a continuous singular increasing function and

$$f = r_f + s_f = F + f_{cs} + s_f.]$$

The fact that an $f \in BV[a,b]$ can be represented as the difference of two increasing functions implies that f is differentiable almost everywhere.

[Note: Therefore a continuous nowhere differentiable function is not of bounded variation.]

11: THEOREM Suppose that $f \in BV[a,b]$ — then for almost all $x \in [a,b]$,

$$|f'(x)| = T_f'[a,x].$$

PROOF Given $n \in N$, choose a partition $P_n \in P[a,b]$ such that

$$\sum_k |f(x_k) - f(x_{k-1})| > T_f[a,b] - 2^{-n}.$$

In the segment $x_{k-1} \leq x \leq x_k$ of P_n, let

$$\begin{cases} f_n(x) = f(x) + c_n^+ \text{ if } f(x_k) - f(x_{k-1}) \geq 0 \\ \quad\text{or} \\ f_n(x) = -f(x) + c_n^- \text{ if } f(x_k) - f(x_{k-1}) \leq 0, \end{cases}$$

where the constants are chosen so that $f_n(a) = 0$ and the values of f_n at x_k agree — then

$$f_n(x_k) - f_n(x_{k-1}) = |f(x_k) - f(x_{k-1})|,$$

so

$$\begin{aligned} T_f[a,b] - f_n(b) &= T_f[a,b] - \sum_k (f_n(x_k) - f_n(x_{k-1})) \\ &= T_f[a,b] - \sum_k |f(x_k) - f(x_{k-1})| \\ &\leq 2^{-n}. \end{aligned}$$

On the other hand, the function

$$x \to T_f[a,x] - f_n(x)$$

is increasing, hence

$$T_f[a,x] - f_n(x) \le T_f[a,b] - f_n(b)$$

$$\le 2^{-n}$$

=>

$$\sum_{n=1}^{\infty} (T_f[a,x] - f_n(x)) \le \sum_{n=1}^{\infty} 2^{-n} < +\infty.$$

The series

$$\sum_{n=1}^{\infty} (T_f[a,x] - f_n(x))$$

is therefore pointwise convergent, thus by Fubini's lemma, the derived series converges almost everywhere, thus

$$T_f'[a,x] - f_n'(x) \to 0$$

almost everywhere. But

$$f_n'(x) = \pm f'(x).$$

Since $T_f'[a,x] \ge 0$ ($T_f[a,x]$ being increasing), the upshot is that

$$|f'(x)| = T_f'[a,x]$$

almost everywhere.

<u>12:</u> APPLICATION

$$f \in BV[a,b] \Rightarrow f' \in L^1[a,b].$$

[For

$$\int_a^b |f'| = \int_a^b T_f'[a,—]$$

$$\le T_f[a,b] - T_f[a,a]$$

$$= T_f[a,b] < +\infty.]$$

<u>13:</u> THEOREM Given an $f \in L^1[a,b]$, put

$$F(x) = \int_a^x f.$$

Then

$$T_F[a,b] = ||f||_{L^1}.$$

PROOF Given a $P \in \mathcal{P}[a,b]$,

$$\sum_{k=1}^{n} |F(x_k) - F(x_{k-1})|$$

$$= \sum_{k=1}^{n} |\int_{x_{k-1}}^{x_k} f| \leq \int_a^b |f| < +\infty$$

=>

$$T_F[a,b] \leq ||f||_{L^1}.$$

To reverse this, recall that $F \in AC[a,b]$, that $F' = f$ almost everywhere, and that

$$|F'| = T_F'[a,—]$$

almost everywhere. Therefore

$$||f||_{L^1} = \int_a^b |F'|$$

$$= \int_a^b T_F'[a,—]$$

$$\leq T_F[a,b] - T_F[a,a]$$

$$= T_F[a,b].$$

<u>14:</u> LEMMA Suppose that $f:[a,b] \rightarrow R$ is increasing -- then $f \in AC[a,b]$ iff

$$\int_a^b f' = f(b) - f(a).$$

PROOF If $f \in AC[a,b]$, then

$$f(x) = f(a) + \int_a^x f' \quad (a \leq x \leq b)$$

=>

$$f(b) - f(a) = \int_a^b f'.$$

Conversely, write

$$f(x) = \int_a^x f' + f_{cs}(x) + s_f(x).$$

Then

$$f(x) = f(a) + \int_a^x f' + g(x),$$

where

$$f_{cs}(x) + s_f(x) = f(a) + g(x),$$

=>

$$f_{cs}(a) + s_f(a) = f(a) + g(a)$$

=>

$$r_f(a) - F(a) + s_f(a) = f(a) + g(a)$$

=>

$$r_f(a) + s_f(a) = f(a) + g(a)$$

=>

$$(f - s_f)(a) + s_f(a) = f(a) + g(a)$$

=>

$$f(a) = f(a) + g(a)$$

=>

$$g(a) = 0.$$

In addition, the assumption that

$$\int_a^b f' = f(b) - f(a)$$

implies that

$$g(b) = f(b) - f(a) - \int_a^b f'$$
$$= 0.$$

Since g is increasing, it follows that $g(x) = 0$ for all $x \in [a,b]$, hence

$$f(x) = f(a) + \int_a^x f'.$$

15: THEOREM Suppose that $f \in BV[a,b]$ — then $f \in AC[a,b]$ iff

$$T_f[a,b] = \int_a^b |f'|.$$

PROOF On the one hand,

$$f \in AC[a,b] \Rightarrow f' \in L^1[a,b]$$
$$\Rightarrow T_f[a,b] = \int_a^b |f'|.$$

On the other hand, assume the stated relation. Since for almost all x in [a,b],

$$|f'(x)| = T_f'[a,x],$$

we have

$$T_f[a,b] = \int_a^b T_f'[a,—]$$

or still,

$$T_f[a,b] - T_f[a,a] = \int_a^b T_f'[a,—].$$

But $T_f[a,—]$ is increasing, thus in view of the lemma, $T_f[a,—]$ is absolutely continuous, which in turn implies that f is absolutely continuous.

§11. *ESTIMATE OF THE IMAGE*

1: RAPPEL

$$\left[\begin{array}{l}\lambda \ = \text{Lebesgue measure} \\ \lambda^* = \text{outer Lebesgue measure.}\end{array}\right.$$

2: LEMMA Let $f:[a,b] \to R$. Suppose that $E \subset [a,b]$ is a subset in which f' exists, subject to $|f'| \leq K$ --- then

$$\lambda^*(f(E)) \leq K\lambda^*(E).$$

The proof will be carried out in seven steps.

Step 1: Given $x \in E$, $f'(x)$ exists and

$$|f'(x)| = \left|\lim_{y \to x} \frac{f(y) - f(x)}{y - x}\right| \leq K.$$

So, $\forall\ x \in E,\ \exists\ \delta > 0$:

$$|f(y) - f(x)| \leq K|y - x| \quad (y \in \,]x - \delta, x + \delta[\, \cap [a,b]).$$

If now for $n = 1,2,\ldots,$

$$E_n = \{x \in E: |f(y) - f(x)| \leq K|y - x| \quad (y \in \,]x - \frac{1}{n}, x + \frac{1}{n}[\,)\},$$

then each $x \in E$ belongs to E_n $(n >> 0)$, hence

$$E \subset \bigcup_{n=1}^{\infty} E_n.$$

On the other hand, $\forall\ n$, $E_n \subset E$ and $\{E_n\}$ is increasing. Therefore

$$E = \bigcup_{n=1}^{\infty} E_n = \lim_{n \to \infty} E_n.$$

Step 2: Consequently

$$\lim_{n \to \infty} \lambda^*(E_n) = \lambda^*(E).$$

But

$$f(E) = f\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} f(E_n) = \lim_{n \to \infty} f(E_n)$$

=>

$$\lim_{n \to \infty} \lambda^*(f(E_n)) = \lambda^*(f(E)).$$

Step 3: Let $\varepsilon > 0$ be given and let $I_{n,k}$ $(k = 1,2,\ldots)$ be a sequence of open intervals such that

$$\lambda(I_{n,k}) < \frac{1}{n}, \; E_n \subset \bigcup_{k=1}^{\infty} I_{n,k},$$

and

$$\sum_{k=1}^{\infty} \lambda(I_{n,k}) \leq \lambda^*(E_n) + \varepsilon.$$

Step 4:

$$E_n = \bigcup_{k=1}^{\infty} (E_n \cap I_{n,k})$$

and

$$f(E_n) = \bigcup_{k=1}^{\infty} f(E_n \cap I_{n,k}).$$

Step 5: If $x_1, x_2 \in E_n \cap I_{n,k}$, then

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2| \leq K\lambda(I_{n,k})$$

=>

$$\lambda^*(f(E_n \cap I_{n,k})) \leq K\lambda(I_{n,k}).$$

Step 6:

$$\lambda^*(f(E_n)) = \lambda^*\left(\bigcup_{k=1}^{\infty} f(E_n \cap I_{n,k})\right)$$

$$\leq \sum_{k=1}^{\infty} \lambda^*(f(E_n \cap I_{n,k}))$$

$$\leq \sum_{k=1}^{\infty} K\lambda(I_{n,k}) \leq K(\lambda^*(E_n) + \varepsilon).$$

Step 7:

$$\lambda^*(f(E)) = \lim_{n \to \infty} \lambda^*(f(E_n))$$

$$\leq K(\lim_{n \to \infty} \lambda^*(E_n) + \varepsilon)$$

$$= K(\lambda^*(E) + \varepsilon)$$

=>

$$\lambda^*(f(E)) \leq K\lambda^*(E) \quad (\varepsilon \downarrow 0),$$

the assertion of the lemma.

3: THEOREM Let $f:[a,b] \to R$ be Lebesgue measurable. Suppose that $E \subset [a,b]$ is a Lebesgue measurable subset in which f is differentiable — then

$$\lambda^*(f(E)) \leq \int_E |f'(x)|.$$

PROOF Note that $f':E \to R$ is a Lebesgue measurable function. This said, to begin with, assume that in E, $|f'| < M$ (a positive integer). Let

$$E_k^n = \{x \in E: \frac{k-1}{2^n} \leq |f'(x)| < \frac{k}{2^n}\},$$

where

$$k = 1,2,\ldots, M2^n, \ n = 1,2,\ldots .$$

Then for each n,

$$\lambda^*(f(E)) = \lambda^*(f(\bigcup_k E_k^n))$$

$$= \lambda^*(\bigcup_k f(E_k^n))$$

$$\leq \sum_k \lambda^*(f(E_k^n))$$

$$\leq \sum_k \frac{k}{2^n} \lambda(E_k^n)$$

$$= \sum_k \frac{k-1}{2^n} \lambda(E_k^n) + \frac{1}{2^n} \sum_k \lambda(E_k^n).$$

Therefore

$$\lambda^*(f(E)) \leq \lim_{n \to \infty} \left(\sum_k \frac{k-1}{2^n} \lambda(E_k^n) + \frac{1}{2^n} \sum_k \lambda(E_k^n)\right)$$

$$= \int_E |f'|.$$

To treat the case of an unbounded f', let

$$A_k = \{x \in E : k - 1 \leq |f'(x)| < k\} \quad (k = 1,2,\ldots).$$

Then

$$\lambda^*(f(E)) = \lambda^*(f(\bigcup_k A_k))$$

$$\leq \lambda^*(\bigcup_k f(A_k))$$

$$\leq \sum_k \lambda^*(f(A_k))$$

$$\leq \sum_k \int_{A_k} |f'|$$

$$= \int_E |f'|.$$

[Note: In point of fact, $f(E)$ is Lebesgue measurable, so

$$\lambda^*(f(E)) = \lambda(f(E)).]$$

4: N.B. It follows that

$$\lambda^*(f(E)) = 0$$

if $f' = 0$.

[It can be shown conversely that

$$\lambda^*(f(E)) = 0$$

implies that $f' = 0$ almost everywhere in E.]

5: SCHOLIUM Suppose that f has a finite derivative on a set E — then $\lambda^*(f(E)) = 0$ iff $f' = 0$ almost everywhere on E.

§12. *ABSOLUTE CONTINUITY II*

1: THEOREM If $f:[a,b] \to R$ is absolutely continuous and if $f'(x) = 0$ almost everywhere, then f is a constant function.

[Let

$$E = \{x \in [a,b]:f'(x) = 0\}$$

and let

$$E' = [a,b] - E.$$

The assumption that $f \in AC[a,b]$ implies that f has property (N) which in turn implies that f sends Lebesgue measurable sets to Lebesgue measurable sets. In particular: $f(E)$, $f(E')$ are Lebesgue measurable and

$$\lambda(f[a,b]) \leq \lambda(f(E)) + \lambda(f(E')).$$

So first

$$\lambda(f(E)) \leq 0\ \lambda(E) = 0\ (\text{"K"} = 0).$$

And second, E' is a set of Lebesgue measure 0, hence the same is true of $f(E')$. All told then

$$\lambda(f[a,b]) = 0.$$

Owing now to the continuity of f, the image $f([a,b])$ is a point or a closed interval. But the latter is a non-sequitur, thus $f([a,b])$ is a singleton.]

2: MAIN THEOREM Let $f:[a,b] \to R$ — then f is absolutely continuous iff the following four conditions are satisfied:

(1) f is continuous.

(2) f' exists almost everywhere.

(3) $f' \in L^1[a,b]$.

(4) f has property (N).

PROOF An absolutely continuous function has these properties. Conversely, assume that f satisfies the stated conditions. Owing to (3), given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$E \subset [a,b] \ \&\ \lambda(E) < \delta \Rightarrow \int_E |f'| < \varepsilon.$$

Fix

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq b$$

with

$$\sum_{k=1}^{n} (b_k - a_k) < \delta.$$

Then

$$\sum_{k=1}^{n} \int_{[a_k,b_k]} |f'| < \varepsilon.$$

Let

$$A_k = \{x \in [a_k,b_k] : f'(x) \text{ exists}\}.$$

Thanks to (2), $[a_k,b_k] - A_k$ is a set of Lebesgue measure 0, hence thanks to (4), $f([a_k,b_k] - A_k)$ is a set of Lebesgue measure 0. Therefore

$$\begin{aligned}\sum_{k=1}^{n} |f(b_k) - f(a_k)| &\leq \sum_{k=1}^{n} \lambda(f([a_k,b_k])) \quad \text{(by (1))} \\ &= \sum_{k=1}^{n} \lambda(f(A_k)) \\ &\leq \sum_{k=1}^{n} \int_{A_k} |f'| \\ &= \sum_{k=1}^{n} \int_{[a_k,b_k]} |f'| \\ &< \varepsilon.\end{aligned}$$

3: SCHOLIUM If $f \in BV[a,b]$ is continuous and possesses property (N), then $f \in AC[a,b]$.

[One has only to note that if f is of bounded variation, then f' exists almost everywhere and $f' \in L^1[a,b]$.]

4: LEMMA If $f:[a,b] \to R$ has a finite derivative at every point $x \in [a,b]$, then f has property (N).

PROOF Suppose that $\lambda(E) = 0$ ($E \subset [a,b]$). For each positive integer n, let

$$E_n = \{x \in E: |f'(x)| \leq n\}.$$

Then $\lambda(E_n) = 0$ and

$$\lambda^*(f(E_n)) \leq n\lambda^*(E_n)$$
$$n\lambda(E_n) = 0$$

$$\Rightarrow$$

$$\lambda(f(E_n)) = 0.$$

Since

$$E = \bigcup_{n=1}^{\infty} E_n$$

and

$$f(E) = f(\bigcup_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} f(E_n),$$

the conclusion is that

$$\lambda^*(f(E)) \leq \sum_{n=1}^{\infty} \lambda^*(f(E_n))$$

$$= \sum_{n=1}^{\infty} \lambda(f(E_n))$$

$$= 0.$$

I.e.: $\lambda(f(E)) = 0$.

<u>5:</u> EXAMPLE One can construct a continuous function $f:[a,b] \to R$ with a finite derivative almost everywhere which fails to have property (N).

<u>6:</u> THEOREM Let $f:[a,b] \to R$. Assume: $f'(x)$ exists and is finite for all $x \in [a,b]$ and that f' is integrable there --- then f is absolutely continuous.

PROOF Condition (1) of the Main Theorem is satisfied ("differentiability" => "continuity"), conditions (2) and (3) are given, and (4) is satisfied in view of the previous lemma.

The composition of two absolutely continuous functions need not be absolutely continuous. However:

<u>7:</u> FACT Suppose that $f:[a,b] \to [c,d]$ and $g:[c,d] \to R$ are absolutely continuous --- then $g \circ f \in AC[a,b]$ iff $(g' \circ f)f'$ is integrable.

[Note: Interpret $g'(f(x))f'(x)$ to be zero whenever $f'(x) = 0$.]

§13. *MULTIPLICITIES*

Let $f:[a,b] \to R$ be a continuous function. Put

$$m = \min_{[a,b]} f, \quad M = \max_{[a,b]} f.$$

<u>1:</u> NOTATION Define a function $N(f;—):]-\infty, +\infty[\to R$ by stipulating that $N(f;y)$ is the number of times that f assumes the value y in $[a,b]$, i.e., the number of solutions of the equation

$$f(x) = y \quad (a \le x \le b).$$

[Note: $N(f;y)$ is either 0, or a positive integer, or $+\infty$.]

<u>2:</u> DEFINITION $N(f;—)$ is the <u>multiplicity function</u> attached to f.

<u>3:</u> THEOREM $N(f;—)$ is a Borel measurable function and

$$\int_{-\infty}^{+\infty} N(f;—) = T_f[a,b].$$

PROOF Subdivide $[a,b]$ into 2^n equal parts, let

$$I_{ni} = [a, a + (b - a)/2^n], \ i = 1,$$

and let

$$I_{ni} =]a + (i - 1)(b - a)/2^n, a + i(b - a)/2^n], \ i = 2,3,\ldots,2^n.$$

Then f maps each I_{ni} to a segment (closed or not), viz. the segment from m_i to M_i, where

$$m_i = \inf_{I_{ni}} f, \quad M_i = \sup_{I_{ni}} f.$$

The characteristic function χ_{ni} of the set $f(I_{ni})$ is zero for $y > M_i$ & $y < m_i$, one for $m_i < y < M_i$, while it may be zero or one at the two endpoints. Therefore

χ_{ni} is Borel measurable, thus so is the function

$$\chi_n(y) = \sum_{i=1}^{2^n} \chi_{ni}(y) \quad (-\infty < y < +\infty).$$

And

$$\int_{-\infty}^{+\infty} \chi_n = \sum_{i=1}^{2^n} \int_{-\infty}^{+\infty} \chi_{ni}$$

$$= \sum_{i=1}^{2^n} (M_i - m_i)$$

$$= \sum_{i=1}^{2^n} \operatorname{osc}(f;I_{ni}).$$

Moreover

$$\chi_n \geq 0, \ \chi_n \leq \chi_{n+1},$$

which implies that

$$\chi \equiv \lim_{n \to \infty} \chi_n$$

is Borel measurable. Pass then to the limit:

$$\int_{-\infty}^{+\infty} \chi = \lim_{n \to \infty} \int_{-\infty}^{+\infty} \chi_n = T_f[a,b],$$

f being continuous. Matters thereby reduce to establishing that

$$\chi = N(f;—).$$

First

$$\forall\, n,\ \chi_n \leq N(f;—) \Rightarrow \chi \leq N(f;—).$$

Let now q be a natural number not greater than $N(f;y)$, giving rise to q distinct roots

$$x_1 < x_2 < \cdots < x_q$$

of the equation

$$f(x) = y \quad (a \le x \le b).$$

Upon choosing $n >> 0$:

$$\frac{b - a}{2^n} < \min(x_{i+1} - x_i),$$

it follows that all q roots will fall into distinct intervals I_{ni}, hence

$$\chi_n \ge q \Rightarrow \chi \ge q.$$

If $N(f;y) = +\infty$, q can be chosen arbitrarily large, thus $\chi(y) = +\infty$. On the other hand, if $N(f;y)$ is finite, take $q = N(f;y)$ to get

$$\chi(y) \ge N(f;y) \Rightarrow \chi \ge N(f;\text{—}).$$

4: SCHOLIUM A continuous function $f:[a,b] \to R$ is of bounded variation iff its multiplicity function $N(f;\text{—})$ is integrable.

5: N.B. If $f \in BV[a,b] \cap C[a,b]$, then

$$\{y : N(f;y) = +\infty\}$$

is a set of Lebesgue measure 0.

[In fact, $N(f;\text{—})$ is integrable, thus is finite almost everywhere.]

Maintain the assumption that $f:[a,b] \to R$ is continuous.

6: NOTATION Given $J = [c,d] \subset [a,b]$, write

$$\phi(f;J,y) = \begin{cases} +1 & \text{if } f(c) < y < f(d) \\ -1 & \text{if } f(c) > y > f(d) \\ 0 & \text{otherwise,} \end{cases}$$

where $-\infty < y < +\infty$.

7: LEMMA If

$$c = y_0 < y_1 < \cdots < y_m = d$$

is a partition of $J = [c,d]$ into the m intervals $J_j = [y_{j-1}, y_j]$ and $f(y_j) \neq y$ for $j = 0,1,\ldots,m$, then

$$\phi(f;J,y) = \sum_{j=1}^{m} \phi(f;J_j,y).$$

8: NOTATION Given a finite system S of nonoverlapping intervals $J = [c,d]$ in $[a,b]$, put

$$cN(f;y) = \sup_{S} \sum_{J \in S} |\phi(f;J,y)|.$$

9: DEFINITION $cN(f;y)$ is the corrected multiplicity function attached to f.

Obviously

$$0 \leq cN(f;\text{—}) \leq +\infty.$$

10: THEOREM $\forall\, y,\ -\infty < y < +\infty$,

$$0 \leq cN(f;y) \leq N(f;y)$$

and

$$cN(f;y) = N(f;y)$$

for all but countably many y.

Therefore

$$T_f[a,b] = \int_{-\infty}^{+\infty} N(f;\text{—}) = \int_{-\infty}^{+\infty} cN(f;\text{—}).$$

§14. *LOWER SEMICONTINUITY*

1: EXAMPLE (Fatou's Lemma) Suppose given a measure space (X,μ) and a sequence $\{f_n\}$ of nonnegative integrable functions such that $f_n \to f$ almost everywhere -- then

$$\int_X f\,d\mu \leq \liminf_{n \to \infty} \int_X f_n\,d\mu.$$

2: THEOREM Suppose that $f_n:[a,b] \to R$ $(n = 1,2,\ldots)$ is a sequence of functions that converges pointwise to $f:[a,b] \to R$ -- then

$$T_f[a,b] \leq \liminf_{n \to \infty} T_{f_n}[a,b].$$

PROOF Given $\varepsilon > 0$, there exists a partition $P = \{x_0,\ldots,x_m\}$ of $[a,b]$ such that

$$\underset{a}{\overset{b}{V}}(f;P) = \sum_{j=1}^{m} |f(x_j) - f(x_{j-1})|$$

$$> T_f[a,b] - 2^{-1}\varepsilon$$

if $T_f[a,b] < +\infty$ or $> \varepsilon^{-1}$ if $T_f[a,b] = +\infty$. Since $f_n(x_j) \to f(x_j)$ at each of the $m + 1$ points $x_0,\ldots,x_m$, there is an n_ε such that

$$|f(x_j) - f_n(x_j)| < 4^{-1}\,m^{-1}\varepsilon$$

for all $n \geq n_\varepsilon$ and $j = 0,\ldots,m$, hence if $n \geq n_\varepsilon$,

$$|f(x_j) - f(x_{j-1})|$$

$$= |f(x_j) - f_n(x_j) + f_n(x_j) - f_n(x_{j-1}) - f(x_{j-1}) + f_n(x_{j-1})|$$

$$\leq |f(x_j) - f_n(x_j)| + |f(x_{j-1}) - f_n(x_{j-1})|$$

$$+ |f_n(x_j) - f_n(x_{j-1})|$$

=>

$$\sum_{j=1}^{m} |f(x_j) - f(x_{j-1})| \leq 4^{-1}\varepsilon + 4^{-1}\varepsilon + \sum_{j=1}^{m} |f_n(x_j) - f_n(x_{j-1})|$$

or still,

$$\sum_{j=1}^{m} |f(x_j) - f(x_{j-1})| - 2^{-1}\varepsilon$$

$$\leq \sum_{j=1}^{m} |f_n(x_j) - f_n(x_{j-1})|$$

$$\leq T_{f_n}[a,b].$$

<u>Case 1:</u> $T_f[a,b] < +\infty$ -- then

$$\sum_{j=1}^{m} |f(x_j) - f(x_{j-1})|$$

$$> T_f[a,b] - 2^{-1}\varepsilon - 2^{-1}\varepsilon$$

$$= T_f[a,b] - \varepsilon$$

=>

$$T_f[a,b] - \varepsilon < T_{f_n}[a,b] \quad (n \geq n_\varepsilon)$$

=>

$$T_f[a,b] - \varepsilon \leq \liminf_{n \to \infty} T_{f_n}[a,b]$$

=> $(\varepsilon \downarrow 0)$

$$T_f[a,b] \leq \liminf_{n \to \infty} T_{f_n}[a,b].$$

Case 2: $T_f[a,b] = +\infty$ -- then

$$\sum_{j=1}^{m} |f(x_j) - f(x_{j-1})| - 2^{-1}\varepsilon$$

$$> \varepsilon^{-1} - 2^{-1}\varepsilon$$

=>

$$\varepsilon^{-1} - 2^{-1}\varepsilon < T_{f_n}[a,b] \quad (n \geq n_\varepsilon)$$

=>

$$+\infty = T_f[a,b] = \liminf_{n \to \infty} T_{f_n}[a,b].$$

3: REMARK One cannot in general replace pointwise convergence by convergence almost everywhere, i.e., it can happen that under such circumstances

$$\liminf_{n \to \infty} T_{f_n}[a,b] < T_f[a,b].$$

4: EXAMPLE Work on $[0,2\pi]$ and take

$$f_n(x) = \frac{1}{n} \sin(nx),$$

so $f(x) = 0$ -- then $f_n \to f$ uniformly,

$$T_f[0,2\pi] = 0, \ T_{f_n}[0,2\pi] = 4.$$

5: EXAMPLE Work on $[0,2\pi]$ and take

$$f_n(x) = \frac{1}{n} \sin(n^2x),$$

so $f(x) = 0$ -- then $f_n \to f$ uniformly,

$$T_f[0,2\pi] = 0,\ T_{f_n}[0,2\pi] = +\infty.$$

6: THEOREM Let $f:[a,b] \to R$ be a continuous function -- then $cN(f;—)$ is lower semicontinuous in $]-\infty, +\infty[$, i.e., $\forall\ y_0$,

$$cN(f;y_0) \le \liminf_{y \to y_0} cN(f;y).$$

7: THEOREM Suppose that $f_n:[a,b] \to R$ is a sequence of continuous functions that converges pointwise to $f:[a,b] \to R$ -- then $\forall\ y$,

$$cN(f;y) \le \liminf_{n \to \infty} cN(f_n;y).$$

8: REMARK These statements ensure that cN is lower semicontinuous w.r.t. to f and w.r.t. y separately. More is true: cN is lower semicontinuous w.r.t. the pair (f,y), i.e., if $f_n \to f$, $y \to y_0$, then

$$cN(f;y_0) \le \liminf cN(f_n;y)$$

as $f_n \to f$, $y \to y_0$.

9: N.B. In the foregoing, one cannot in general replace cN by N.

§15. *FUNCTIONAL ANALYSIS*

1: THEOREM BV[a,b] is a Banach space under the norm

$$||f||_{BV} = |f(a)| + T_f[a,b].$$

[Note: $T_f[a,b]$ is not a norm since a constant function f has zero total variation, hence the introduction of $|f(a)|$. Recall, however, that

$$T_{f+g}[a,b] \leq T_f[a,b] + T_g(a,b]$$

and

$$T_{cf}[a,b] = |c|T_f[a,b].]$$

As a preliminary to the proof, consider a Cauchy sequence $\{f_k\}$ in BV[a,b]. Given $\varepsilon > 0$, there exists $C_\varepsilon \in \mathbb{N}$ such that

$$||f_k - f_\ell||_{BV} = |f_k(a) - f_\ell(a)| + T_{f_k-f_\ell}[a,b] \leq \varepsilon$$

for all $k,\ell \geq C_\varepsilon$. Therefore

$$||f_k - f_\ell||_\infty \leq \varepsilon,$$

thus the sequence $\{f_k\}$ converges uniformly to a bounded function $f:[a,b] \to \mathbb{R}$, the claim being that $f \in BV[a,b]$.

This said, take a partition $P \in \mathcal{P}[a,b]$ and note that

$$\sum_{i=1}^{n} |(f_k - f_\ell)(x_i) - (f_k - f_\ell)(x_{i-1})| \leq T_{f_k - f_\ell}[a,b] \leq \varepsilon$$

for all $k,\ell \geq C_\varepsilon$. From here, send ℓ to $+\infty$ to get

$$\sum_{i=1}^{n} |(f_k - f)(x_i) - (f_k - f)(x_{i-1})| \leq \varepsilon$$

for all $k \geq C_\varepsilon$, hence

$$T_{f_k - f}[a,b] \leq \varepsilon$$

for all $k \geq C_\varepsilon$. And

$$|f_k(a) - f_\ell(a)| \to |f_k(a) - f(a)| \leq \varepsilon \quad (\ell \to +\infty).$$

Therefore

$$||f_k - f||_{BV} \leq 2\varepsilon$$

for all $k \geq C_\varepsilon$. Moreover

$$T_f[a,b] \leq T_{f-f_k}[a,b] + T_{f_k}[a,b]$$

$$< +\infty.$$

So $f \in BV[a,b]$ and $f_k \to f$ in $BV[a,b]$.

2: REMARK $BV[a,b]$, equipped with the norm $||\cdot||_{BV}$, is not separable. [Take $[a,b] = [0,1]$ and for $f \in BV[0,1]$, $r > 0$, let

$$S(f,r) = \{g \in BV[0,1] : ||g - f||_{BV} < r\}.$$

Call χ_t $(0 < t < 1)$ the characteristic function of $\{t\}$ -- then for $t_1 \neq t_2$,

$$||\chi_{t_1} - \chi_{t_2}||_{BV} = (\chi_{t_1} - \chi_{t_2})(a) + T_{\chi_{t_1} - \chi_{t_2}}[0,1]$$

$$= 0 + T_{\chi_{t_1} - \chi_{t_2}}[0,1]$$

$$= 4.$$

But this implies that

$$S(\chi_{t_1}, 1) \cap S(\chi_{t_2}, 1) = \emptyset.$$

In fact

$$\begin{cases} \|h - \chi_{t_1}\|_{BV} < 1 \\ \|h - \chi_{t_2}\|_{BV} < 1 \end{cases}$$

$\Rightarrow$

$$\begin{aligned} \|\chi_{t_1} - \chi_{t_2}\|_{BV} &= \|\chi_{t_1} - h + h - \chi_{t_2}\|_{BV} \\ &\leq \|\chi_{t_1} - h\|_{BV} + \|\chi_{t_2} - h\|_{BV} \\ &< 1 + 1 = 2. \end{aligned}$$

Accordingly there exists a continuum of disjoint spheres $S(\chi_t,1) \subset S(0,3)$, hence an arbitrary sphere $S(f,r)$ contains a continuum of disjoint spheres $S(r\chi_t/3 + f,r/3)$.]

3: THEOREM BV[a,b] is a complete metric space under the distance function

$$d_{BV}(f,g) = \int_a^b |f - g| + |T_f[a,b] - T_g[a,b]|.$$

The issue is completeness and for this, it suffices to establish that the balls B_M of radius M centered at 0 are compact, the claim being that every sequence $\{f_n\} \subset B_M$ has a subsequence converging to a limit in B_M.

4: N.B. Spelled out, B_M is the set of functions $f \in BV[a,b]$ satisfying the condition

$$d_{BV}(f,0) = \int_a^b |f| + T_f[a,b] \leq M.$$

5: HELLY'S SELECTION THEOREM Let $\mathcal{F}$ be an infinite family of functions in BV[a,b]. Assume that there exists a point $x_0 \in [a,b]$ and a constant $K > 0$

such that $\forall\, f \in \mathcal{F}$,

$$|f(x_0)| + T_f[a,b] \le K.$$

Then there exists a sequence $\{f_n\} \subset \mathcal{F}$ and a function $g \in BV[a,b]$ such that

$$f_n \to g \ (n \to \infty)$$

pointwise in [a,b].

6: LEMMA $\forall\, f \in B_M$,

$$|f(a)| \le M\left(1 + \frac{1}{b - a}\right).$$

PROOF Write

$$f(a) = f(a) - f(x) + f(x)$$

=>

$$|f(a)| \le |f(a) - f(x)| + |f(x)|$$

$$\le T_f[a,b] + |f(x)|$$

=>

$$|f(a)| \int_a^b 1 \le \int_a^b T_f[a,b] + \int_a^b |f|$$

$$\le M(b - a) + M$$

=>

$$|f(a)| \le M\left(1 + \frac{1}{b - a}\right).$$

In the HST, take $\mathcal{F} = \{f_n\}$, $x_0 = a$, and

$$K = M\left(1 + \frac{1}{b - a}\right) + M.$$

Then there exists a subsequence $\{f_{n_k}\}$ and a function $g \in BV[a,b]$ such that

$$f_{n_k} \to g \ (k \to \infty)$$

pointwise in [a,b].

<u>7:</u> LEMMA $\forall\, n_k$, $\forall\, x \in [a,b]$,

$$|f_{n_k}(x)| \leq |f_{n_k}(a)| + T_{f_{n_k}}[a,b] < +\infty.$$

The f_{n_k} are therefore bounded, hence by dominated convergence,

$$f_{n_k} \to g \quad (k \to \infty)$$

in $L^1[a,b]$.

Consider now the numbers

$$T_{f_{n_k}}[a,b] \quad (k = 1,2,\ldots).$$

They constitute a bounded set, hence there exists a subsequence $\{T_{f_{n_k}}[a,b]\}$ (not relabeled) which converges to a limit τ. Since f_{n_k} tends to g pointwise, on the basis of lower semicontinuity, it follows that

$$T_g[a,b] \leq \lim_{k \to \infty} T_{f_{n_k}}[a,b],$$

which implies that

$$T_g[a,b] \leq \tau.$$

Adjusting g at a if necessary, matters can be arranged so as to ensure that $T_g[a,b] = \tau$.

Consequently

$$d_{BV}(f_{n_k},g) = \int_0^1 |f_{n_k} - g| + |T_{f_{n_k}}[a,b] - T_g[a,b]|.$$

$$\begin{array}{cc} \downarrow\ (k \to \infty) & \downarrow\ (k \to \infty) \\ 0 & |\tau - \tau| \end{array}$$

I.e.:

$$\lim_{k \to \infty} d_{BV}(f_{n_k}, g) = 0.$$

The final detail is the verification that $g \in B_M$. To this end, fix $\varepsilon > 0$ -- then for $k >> 0$,

$$d_{BV}(g,0) \leq d_{BV}(g,f_{n_k}) + d_{BV}(f_{n_k},0)$$

$$\leq \varepsilon + M.$$

<u>8:</u> LEMMA In the d_{BV} metric, $BV[a,b]$ is separable.

<u>9:</u> LEMMA $\forall\, a \in R, \forall\, f,g \in BV[a,b]$,

$$d_{BV}(af,ag) = |a|\, d_{BV}(f,g).$$

<u>10:</u> THEOREM Let $\alpha \in L^1[a,b]$ -- then the assignment

$$f \to \int_a^b f\alpha \equiv \Lambda_\alpha(f)$$

is a continuous linear functional on $BV[a,b]$ when equipped with the d_{BV} metric.

PROOF To establish the continuity, take an $f \in BV[a,b]$ and suppose that $\{f_n\}$ is a sequence in $BV[a,b]$ such that

$$d_{BV}(f_n,f) \to 0 \quad (n \to \infty),$$

the objective being to show that if $\varepsilon > 0$ be given, then

$$|\Lambda_\alpha(f_n) - \Lambda_\alpha(f)| < \varepsilon$$

provided $n >> 0$.

So fix a constant $C > 0$: $\forall\, n$,

$$\int_0^1 |f_n - f| + |T_{f_n}[a,b] - T_f[a,b]| \leq C.$$

For each n choose a point $\bar{x}_n$ such that

$$|f_n(\bar{x}_n) - f(\bar{x}_n)| \leq C$$

and note that for all $x \in [a,b]$,

$$\begin{bmatrix} |f_n(x) - f_n(\bar{x}_n)| \leq T_{f_n}[a,b] \\ |f(x) - f(\bar{x}_n)| \leq T_f[a,b] \end{bmatrix}$$

and

$$T_{f_n}[a,b] \leq T_f[a,b] + C$$

$$\Rightarrow$$

$$\begin{aligned} &|f_n(x) - f(x)| \\ &\leq |f_n(x) - f_n(\bar{x}_n) + f_n(\bar{x}_n) - f(\bar{x}_n) + f(\bar{x}_n) - f(x)| \\ &\leq |f_n(x) - f_n(\bar{x}_n)| + |f(x) - f(\bar{x}_n)| + |f_n(\bar{x}_n) - f(\bar{x}_n)| \\ &\leq T_{f_n}[a,b] + T_f[a,b] + |f_n(\bar{x}_n) - f(\bar{x}_n)| \\ &\leq T_f[a,b] + C + T_f[a,b] + C \\ &= 2T_f[a,b] + 2C \\ &\equiv K. \end{aligned}$$

On general grounds (absolute continuity of the integral), given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_E K|\alpha| < \varepsilon/2$$

if $\lambda(E) < \delta$. Take now $N >> 0$:

$$\lambda(E_N) < \delta \quad (E_N = \{x; |\alpha(x)| > N\}).$$

Then

$$|\Lambda_\alpha(f_n) - \Lambda_\alpha(f)|$$

$$= |\int_a^b f_n\alpha - \int_a^b f\alpha|$$

$$\leq \int_a^b |f_n - f|\ |\alpha|$$

$$= \int_{E_N} |f_n - f|\ |\alpha| + \int_{E_N^c} |f_n - f|\ |\alpha|$$

$$\leq \int_{E_N} K|\alpha| + \int_{E_N^c} |f_n - f|\ |\alpha|$$

$$< \varepsilon/2 + \int_{E_N^c} |f_n - f|\ |\alpha|.$$

And

$$x \in E_N^c \Rightarrow |\alpha(x)| \leq N$$

$$\Rightarrow \int_{E_N^c} |f_n - f|\ |\alpha| \leq N\int_{E_N^c} |f_n - f|$$

$$\leq N\int_a^b |f_n - f| < \varepsilon/2 \quad (n >> 0).$$

Therefore in the end

$$|\Lambda_\alpha(f_n) - \Lambda_\alpha(f)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all n sufficiently large.

<u>11:</u> <u>N.B.</u>

$$\Lambda_{\alpha_1} = \Lambda_{\alpha_2}$$

iff $\alpha_1 = \alpha_2$ almost everywhere.

[Suppose that $\Lambda_{\alpha_1} = \Lambda_{\alpha_2}$. Define $f_t \in BV[a,b]$ by the prescription

$$f_t(x) = \begin{cases} 1 & (0 \leq x \leq t) \\ 0 & (t < x \leq 1). \end{cases}$$

Then

$$\int_a^b f_t\alpha_1 = \int_a^b f_t\alpha_2$$

=>

$$\int_0^t \alpha_1 = \int_0^t \alpha_2$$

=>

$$\alpha_1 = \alpha_2$$

almost everywhere.]

§16. *DUALITY*

In the abstract theory, take $X = [a,b]$ -- then there is an isometric isomorphism

$$\Lambda: M([a,b]) \to C[a,b]^*,$$

viz. the rule that sends a finite signed measure μ to the bounded linear functional

$$f \to \int_{[a,b]} f \, d\mu.$$

On the other hand, it is a point of some importance that there is another description of $C[a,b]^*$ which does not involve any measure theory at all.

1: RAPPEL If f is continuous on $[a,b]$ and if $g \in BV[a,b]$, then the Stieltjes integral

$$\int_a^b f(x) \, dg(x)$$

exists.

2: NOTATION $C[a,b]$ is the set of continuous functions on $[a,b]$ equipped with the supremum norm:

$$||f||_\infty = \sup_{[a,b]} |f|,$$

and $C[a,b]^*$ is its dual.

3: LEMMA Let $g \in BV[a,b]$ -- then the assignment

$$f \to \int_a^b f(x) \, dg(x)$$

defines a bounded linear functional $\Lambda_g \in C[a,b]^*$.

[Note:

$$\forall f, \ |\Lambda_g(f)| \le T_g[a,b] \, ||f||_\infty,$$

hence

$$||\Lambda_g|| \leq T_g[a,b].]$$

4: RIESZ REPRESENTATION THEOREM If Λ is a bounded linear functional on $C[a,b]$, then there exists a $g \in BV[a,b]$ such that

$$\Lambda(f) = \int_a^b f(x)\, dg(x) \quad (= \Lambda_g(f))$$

for all $f \in C[a,b]$. And:

$$||\Lambda|| = T_g[a,b].$$

PROOF Extend Λ to $L^\infty[a,b] \supset C[a,b]$ without increasing its norm (Hahn-Banach). Given $x \in [a,b]$, let

$$u_x(t) = \begin{cases} 1 & (a \leq t \leq x) \\ 0 & (x < t \leq b) \end{cases}$$

and put

$$g(x) = \Lambda(u_x).$$

Claim: $g \in BV[a,b]$ and in fact

$$T_g[a,b] \leq ||\Lambda||.$$

Thus take a partition $P \in \mathcal{P}[a,b]$ and let

$$\varepsilon_i = \text{sgn}(g(x_i) - g(x_{i-1})) \quad (i = 1,\ldots,n).$$

Then

$$\sum_{i=1}^n |g(x_i) - g(x_{i-1})| = \sum_{i=1}^n \varepsilon_i (g(x_i) - g(x_{i-1}))$$

$$= \sum_{i=1}^n \varepsilon_i (\Lambda(u_{x_i}) - \Lambda(u_{x_{i-1}}))$$

$$= \Lambda(\sum_{i=1}^{n} \varepsilon_i (u_{x_i} - u_{x_{i-1}}))$$

$$\leq ||\Lambda|| \, ||\sum_{i=1}^{n} \varepsilon_i (u_{x_i} - u_{x_{i-1}})||$$

$$\leq ||\Lambda||.$$

Therefore

$$T_g[a,b] \leq ||\Lambda|| < +\infty \Rightarrow g \in BV[a,b].$$

Suppose next that $f \in C[a,b]$ and let

$$x_i = a + \frac{i(b-a)}{n} \quad (i = 0,\ldots,n).$$

Define

$$f_n(x) = \sum_{i=1}^{n} f(x_i)(u_{x_i}(x) - u_{x_{i-1}}(x)).$$

Then

$$||f - f_n||_\infty = \sup_{[a,b]} |f - f_n|$$

$$\leq \max_{1\leq i\leq n} \sup\{|f(x) - f(x_i)|:x_{i-1} \leq x \leq x_i\}.$$

Invoking uniform continuity, it follows that

$$||f - f_n||_\infty \to 0 \quad (n \to +\infty),$$

i.e.,

$$f_n \to f \Rightarrow \Lambda(f) = \lim_{n\to\infty} \Lambda(f_n)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)(\Lambda(u_{x_i}) - \Lambda(u_{x_{i-1}}))$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)(g(x_i) - g(x_{i-1}))$$

$$= \int_a^b f(x)\, dg(x) = \Lambda_g(f).$$

From the above,

$$T_g[a,g] \leq ||\Lambda||$$

and

$$||\Lambda|| \leq T_g[a,b].$$

So

$$||\Lambda|| = T_g[a,b],$$

as contended.

The "g" that figures in this theorem is definitely not unique. To remedy this, proceed as follows.

<u>5:</u> DEFINITION $g \in BV[a,b]$ is <u>normalized</u> if $g(a) = 0$ and $g(x+) = g(x)$ when $a < x < b$.

[Note: Since $g(a) = 0$,

$$||g||_{BV} = T_g[a,b].$$

Observe too that by definition, the right continuous modification g_r of g in $]a,b[$ is given by the formula

$$g_r(x) = g(x+),$$

so the assumption is that $g_r = g$, i.e., in $]a,b[$, g is right continuous.]

<u>6:</u> NOTATION Write NBV[a,b] for the linear subspace of BV[a,b] whose elements are normalized.

<u>7:</u> THEOREM The arrow

$$NBV[a,b] \rightarrow C[a,b]^*$$

that sends g to Λ_g is an isometric isomorphism:

$$||g||_{BV} = T_g[a,b] = ||\Lambda_g||.$$

Here is a sketch of the proof.

Step 1: Define an equivalence relation in BV[a,b] by writing $g_1 \sim g_2$ iff $\Lambda_{g_1} = \Lambda_{g_2}$.

Step 2: Note that

$$g \sim 0 \Rightarrow 0 = \int_a^b dg(x) = g(b) - g(a)$$

$$\Rightarrow g(a) = g(b).$$

Step 3: Establish that

$$g \sim 0$$

$$\Rightarrow$$

$$g(a) = g(c+) = g(c-) = g(b)$$

if $a < c < b$.

[Suppose that

$$a \leq c < b, \ 0 < h < b - c$$

and define

$$f(x) = \begin{cases} 1 & (a \leq x \leq c) \\ 1 - \dfrac{x - c}{h} & (c \leq x \leq c + h) \\ 0 & (c + h \leq x \leq b). \end{cases}$$

Then

$$g \sim 0$$

$$\Rightarrow$$

$$0 = \int_a^b f(x)\ dg(x) = g(c) - g(a) + \int_c^{c+h} f(x)\ dg(x).$$

Integrate

$$\int_c^{c+h} f(x)\ dg(x)$$

by parts to get

$$-\ g(c)\ +\frac{1}{h}\int_c^{c+h} g(x)\ dx$$

$\Rightarrow$ $(h \to 0)$

$$0 = g(c) - g(a) - g(c) - g(c+)$$

$\Rightarrow$

$$g(a) = g(c+).$$

Analogously

$$a < c \leq b \Rightarrow g(b) = g(c-).]$$

Step 4: Establish that if $g \in BV[a,b]$ and if

$$g(a) = g(c+) = g(c-) = g(b)$$

when $a < c < b$, then $g \sim 0$.

[In fact, $g(x) = g(a)$ at $x = a$, $x = b$, and at all interior points of $[a,b]$ at which g is continuous, thus $\forall\ f \in C[a,b]$,

$$\int_a^b f(x)\ dg(x) = \int_a^b f(x)\ dh(x) = 0,$$

where $h(x) \equiv g(a)$.]

Step 5: Every equivalence class contains at most one normalized function.

[If $g_1, g_2 \in NBV[a,b]$ and if $g_1 \sim g_2$, then $g \equiv g_1 - g_2 \sim 0$. By hypothesis, $g_1(a) = 0$, $g_2(a) = 0$, so

$$(g_1 - g_2)(a) = 0 \Rightarrow (g_1 - g_2)(b) = 0$$

$$\Rightarrow g_1(b) - g_2(b) = 0 \Rightarrow g_1(b) = g_2(b).$$

Moreover

$$g(c+) = g(a) = 0$$

$$\Rightarrow g_1(c+) - g_2(c+) = 0$$

$$\Rightarrow g_1(c+) = g_2(c+).$$

On the other hand,

$$\left[\begin{array}{l} g_1 \in NBV[a,b] \Rightarrow g_1(c+) = g_1(c) \\ \\ g_2 \in NBV[a,b] \Rightarrow g_2(c+) = g_2(c) \end{array}\right. \Rightarrow g_1(c) = g_2(c).$$

I.e.: $g_1 = g_2$.]

<u>Step 6:</u> Every equivalence class contains at least one normalized function.

[Given $g \in BV[a,b]$, define $g^* \in BV[a,b]$ as follows:

$$g^*(a) = 0,\ g^*(b) = g(b) - g(a)$$

$$g^*(x) = g(x+) - g(a) \quad (a < x < b).$$

Then $g^* \in NBV[a,b]$ and $g^* \sim g$. The verification that $g^* \in NBV[a,b]$ is immediate. There remains the claim that $g^* - g \sim 0$.

- $(g^* - g)(a) = g^*(a) - g(a) = - g(a).$
- $(g^* - g)(b) = g^*(b) - g(b) = g(b) - g(a) - g(b) = - g(a).$

When $a < x < b$,

$$g^*(x) = g_r(x) - g(a).$$

And for $c \in]a,b[$,

$$\left[\begin{array}{l} \lim\limits_{x \downarrow c} g_r(x) = \lim\limits_{x \downarrow c} g(x) \\ \\ \lim\limits_{x \uparrow c} g_r(x) = \lim\limits_{x \uparrow c} g(x). \end{array}\right.$$

- $(g^* - g)(c+)$

$$= g^*(c+) - g(c+)$$

$$= g_r(c+) - g(a) - g(c+)$$

$$= \lim_{x \downarrow c} g_r(x) - g(a) - g(c+)$$

$$= \lim_{x \downarrow c} g(x) - g(a) - g(c+)$$

$$= g(c+) - g(a) - g(c+)$$

$$= - g(a).$$

- $(g^* - g)(c-)$

$$= g^*(c-) - g(c-)$$

$$= g_r(c-) - g(a) - g(c-)$$

$$= \lim_{x \uparrow c} g_r(x) - g(a) - g(c-)$$

$$= \lim_{x \uparrow c} g(x) - g(a) - g(c-)$$

$$= g(c-) - g(a) - g(c-)$$

$$= - g(a).$$

Therefore

$$g^* - g \sim 0 \Rightarrow g^* \sim g.]$$

<u>Step 7:</u>

$$T_{g^*}[a,b] \leq T_g[a,b].$$

[Let $P \in \mathcal{P}[a,b]$:

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Given $\varepsilon > 0$, choose points $y_1, \ldots, y_{n-1}$ at which g is continuous with y_i so close to x_i (on the right) that

$$|g(x_i+) - g(y_i)| < \frac{\varepsilon}{2n}.$$

Taking $y_\theta = a$, $y_n = b$, there follows

$$\sum_{i=1}^{n} |g^*(x_i) - g^*(x_{i-1})|$$

$$= \sum_{i=1}^{n} |g(x_i+) - g(a) - g(x_{i-1}+) + g(a)|$$

$$\leq \sum_{i=1}^{n} |g(x_i+) - g(y_i)|$$

$$+ \sum_{i=1}^{n} |g(x_{i-1}+) - g(y_{i-1})|$$

$$+ \sum_{i=1}^{n} |g(y_i) - g(y_{i-1})|$$

$$\leq \sum_{i=1}^{n} |g(y_i) - g(y_{i-1})| + \varepsilon$$

=>

$$T_{g^*}[a,b] \leq T_g[a,b] + \varepsilon$$

=> ($\varepsilon \to 0$)

$$T_{g^*}[a,b] \leq T_g[a,b].$$

Consider now the arrow

$$NBV[a,b] \to C[a,b]^*$$

that sends g to Λ_g. To see that it is surjective, let $\Lambda \in C[a,b]^*$ and choose a $g \in BV[a,b]$ such that

$$\Lambda_g = \Lambda.$$

The equivalence class to which g belongs contains a unique normalized element g^*, so $g^* \sim g$

$$\Rightarrow$$

$$\Lambda_{g^*} = \Lambda_g = \Lambda.$$

Finally, as regards the norms,

$$||\Lambda|| = ||\Lambda_g|| = ||\Lambda_{g^*}||$$

$$\leq T_{g^*}[a,b] \leq T_g[a,b] = ||\Lambda||.$$

Meanwhile

$$T_{g^*}[a,b] = ||g^*||_{BV} \Rightarrow ||\Lambda|| = ||g^*||_{BV}.$$

§17. *INTEGRAL MEANS*

To simplify the notation, work in [0,1] (the generalization to [a,b] being straightforward).

1: NOTATION $I = [0,1]$, $0 < \delta < 1$, $I_\delta = [0,1 - \delta]$ ($\Rightarrow 1 - \delta > 0$), $0 < h < \delta$ ($\Rightarrow 1 - h > 1 - \delta$).

2: DEFINITION Let $f \in BV[0,1]$ and suppose that f is continuous -- then its <u>integral mean</u> is the function f^h on $[0,1 - \delta]$ defined by the prescription

$$f^h(x) = \frac{1}{h}\int_0^h f(x + t)\,dt \quad (0 \leq x \leq 1 - \delta).$$

3: LEMMA $f^h \in C[I_\delta]$ and

$$f^h \to f \quad (h \to 0)$$

uniformly in I_δ.

4: LEMMA The derivative of f^h exists in $]0,1 - \delta[$ and is given there by the formula

$$(f^h)'(x) = \frac{f(x + h) - f(x)}{h}.$$

[Note: Therefore f^h has a continuous first derivative in the interior of I_δ.]

5: LEMMA

$$f^h \in AC[0, 1 - \delta].$$

PROOF Let

$$M = \sup_{[0,1]} |f|.$$

Then for fixed h,

$$|(f^h)'(x)| = \left|\frac{f(x+h) - f(x)}{h}\right| \quad (0 < x < 1 - \delta)$$

$$\leq \frac{2M}{h}.$$

Choose a < b such that

$$0 < a < b < 1 - \delta.$$

Then

$$f^h(b) - f^h(a) = \int_a^b (f^h)'(x)\,dx$$

=>

$$|f^h(b) - f^h(a)| \leq \frac{2M(b-a)}{h} \quad (0 < a < b < 1 - \delta)$$

or still, by continuity,

$$|f^h(b) - f^h(a)| \leq \frac{2M(b-a)}{h} \quad (0 \leq a < b \leq 1 - \delta).$$

And this implies that f^h is absolutely continuous.

[In the usual notation,

$$\sum_{k=1}^{n} |f^h(b_k) - f^h(a_k)|$$

$$\leq \frac{2M}{h} \sum_{k=1}^{n} (b_k - a_k).]$$

6: LEMMA Let

$$[a,b] \subset I_\delta.$$

Then

$$T_{f^h}[a,b] \leq T_f[a, b+\delta] \quad (0 < h < \delta).$$

PROOF Take a finite system of intervals $[a_i, b_i]$ $(1 \leq i \leq n)$ without common interior points in $[a,b]$ — then

$$\sum_{i=1}^{n} |f(b_i + t) - f(a_i + t)| \leq T_f[a,b + \delta]$$

=>

$$\sum_{i=1}^{n} |f^h(b_i) - f^h(a_i)|$$

$$\leq \frac{1}{h} \int_0^h T_f[a,b + \delta]dt$$

$$= T_f[a,b + \delta]$$

=>

$$T_{f^h}[a,b] \leq T_f[a,b + \delta] \quad (0 < h < \delta).$$

<u>7:</u> THEOREM Let

$$[a,b] \subset I_\delta.$$

Then

$$T_{f^h}[a,b] \to T_f[a,b] \quad (0 < h \to 0).$$

PROOF

$$T_{f^h}[a,b] \leq T_f[a,b + \delta] \quad (0 < h < \delta)$$

=>

$$\limsup_{h \to 0} T_{f^h}[a,b] \leq T_f[a,b + \delta].$$

Since

$$T_f[a,b + \delta] \to T_f[a,b] \quad (\delta \to 0),$$

it follows that

$$\limsup_{h \to 0} T_{f^h}[a,b] \leq T_f[a,b].$$

By hypothesis, $[a,b] \subset I_\delta$ and in I_δ,

$$f^h \to f \quad (h \to 0)$$

uniformly, hence pointwise. Therefore

$$\liminf_{h \to 0} T_{f^h}[a,b] \geq T_f[a,b].$$

<u>8:</u> SCHOLIUM Owing to the absolute continuity of f^h in I_δ, for any $[a,b] \subset I_\delta$, we have

$$T_{f^h}[a,b] = \int_a^b |(f^h)'(x)|\, dx$$

$$= \int_a^b \left|\frac{f(x + h) - f(x)}{h}\right| dx$$

and

$$\int_a^b \left|\frac{f(x + h) - f(x)}{h}\right| dx \to T_f[a,b] \qquad (0 < h \to 0).$$

§18. *ESSENTIAL VARIATION*

1: DEFINITION $BVL^1]a,b[$ is the subset of $L^1]a,b[$ consisting of those f whose distributional derivative Df is represented by a finite signed Radon measure in $]a,b[$ of finite total variation, i.e., if

$$\int_{]a,b[} f\phi' = - \int_{]a,b[} \phi \, dDf \qquad (\forall \phi \in C_c^\infty]a,b[)$$

for some finite signed Radon measure Df with

$$|Df|]a,b[< + \infty.$$

[Note: Two L^1-functions which are equal almost everywhere define the same distribution (and so have the same distributional derivative).]

2: N.B. A smoothing argument shows that the integration by parts formula is still true for all $\phi \in C_c^1]a,b[$.

Of course it may happen that Df is a function, say Df = gdx, hence $\forall \phi \in C_c^1]a,b[$,

$$\int_{]a,b[} f\phi' = - \int_{]a,b[} \phi \, g dx.$$

3: EXAMPLE Work in $]0,2[$ and let

$$f(x) = \begin{cases} x & (0 < x \leq 1) \\ 1 & (1 < x < 2). \end{cases}$$

Put

$$g(x) = \begin{cases} 1 & (0 < x \leq 1) \\ 0 & (1 < x < 2). \end{cases}$$

Then $Df = gdx$. In fact, $\forall\ \phi \in C_c^1]0,2[$,

$$\int_0^2 f\phi'\ dx = \int_0^1 x\phi'\ dx + \int_1^2 \phi'\ dx$$

$$= -\int_0^1 \phi\ dx + \phi(1) - \phi(1)$$

$$= -\int_0^1 \phi\ dx = -\int_0^2 \phi\ gdx.$$

<u>4:</u> EXAMPLE Let μ be a finite signed Radon measure in $]a,b[$. Put $f(x) = \mu(]a,x[)$ -- then the distributional derivative of f is μ.

[$\forall\ \phi \in C_c^1]a,b[$,

$$\int_{]a,b[} f(x)\phi'(x)\ dx = \int_{]a,b[} \int_{]a,x[} \phi'(x)\ d\mu(y)dx$$

$$= \int_{]a,b[} \int_{]y,b[} \phi'(x)\ dxd\mu(y)$$

$$= -\int_{]a,b[} \phi(y)\ d\mu(y).]$$

<u>5:</u> NOTATION Let $f:]a,b[\to R$ -- then the <u>total variation</u> $T_f]a,b[$ of f in $]a,b[$ is the supremum of the total variations of f in the closed subintervals of $]a,b[$.

<u>6:</u> FACT If $f:[a,b] \to R$, then

$$T_f[a,b] = T_f]a,b[$$
$$+ |f(a+) - f(a)| + |f(b-) - f(b)|.$$

<u>7:</u> <u>N.B.</u> Therefore

$$T_f[a,b] = T_f]a,b[$$

whenever f is continuous.

8: DEFINITION A function $f:]a,b[\to R$ is of bounded variation in $]a,b[$ provided

$$T_f]a,b[< +\infty.$$

9: NOTATION $BV]a,b[$ is the set of functions of bounded variation in $]a,b[$.

10: N.B. Elements of $BV]a,b[$ are bounded, hence are integrable:

$$BV]a,b[\subset L^1]a,b[.$$

Moreover, $\forall f \in BV]a,b[$,

$$\begin{bmatrix} f(a+) \\ \\ f(b-) \end{bmatrix} \quad \text{exist.}$$

11: EXAMPLE Take $]a,b[=]0,1[$ — then

$$f(x) = \frac{1}{1-x}$$

is increasing and of bounded variation in every closed subinterval of $]0,1[$, yet $f \notin BV]0,1[$.

The initial step in the theoretical development is to characterize the elements of $BVL^1]a,b[$.

12: FACT Let μ be a finite signed Radon measure in $]a,b[$ — then for any open set $S \subset]a,b[$,

$$|\mu|(S) = \sup\{\int_{]a,b[} \phi \, d\mu : \phi \in C_c(S), \ ||\phi||_\infty \leq 1\}.$$

13: DEFINITION Given $f \in L^1]a,b[$, let

$$V(f;]a,b[) = \sup\{\int_{]a,b[} f\phi' : \phi \in C_c^1]a,b[, \ ||\phi||_\infty \leq 1\}.$$

<u>14:</u> THEOREM Let $f \in L^1]a,b[$ — then $f \in BVL^1]a,b[$ iff

$$V(f;]a,b[) < +\infty.$$

And when this is so,

$$V(f;]a,b[) = |Df|]a,b[.$$

PROOF Suppose first that $f \in BVL^1]a,b[$ — then

$$\begin{aligned}
&V(f;]a,b[) \\
&= \sup\{-\int_{]a,b[} \phi\, dDf : \phi \in C_c^1]a,b[, \ ||\phi||_\infty \leq 1\} \\
&= \sup\{-\int_{]a,b[} \phi\, dDf : \phi \in C_c]a,b[, \ ||\phi||_\infty \leq 1\} \\
&= |-Df|]a,b[\\
&= |Df|]a,b[< +\infty.
\end{aligned}$$

Conversely assume that

$$V(f;]a,b[) < +\infty.$$

Then

$$\left|\int_{]a,b[} f\phi'\right| \leq V(f;]a,b[)\,||\phi||_\infty.$$

Since $C_c^1]a,b[$ is dense in $C_0]a,b[$, the linear functional

$$\Lambda : C_c^1]a,b[\to R$$

defined by the rule

$$\phi \to \int_{]a,b[} f\phi'$$

can be extended uniquely to a continuous linear functional

$$\Lambda : C_0]a,b[\to R,$$

where

$$||\Lambda||^* \leq V(f;]a,b[).$$

Thanks to the "C_0" version of the RRT, there exists a finite signed Radon measure μ in $]a,b[$ such that

$$||\Lambda||^* = |\mu|(]a,b[)$$

and

$$\Lambda(\phi) = \int_{]a,b[} \phi \, d\mu \quad (\forall \phi \in C_0]a,b[).$$

Definition:

$$Df = \mu$$

$$\Rightarrow$$

$$|Df|]a,b[= |\mu|(]a,b[)$$

$$= ||\Lambda||^*$$

$$\leq V(f;]a,b[) < +\infty.$$

15: LEMMA The map

$$f \to V(f;]a,b[)$$

is lower semicontinuous in the $L^1_{loc}]a,b[$ topology.

16: APPLICATION The map

$$f \to |Df|]a,b[$$

is lower semicontinuous in the $L^1_{loc}]a,b[$ topology.

17: SUBLEMMA Any element of BV]a,b[can be represented as the difference of two bounded increasing functions.

18: LEMMA $\forall f \in BV]a,b[$,

$$V(f;]a,b[) \leq T_f]a,b[\quad (< +\infty).$$

PROOF Construct a sequence χ_n of step functions such that

$$\chi_n \to f \quad (n \to \infty)$$

in $L^1_{loc}]a,b[$ and $\forall\, n$,

$$V(\chi_n;]a,b[) \leq T_f]a,b[.$$

Thanks now to lower semicontinuity,

$$V(f;]a,b[) \leq \liminf_{n \to \infty} V(\chi_n;]a,b[)$$

$$\leq T_f]a,b[.$$

19: SCHOLIUM

$$BV]a,b[\subset BVL^1]a,b[.$$

[Note: If $f:[a,b] \to \mathbb{R}$ is in $BV[a,b]$, then its restriction to $]a,b[$ is in $BV]a,b[$, hence is in $BVL^1]a,b[$.]

20: DEFINITION Let $f \in L^1]a,b[$ -- then the essential variation of f, denoted $e - T_f]a,b[$, is the set

$$\inf\{T_g]a,b[:g = f \text{ almost everywhere}\}.$$

[Note: If $f_1,f_2 \in L^1]a,b[$ and if $f_1 = f_2$ almost everywhere, then

$$e - T_{f_1}]a,b[= e - T_{f_2}]a,b[.]$$

21: LEMMA Let $f \in L^1]a,b[$ -- then

$$e - T_f]a,b[= V(f;]a,b[).$$

Consequently

22: THEOREM Let $f \in L^1]a,b[$ -- then

$$e - T_f]a,b[< +\infty \Leftrightarrow f \in BVL^1]a,b[.$$

And then

$$|Df|]a,b[= e - T_f]a,b[.$$

23: LEMMA Let $f \in BVL^1]a,b[$. Assume: $Df = 0$ -- then f is (equivalent to) a unique constant.

Assuming still that $f \in BVL^1]a,b[$, let $\mu = Df$ and put $w(x) = \mu(]a,x[)$ — then $Dw = \mu$, thus $D(f-w) \doteq 0$, so there exists a unique constant C such that

$$f = C + w$$

almost everywhere.

24: LEMMA

$$T_{C+w}]a,b[= e - T_f]a,b[.$$

PROOF Take points

$$x_0 < x_1 < \cdots < x_n$$

in $]a,b[$ -- then

$$\sum_{i=1}^{n} |(C+w)(x_i) - (C+w)(x_{i-1})| \leq |\mu|(]a,b[)$$

$$\Rightarrow$$

$$T_{C+w}]a,b[\leq V(f;]a,b[) = e - T_f]a,b[.$$

25: DEFINITION Given $f \in BVL^1]a,b[$, a function $g \in L^1]a,b[$ such that $g = f$ almost everywhere is admissible if

$$T_g]a,b[= e - T_f]a,b[.$$

[Note: Since

$$e - T_f]a,b[< +\infty \Rightarrow T_g]a,b[< +\infty,$$

this says that f is equivalent to g, where $g \in BV]a,b[$.]

So, in this terminology, C+w is admissible, i.e.,

$$f^{\ell}(x) \equiv C + Df]a,x[$$

is admissible, the same being the case of

$$f^{r}(x) \equiv C + Df]a,x].$$

<u>26:</u> LEMMA

$$\begin{cases} f^{\ell} \text{ is left continuous} \\ f^{r} \text{ is right continuous.} \end{cases}$$

<u>27:</u> REMARK

$$\begin{cases} f^{\ell}(x) - f^{\ell}(y) = Df[y,x[\\ f^{r}(x) - f^{r}(y) = Df]y,x] \end{cases} \quad (a < y < x < b).$$

<u>28:</u> THEOREM A function $g \in L^1]a,b[$ is admissible iff

$$g \in \{\theta f^{\ell} + (1-\theta)f^{r} : 0 \le \theta \le 1\}.$$

<u>29:</u> <u>N.B.</u> Denote by AT_f the atoms of the theory, i.e., the $x \in]a,b[$ such that $Df(\{x\}) \neq 0$ –– then $f^{\ell} = f^{r}$ in $]a,b[- AT_f$ and every admissible g is continuous in $]a,b[- AT_f$.

<u>30:</u> LEMMA Suppose that $g \in L^1]a,b[$ is admissible –– then g is differentiable almost everywhere and its derivative g' is the density of Df w.r.t. Lebesgue measure.

There is a characterization of the essential variation which is purely internal.

31: NOTATION Given an $f \in L^1]a,b[$, let $C_{ap}(f)$ stand for its set of points of approximate continuity.

[Recall that $C_{ap}(f)$ is a subset of $]a,b[$ of full measure.]

32: LEMMA

$$e - T_f]a,b[= \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

where the supremum is taken over all finite collections of points $x_i \in C_{ap}(f)$ subject to

$$a < x_0 < x_1 < \cdots < x_n < b.$$

§19. *BVC*

1: NOTATION Given a subset $M \subset]a,b[$ of Lebesgue measure 0, denote by $\mathcal{P}_M]a,b[$ the set of all sequences

$$P: x_0 < x_1 < \cdots < x_n,$$

where

$$\begin{cases} a < x_0 \\ x_n < b \end{cases}$$

and

$$x_i \in]a,b[- M \quad (i = 0,1,\ldots,n).$$

[Note: The possibility that $M = \emptyset$ is not excluded.]

2: NOTATION Given a function $f:]a,b[\to R$, let f_M be the resriction of f to $]a,b[- M$.

3: NOTATION Given an element $P \in \mathcal{P}_M]a,b[$, put

$$\overset{b}{\underset{a}{V}} (f_M;P) = \sum_{i=1}^{n} |f_M(x_i) - f_M(x_{i-1})|.$$

4: NOTATION Given a function $f:]a,b[\to R$, put

$$T_{f_M}]a,b[= \sup_{P \in \mathcal{P}_M]a,b[} \overset{b}{\underset{a}{V}} (f_M;P).$$

5: DEFINITION $T_{f_M}]a,b[$ is the total variation of f_M in $]a,b[- M$.

6: DEFINITION A function $f \in L^1]a,b[$ is said to be of bounded variation in the sense of Cesari if there exists a subset $M \subset \,]a,b[$ of Lebesgue measure 0 such that

$$T_{f_M}\,]a,b[\; < +\infty.$$

7: NOTATION $BVC]a,b[$ is the set of functions which are of bounded variation in the sense of Cesari.

8: EXAMPLE

$$BV]a,b[\; \subset BVC]a,b[\quad (M = \emptyset).$$

9: THEOREM

$$BVC]a,b[\; = BVL^1]a,b[.$$

Proceed via a couple of lemmas.

10: LEMMA Suppose that $f \in BVL^1]a,b[$ -- then $f \in BVC]a,b[$.

PROOF The assumption that

$$f \in BVL^1]a,b[\; \Rightarrow e - T_f]a,b[\; < +\infty.$$

So there exists a g: $g = f$ almost everywhere and

$$T_g]a,b[\; < +\infty.$$

Take now for M the set of x such that $g(x) \neq f(x)$, the complement $]a,b[- M$ being the set of x where $g(x) = f(x)$. Consider a typical sum

$$\sum_{i=1}^{n} |f_M(x_i) - f_M(x_{i-1})|$$

which is equal to

$$\sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$$

which is less than or equal to

$$T_g]a,b[< +\infty.$$

Therefore $f \in BVC]a,b[$.

<u>11:</u> SUBLEMMA If $T_{f_M}]a,b[< +\infty$, then there exists a $g:]a,b[\to R$ such that $g_M = f_M$ and

$$T_g]a,b[= T_{f_M}]a,b[.$$

<u>12:</u> LEMMA Suppose that $f \in BVC]a,b[$ — then $f \in BVL^1]a,b[$.

PROOF The assumption that $f \in BVC]a,b[$ produces an "M" and from the preceding consideration,

$$g_M = f_M \Rightarrow g|]a,b[- M = f|]a,b[- M,$$

hence $g = f$ almost everywhere. But

$$T_{f_M}]a,b[< +\infty \Rightarrow T_g]a,b[< +\infty$$

$$\Rightarrow g \in BV]a,b[\Rightarrow g \in BVL^1]a,b[.$$

Since $g = f$ almost everywhere, they have the same distributional derivative, thus $f \in BVL^1]a,b[$.

Let $\mathcal{M}$ be the set of all subsets of $]a,b[$ of Lebesgue measure 0.

<u>13:</u> NOTATION Given an $f \in BVL^1]a,b[$, put

$$\varphi(f) = \inf_{M \in \mathcal{M}} T_{f_M}]a,b[.$$

<u>14:</u> THEOREM

$$e - T_f]a,b[= \varphi(f).$$

PROOF To begin with,

$$f \in BVL^1]a,b[\Rightarrow e - T_f]a,b[< +\infty.$$

On the other hand, $f \in BVC]a,b[$, so there exists $M \in \mathcal{M}$:

$$T_{f_M}]a,b[< +\infty \Rightarrow \varphi(f) < +\infty.$$

- $e - T_f]a,b[\leq \varphi(f)$.

[Denote by $\mathcal{M}_f$ the subset of $\mathcal{M}$ consisting of those M such that $T_{f_M}]a,b[< +\infty$.

Assign to each $M \in \mathcal{M}_f$ a function $g:]a,b[\to R$ such that $g_M = f_M$ and

$$T_g]a,b[= T_{f_M}]a,b[.$$

Therefore

$$\{T_{f_M}]a,b[: M \in \mathcal{M}_f\}$$

$$\subset \{T_g]a,b[: g = f \text{ almost everywhere}\}$$

$$\Rightarrow$$

$$\varphi(f) = \inf_{M \in \mathcal{M}_f} T_{f_M}]a,b[$$

$$\geq e - T_f]a,b[.]$$

- $\varphi(f) \leq e - T_f]a,b[$.

[Denote by $\mathcal{M}_E$ the subset of $\mathcal{M}$ consisting of those M that arise from the elements $T_g]a,b[$ in the set defining $e - T_f]a,b[$ (i.e., per the requirement that $g = f$ almost everywhere) -- then

$$T_{f_M}]a,b[\leq T_g]a,b[\quad (M \in \mathcal{M}_E),$$

hence

$$\varphi(f) = \inf_{M \in \mathcal{M}} T_{f_M}]a,b[$$

$$\leq \inf_{M \in \mathcal{M}_E} T_{f_M}]a,b[$$

$$\leq \inf\{T_g]a,b[: g = f \text{ almost everywhere}\}$$

$$= e - T_f]a,b[.]$$

<u>15:</u> THEOREM Let $f \in BVL^1]a,b[$ — then there exists a $g \in BV]a,b[$ which is equal to f almost everywhere and has the property that

$$\varphi(f) = T_g]a,b[.$$

PROOF Take g admissible:

$$T_g]a,b[= e - T_f]a,b[= \varphi(f).$$

§20. *ABSOLUTE CONTINUITY III*

1: DEFINITION A function $f:]a,b[\to R$ is said to be absolutely continuous in $]a,b[$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any collection of non overlapping closed intervals

$$[a_1,b_1] \subset]a,b[, \ldots, [a_n,b_n] \subset]a,b[,$$

then

$$\sum_{k=1}^{n} (b_k - a_k) < \delta \Rightarrow \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon.$$

2: NOTATION $AC]a,b[$ is the set of absolutely continuous functions in $]a,b[$.

3: N.B. An absolutely continuous function $f:]a,b[\to R$ is uniformly continuous.

4: RAPPEL A uniformly continuous function $f:]a,b[\to R$ can be extended uniquely to $[a,b]$ in such a way that the extended function remains uniformly continuous.

5: LEMMA If $f \in AC]a,b[$, then its extension to $[a,b]$ belongs to $AC[a,b]$.

6: THEOREM Let $f:]a,b[\to R$ -- then f is absolutely continuous iff the following four conditions are satisfied.

(1) f is continuous.

(2) f' exists almost everywhere.

(3) $f' \in L^p]a,b[$ for some $1 \leq p < +\infty$.

(4) $\forall\, x,x_0 \in]a,b[$,

$$f(x) = f(x_0) + \int_{x_0}^{x} f' \, dL^1.$$

Here (and infra), L^1 is Lebesgue measure on $]a,b[$.

7: N.B. For the record,

$$L^p]a,b[\subset L^1]a,b[\quad (1 \leq p < +\infty).$$

8: DEFINITION Let $1 \leq p < +\infty$ — then a function $f \in L^1_{\ell oc}]a,b[$ admits a weak derivative in $L^p]a,b[$ if there exists a function $\frac{df}{dx} \in L^p]a,b[$ such that $\forall \phi \in C_c^\infty]a,b[$,

$$\int_{]a,b[} \phi \frac{df}{dx} \, dL^1 = - \int_{]a,b[} \phi' \, f dL^1.$$

9: CRITERION If $f \in L^1_{\ell oc}]a,b[$ and if $\forall \phi \in C_c^\infty]a,b[$,

$$\int_{]a,b[} \phi f \, dL^1 = 0,$$

then $f = 0$ almost everywhere.

10: SCHOLIUM A weak derivative of f in $L^p]a,b[$, if it exists at all, is unique up to a set of Lebesgue measure 0. For suppose you have two weak derivatives u,v in $L^p]a,b[$, thus $\forall \phi \in C_c^\infty]a,b[$,

$$\begin{cases} \int_{]a,b[} \phi u \, dL^1 = - \int_{]a,b[} \phi' \, f dL^1 \\ \int_{]a,b[} \phi v \, dL^1 = - \int_{]a,b[} \phi' \, f dL^1 \end{cases}$$

$\Rightarrow$

$$\int_{]a,b[} \phi(u - v) \, dL^1 = 0$$

and so $u = v$ almost everywhere, $\phi \in C_c^\infty]a,b[$ being arbitrary.

11: N.B. If $f,g \in L^1_{\ell oc}]a,b[$ are equal almost everywhere, then they have the "same" weak derivative.

[$\forall\ \phi \in C_c^\infty]a,b[$,

$$\int_{]a,b[} \phi \frac{df}{dx}\, dL^1 = - \int_{]a,b[} \phi' \ f dL^1$$

$$= - \int_{]a,b[} \phi' \ g dL^1$$

$$= \int_{]a,b[} \phi \frac{dg}{dx}\, dL^1,$$

so

$$\frac{df}{dx} = \frac{dg}{dx}$$

almost everywhere.]

<u>12:</u> LEMMA Let $f, g \in L^1_{\ell oc}]a,b[$ and suppose that each of them admits a weak derivative -- then $f + g$ admits a weak derivative and

$$\frac{d}{dx}\,(f + g) = \frac{df}{dx} + \frac{dg}{dx}\,.$$

PROOF $\forall\ \phi \in C_c^\infty]a,b[$,

$$\int_{]a,b[} \phi \left(\frac{df}{dx} + \frac{dg}{dx}\right)\, dL^1$$

$$= \int_{]a,b[} \phi \frac{df}{dx}\, dL^1 + \int_{]a,b[} \phi \frac{dg}{dx}\, dL^1$$

$$= - \int_{]a,b[} \phi' \ f dL^1 - \int_{]a,b[} \phi' \ g dL^1$$

$$= - \int_{]a,b[} \phi' (f + g) dL^1.$$

<u>13:</u> LEMMA If $\psi \in C_c^\infty]a,b[$ and if f admits a weak derivative $\frac{df}{dx}$, then

ψf admits a weak derivative and

$$\frac{d}{dx}(\psi f) = \psi' f + \psi \frac{df}{dx}.$$

PROOF $\forall\ \phi \in C_c^\infty]a,b[$,

$$\int_{]a,b[} \phi'(\psi f)\,dL^1 = \int_{]a,b[} (f(\psi\phi)' - f(\psi'\phi))\,dL^1$$

$$= -\int_{]a,b[} \phi\left(\psi \frac{df}{dx} + f\psi'\right)dL^1.$$

<u>14:</u> SUBLEMMA Given $\phi \in C_c^\infty]a,b[$, let

$$\Phi(x) = \int_{]a,x[} \phi\ dL^1$$

and suppose that

$$\int_{]a,b[} \phi\ dL^1 = 0.$$

Then $\Phi \in C_c^\infty]a,b[$.

<u>15:</u> LEMMA Let $f \in L^1_{\ell oc}]a,b[$ and assume that f has weak derivative 0 -- then f coincides almost everywhere in $]a,b[$ with a constant function.

PROOF Fix $\psi_0 \in C_c^\infty]a,b[: \int_{]a,b[} \psi_0\ dL^1 = 1$, and given any $\phi \in C_c^\infty]a,b[$, put $I(\phi) = \int_{]a,b[} \phi\ dL^1$ -- then

$$I(\phi - I(\phi)\psi_0) = I(\phi) - I(\phi)I(\psi_0) = 0,$$

hence

$$\Psi(x) = \int_{]a,x[} (\phi - I(\phi)\psi_0)\,dL^1 \in C_c^\infty]a,b[.$$

Since f has weak derivative 0,

$$\int_{]a,b[} \Psi \frac{df}{dx}\,dL^1 = 0,$$

=>

$$0 = \int_{]a,b[} \Psi' \, f dL^1$$

$$= \int_{]a,b[} (\phi - I(\phi)\psi_0) f dL^1$$

$$= \int_{]a,b[} \phi \, f dL^1 - (\int_{]a,b[} \phi \, dL^1)(\int_{]a,b[} f\psi_0 dL^1)$$

$$= \int_{]a,b[} \phi(f - C_0) dL^1,$$

where

$$C_0 = \int_{]a,b[} f\psi_0 dL^1.$$

Therefore $f - C_0 = 0$ almost everywhere or still, $f = C_0$ almost everywhere.

16: NOTATION Let $1 \leq p < +\infty$ -- then $W^{1,p}]a,b[$ is the set of all functions $f \in L^p]a,b[$ which possess a weak derivative $\frac{df}{dx}$ in $L^p]a,b[$.

17: N.B. $W^{1,1}]a,b[$ is contained in $BVL^1]a,b[$.

[Take an $f \in W^{1,1}]a,b[$ and consider

$$Df(E) = \int_E \frac{df}{dx} \, dL^1 \quad (E \in BO]a,b[),$$

i.e.,

$$dDf = \frac{df}{dx} \, dL^1.$$

Then $\forall \ \phi \in C_c^\infty]a,b[$,

$$\int_{]a,b[} \phi \, dDf = \int_{]a,b[} \phi \, \frac{df}{dx} \, dL^1$$

$$= - \int_{]a,b[} \phi' \, f dL^1,$$

so by definition, $f \in BVL^1]a,b[$.]

[Note: The containment is strict.]

18: THEOREM Let $1 \leq p < +\infty$ -- then a function $f:]a,b[\to R$ belongs to $W^{1,p}]a,b[$ iff it admits an absolutely continuous representative $\bar{f}:]a,b[\to R$ such that $\bar{f}$ and its ordinary derivative $\bar{f}'$ belong to $L^p]a,b[$.

19: LEMMA If $f \in AC]a,b[$, then $\forall\ \phi \in C_c^\infty]a,b[$,

$$\int_{]a,b[} \phi\, f' dL^1 = - \int_{]a,b[} \phi' \, f dL^1,$$

there being no boundary term in the (implicit) integration by parts since ϕ has compact support in $]a,b[$.

20: SCHOLIUM If f is absolutely continuous, then its ordinary derivative f' is a weak derivative.

One direction of the theorem is immediate. For suppose that $f:]a,b[\to R$ admits an absolutely continuous representative $\bar{f}:]a,b[\to R$ such that $\bar{f}$ and $\bar{f}'$ are in $L^p]a,b[$ -- then the claim is that $f \in W^{1,p}]a,b[$. Of course, $f \in L^p]a,b[$. As for the existence of the weak derivative $\frac{df}{dx}$, note that $\forall\ \phi \in C_c^\infty]a,b[$,

$$\int_{]a,b[} \phi\, \bar{f}' dL^1 = - \int_{]a,b[} \phi' \, \bar{f} dL^1$$

or still,

$$\int_{]a,b[} \phi\, \bar{f}' dL^1 = - \int_{]a,b[} \phi' \, f dL^1,$$

since $\bar{f} = f$ almost everywhere. Therefore $\bar{f}'$ is a weak derivative of f in $L^p]a,b[$.

Turning to the converse, let $f \in W^{1,p}]a,b[$, fix a point $x_0 \in\ ]a,b[$, and put

$$\bar{f}(x) = f(x_0) + \int_{x_0}^{x} \frac{df}{dx}\, dL^1 \qquad (x \in]a,b[).$$

Then $\bar{f} \in AC]a,b[$ and almost everywhere,

$$\bar{f}' = \frac{df}{dx} \; (\in L^p]a,b[)$$

i.e., almost everywhere,

$$\bar{f}' - \frac{df}{dx} = 0,$$

or still, almost everywhere,

$$\frac{d}{dx}(\bar{f} - f) = 0,$$

which implies that there exists a constant C such that $\bar{f} - f = C$ almost everywhere, thus f has an absolutely continuous representative $\bar{f}$ such that it and its ordinary derivative belong to $L^p]a,b[$.

21: REMARK Matters simplify slightly when $p = 1$: $f \in W^{1,1}]a,b[$ iff f admits an absolutely continuous representative $\bar{f}$.