Analysis 101:

Curves and Length

ABSTRACT

In addition to providing a systematic account of the classical theorems of Jordan and Tonelli, I have also provided an introduction to the theory of the Weierstrass integral which in its definitive form is due to Cesari.

CURVES AND LENGTH

§1. FUNDAMENTALS

1: NOTATION Given

$$\underline{x} = (x_1,\ldots,x_M) \in R^M \quad (M = 1,2,\ldots),$$

put

$$||\underline{x}|| = (x_1^2 + \cdots + x_M^2)^{1/2},$$

hence

$$|x_m| \leq ||\underline{x}|| \leq |x_1| + \cdots + |x_M| \quad (m = 1,\ldots,M).$$

2: DEFINITION A function $\underline{f}$: [a,b] → R^M is said to be a curve C, denoted C <—>$\underline{f}$, where

$$\underline{f}(x) = (f_1(x),\ldots,f_M(x)) \quad (a \leq x \leq b).$$

3: EXAMPLE Every function f: [a,b] → R gives rise to a curve C in R^2, viz. the arrow x → (x,f(x)).

4: DEFINITION The graph of C, denoted [C], is the range of $\underline{f}$.

5: EXAMPLE Take M = 2, let k = 1,2,..., and put

$$\underline{f}_k(x) = (\sin^2(kx),0) \quad (0 \leq x \leq \tfrac{\pi}{2}).$$

Then the $\underline{f}_k$ all have the same range, i.e., $[C_1] = [C_2] = \cdots$ if C_k <—> $\underline{f}_k$ but the C_k are different curves.

6: REMARK If C is a continuous curve, then its graph [C] is closed, bounded, connected, and uniformly locally connected. Owing to a theorem of Hahn and Mazurkiewicz, these properties are characteristic: Any such set is the graph of a continuous curve. So, e.g., a square in R^2 is the graph of a continuous curve, a cube in R^3 is the graph of a continuous curve etc.

<u>7:</u> DEFINITION The <u>length</u> of a curve C, denoted $\ell(C)$, is

$$T_{\underline{f}}[a,b] \equiv \sup_{P\in\mathcal{P}[a,b]} \sum_{i=1}^{n} ||\underline{f}(x_i) - \underline{f}(x_{i-1})||,$$

C being termed <u>rectifiable</u> if $\ell(C) < +\infty$.

[Note: If C is continuous and rectifiable, then $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$:

$$||P|| < \delta \Rightarrow \overset{b}{\underset{a}{V}} (\underline{f};P) \equiv \sum_{i=1}^{n} ||\underline{f}(x_i) - \underline{f}(x_{i-1})|| > \ell(C) - \varepsilon.]$$

<u>8:</u> LEMMA Given a curve C,

$$T_{f_m}[a,b] \leq \ell(C) \leq T_{f_1}[a,b] + \cdots + T_{f_M}[a,b] \quad (1 \leq m \leq M).$$

<u>9:</u> SCHOLIUM C is rectifiable iff

$$f_1 \in BV[a,b],\ldots,f_M \in BV[a,b].$$

<u>10:</u> THEOREM Let

$$\left[\begin{array}{l} C_n \longleftrightarrow \underline{f_n}:[a,b] \rightarrow R^M \\ \\ C \longleftrightarrow \underline{f}:[a,b] \rightarrow R^M \end{array}\right.$$

and assume that $\underline{f_n}$ converges pointwise to $\underline{f}$ -- then

$$\ell(C) \leq \liminf_{n\rightarrow\infty} \ell(C_n).$$

A continuous curve

$$\Gamma \longleftrightarrow \gamma : [a,b] \to R^M$$

is said to be a <u>polygonal line</u> (and $\underline{\gamma}$ <u>quasi linear</u> in [a,b]) if there exists a $P \in \mathcal{P}[a,b]$ in each segment of which $\underline{\gamma}$ is linear or a constant.

<u>11:</u> DEFINITION The <u>elementary length</u> $\ell_e(\Gamma)$ of Γ is the sum of the lengths of these segments, hence $\ell_e(\Gamma) = \ell(\Gamma)$.

<u>12:</u> NOTATION Given a continuous curve C, denote by $\Gamma(C)$ the set of all sequences

$$\Gamma_n \longleftrightarrow \underline{\gamma_n} : [a,b] \to R^M$$

of polygonal lines such that

$$\gamma_n \to \underline{f} \quad (n \to \infty)$$

uniformly in [a,b].

Therefore

$$\ell(C) \leq \liminf_{n \to \infty} \ell(\Gamma_n) = \liminf_{n \to \infty} \ell_e(\Gamma_n).$$

On the other hand, by definition, there is some $\{\Gamma_n\} \in \Gamma(C)$ such that

$$\ell_e(\Gamma_n) \to \ell(C) \quad (n \to \infty).$$

<u>13:</u> SCHOLIUM If C is a continuous curve, then

$$\ell(C) = \inf_{\{\Gamma_n\} \in \Gamma(C)} [\liminf_{n \to \infty} \ell_e(\Gamma_n)].$$

<u>14:</u> REMARK Let

$$C \longleftrightarrow f: [a,b] \to R^M.$$

Assume: C is continuous and rectifiable -- then f can be decomposed as a sum $f = f_{AC} + f_C$, where f_{AC} is absolutely continuous and f_C is continuous and singular. Therefore

$$\ell(C) = T_{f_{AC}}[a,b] + T_{f_C}[a,b].$$

§2. *ESTIMATES*

1: NOTATION Write

$$T_{\underline{f}}[a,b]$$

in place of $\ell(C)$.

2: DEFINITION Assume that C is rectifiable — then the arc length function

$$s:[a,b] \to R$$

is defined by the prescription

$$s(x) = T_{\underline{f}}[a,x] \quad (a \leq x \leq b).$$

Obviously

$$s(a) = 0, \ s(b) = \ell(C),$$

and s is an increasing function.

3: LEMMA If C is continuous and rectifiable, then s is continuous as are the $T_{f_m}[a,—]$ $(m = 1,\ldots,M)$.

4: LEMMA If C is continuous and rectifiable, then s is absolutely continuous iff all the $T_{f_m}[a,—]$ $(m = 1,\ldots,M)$ are absolutely continuous, hence iff all the f_m $(m = 1,\ldots,M)$ are absolutely continuous.

If C is continuous and rectifiable, then the $f_m \in BV[a,b]$, thus the derivatives f_m' exist almost everywhere in [a,b] and are Lebesgue integrable. On the other hand, s is an increasing function, thus it too is differentiable almost everywhere in [a,b] and is Lebesgue integrable.

5: SUBLEMMA The connection between $\underline{f}'$ and s' is given by the relation

$$||\underline{f}'|| \leq s'$$

almost everywhere in [a,b].

[For any subinterval $[\alpha,\beta] \subset [a,b]$,

$$||\underline{f}(\beta) - \underline{f}(\alpha)|| \leq s(\beta) - s(\alpha).]$$

6: LEMMA

$$\ell(C) = s(b) - s(a) \geq \int_a^b s' \geq \int_a^b ||\underline{f}'||.$$

I.e.: Under the assumption that C is continuous and rectifiable,

$$\ell(C) \geq \int_a^b ||\underline{f}'||.$$

7: THEOREM

$$\ell(C) = \int_a^b ||\underline{f}'||$$

iff all the f_m ($m = 1,\ldots,M$) are absolutely continuous.

This is established in the discussion to follow.

- Suppose that the equality sign obtains, hence

$$s(b) - s(a) = \int_a^b s'.$$

But also

$$s(x) - s(a) \geq \int_a^x s', \quad s(b) - s(x) \geq \int_x^b s'.$$

If

$$s(x) - s(a) > \int_a^x s', \quad s(b) - s(x) \geq \int_x^b s',$$

then

$$s(b) - s(a) > \int_a^b s',$$

a contradiction. Therefore

$$s(x) - s(a) = \int_a^x s'$$

$$\Rightarrow s \in AC[a,b] \Rightarrow f_m \in AC[a,b] \quad (m = 1,\ldots,M).$$

- Consider the other direction, i.e., assume that the $f_m \in AC[a,b]$, the claim being that

$$\ell(C) = \int_a^b ||\underline{f}'||.$$

Given $P \in \mathcal{P}[a,b]$, write

$$\sum_{i=1}^{n} ||\underline{f}(x_i) - \underline{f}(x_{i-1})||$$

$$= \sum_{i=1}^{n} [\sum_{m=1}^{M} (f_m(x_i) - f_m(x_{i-1}))^2]^{1/2}$$

$$= \sum_{i=1}^{n} [\sum_{m=1}^{M} (\int_{x_{i-1}}^{x_i} f_m')^2]^{1/2}$$

$$\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (\sum_{m=1}^{M} (f_m')^2)^{1/2}$$

$$= \int_a^b ||\underline{f}'||.$$

Taking the sup of the first term over all P then gives

$$\ell(C) \leq \int_a^b ||\underline{f}'|| \ (\leq \ell(C))$$

$$\Rightarrow$$

$$\ell(C) = \int_a^b ||\underline{f}'||.$$

<u>8:</u> <u>N.B.</u> Under canonical assumptions,

$$((\int_X \phi_1)^2 + \cdots + (\int_X \phi_n)^2)^{1/2}$$

$$\leq \int_X (\phi_1^2 + \cdots + \phi_n^2)^{1/2}.$$

9: RAPPEL Suppose that $f \in BV[a,b]$ -- then for almost all $x \in [a,b]$,

$$|f'(x)| = T_f'[a,x].$$

10: LEMMA Suppose that C is continuous and rectifiable -- then

$$s' = ||\underline{f}'||$$

almost everywhere in $[a,b]$.

PROOF Since

$$||\underline{f}'|| \leq s',$$

it suffices to show that

$$s' \leq ||\underline{f}'||.$$

Let $E_0 \subset [a,b]$ be the set of x such that $\underline{f}$ and s are differentiable at x and $s'(x) > ||\underline{f}'(x)||$ and for $k = 1,2,\ldots$, let E_k be the set of $x \in E_0$ such that

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1} \geq \frac{||\underline{f}(t_2) - \underline{f}(t_1)||}{t_2 - t_1} + \frac{1}{k}$$

for all intervals $[t_1,t_2]$ such that $x \in [t_1,t_2]$ and $0 < t_2 - t_1 \leq \frac{1}{k}$. So, by construction,

$$E_0 = \bigcup_{k=1}^{\infty} E_k$$

and matters reduce to establishing that $\forall\, k$, $\lambda(E_k) = 0$. To this end, let $\varepsilon > 0$ and choose $P \in \mathcal{P}[a,b]$:

$$\sum_{i=1}^{n} ||\underline{f}(x_i) - \underline{f}(x_{i-1})|| > T_{\underline{f}}[a,b] - \varepsilon.$$

Expanding P if necessary, it can be assumed without loss of generality that

$$0 < x_i - x_{i-1} \le \frac{1}{k} \quad (i = 1,\ldots,n).$$

For each i, either $[x_{i-1},x_i] \cap E_k \neq \emptyset$ and then

$$s(x_i) - s(x_{i-1}) \ge ||\underline{f}(x_i) - \underline{f}(x_{i-1})|| + \frac{x_i - x_{i-1}}{k},$$

or $[x_{i-1},x_i] \cap E_k = \emptyset$ and then

$$s(x_i) - s(x_{i-1}) \ge ||\underline{f}(x_i) - \underline{f}(x_{i-1})||.$$

Consequently

$$\begin{aligned} T_{\underline{f}}[a,b] &= s(b) = s(x_n) \\ &= \sum_{i=1}^{n} (s(x_i) - s(x_{i-1})) \quad (s(x_0) = s(a) = 0) \\ &\ge \sum_{i=1}^{n} ||\underline{f}(x_i) - \underline{f}(x_{i-1})|| + \frac{1}{k}\lambda^*(E_k) \\ &\ge T_{\underline{f}}[a,b] - \varepsilon + \frac{1}{k}\lambda^*(E_k) \end{aligned}$$

$\Rightarrow$

$$\lambda^*(E_k) \le k\varepsilon \Rightarrow \lambda(E_k) = 0 \quad (\varepsilon \downarrow 0).$$

<u>11:</u> THEOREM Suppose that C is continuous and rectifiable. Assume: $M > 1$ -- then the M-dimensional Lebesgue measure of [C] is equal to 0.

<u>12:</u> NOTATION Let

$$C \longleftrightarrow \underline{f}:[a,b] \to R^M$$

be a continuous curve. Given $\underline{x} \in [C]$, let $N(\underline{f};\underline{x})$ be the number of points $x \in [a,b]$ (finite or infinite) such that $f(x) = \underline{x}$ and let $N(\underline{f};—) = 0$ in the complement $R^M - [C]$ of [C].

13: THEOREM

$$\ell(C) = \int_{R^M} N(\underline{f};\text{—})\,dH^1.$$

[Note: H^1 is the 1-dimensional Hausdorff outer measure in R^M and

$$H^1([C]) = \int_{R^M} \chi_{[C]}\,dH^1 \leq \int_{R^M} N(\underline{f};\text{—})\,dH^1,$$

i.e.,

$$H^1([C]) \leq \ell(C)$$

and it can happen that

$$H^1([C]) < \ell(C).]$$

14: N.B. If $\underline{f}$ is one-to-one, then

$$N(\underline{f};\text{—}) = \chi_{[C]}$$

and when this is so,

$$H^1([C]) = \ell(C).$$

§3. *EQUIVALENCES*

In what follows, by <u>interval</u> we shall understand a finite closed interval $\subset R$.

[Note: If I,J are intervals and if $\partial I = \{a,b\}$, $\partial J = \{c,d\}$, then the agreement is that a homeomorphism $\phi: I \to J$ is sense preserving, i.e., sends a to c and b to d.]

<u>1:</u> DEFINITION Suppose given intervals I,J, and curves $\underline{f}: I \to R^M$, $\underline{g}: J \to R^M$ -- then $\underline{f}$ and $\underline{g}$ are said to be <u>Lebesgue equivalent</u> if there exists a homeomorphism $\phi: I \to J$ such that $\underline{f} = \underline{g} \circ \phi$.

<u>2:</u> LEMMA If

$$\begin{cases} \underline{f}: [a,b] \to R^M \\ \underline{g}: [a,b] \to R^M \end{cases}$$

are Lebesgue equivalent and if

$$\begin{cases} C \longleftrightarrow \underline{f} \\ D \longleftrightarrow \underline{g}, \end{cases}$$

then

$$\ell(C) = \ell(D).$$

PROOF The homeomorphism $\phi: [a,b] \to [c,d]$ induces a bijection

$$\begin{cases} \mathcal{P}[a,b] \to \mathcal{P}[c,d] \\ \quad P \to Q. \end{cases}$$

Therefore

$$\ell(C) = \sup_{P \in \mathcal{P}[a,b]} \sum_{i=1}^{n} ||\underline{f}(x_i) - \underline{f}(x_{i-1})||$$

$$= \sup_{P \in \mathcal{P}[a,b]} \sum_{i=1}^{n} ||\underline{g}(\phi(x_i)) - \underline{g}(\phi(x_{i-1}))||$$

$$= \sup_{Q \in \mathcal{P}[c,d]} \sum_{i=1}^{n} ||\underline{g}(y_i) - \underline{g}(y_{i-1})||$$

$$= \ell(D).$$

3: DEFINITION Suppose given intervals I,J and curves $\underline{f}:I \to R^M$, $\underline{g}:J \to R^M$ -- then $\underline{f}$ and $\underline{g}$ are said to be Fréchet equivalent if for every $\varepsilon > 0$ there exists a homeomorphism $\phi:I \to J$ such that

$$||\underline{f}(x) - \underline{g}(\phi(x))|| < \varepsilon \quad (x \in I).$$

4: REMARK It is clear that two Lebesgue equivalent curves are Fréchet equivalent but two Fréchet equivalent curves need not be Lebesgue equivalent.

5: LEMMA If

$$\begin{cases} \underline{f}:[a,b] \to R^M \\ \underline{g}:[a,b] \to R^M \end{cases}$$

are Fréchet equivalent and if

$$\begin{cases} C \longleftrightarrow \underline{f} \\ D \longleftrightarrow \underline{g}, \end{cases}$$

then

$$\ell(C) = \ell(D).$$

PROOF For each n = 1,2,..., there is a homeomorphism $\phi_n:[a,b] \to [c,d]$ such that $\forall\, x \in [a,b]$,

$$||\underline{f}(x) - \underline{g}(\phi_n(x))|| < \frac{1}{n}.$$

Put $\underline{f}_n = \underline{g} \circ \phi_n$, hence $\underline{f}_n$ is Lebesgue equivalent to $\underline{g}$ (viz. $\underline{g} \circ \phi_n = \underline{g} \circ \phi_n \ldots$),

thus if

$$C_n \longleftrightarrow \underline{f}_n, \; D \longleftrightarrow \underline{g},$$

then from the above

$$\ell(C_n) = \ell(D).$$

But $\forall\, x \in [a,b]$,

$$||\underline{f}(x) - \underline{f}_n(x)|| < \frac{1}{n},$$

i.e., $\underline{f}_n \to \underline{f}$ pointwise, so

$$\begin{aligned} \ell(C) &\leq \liminf_{n \to \infty} \ell(C_n) \\ &= \liminf_{n \to \infty} \ell(D) \\ &= \ell(D). \end{aligned}$$

Analogously

$$\ell(D) \leq \ell(C).$$

Therefore

$$\ell(C) = \ell(D).$$

§4. FRÉCHET DISTANCE

Let

$$\begin{cases} C \Longleftrightarrow \underline{f}: [a,b] \rightarrow R^M \\ D \Longleftrightarrow \underline{g}: [a,b] \rightarrow R^M \end{cases}$$

be two continuous curves.

1: NOTATION H is the set of all homeomorphisms $\phi: [a,b] \rightarrow [c,d]$ $(\phi(a) = c, \phi(b) = d)$.

Given $\phi \in H$, the expression

$$||\underline{f}(x) - \underline{g}(\phi(x))|| \quad (a \leq x \leq b)$$

has an absolute maximum $M(\underline{f},\underline{g};\phi)$.

2: DEFINITION The Fréchet distance between C and D, denoted $||C,D||$, is

$$\inf_{\phi \in H} M(\underline{f},\underline{g};\phi).$$

[Note: In other words, $||C,D||$ is the infimum of all numbers $\varepsilon \geq 0$ with the property that there exists a homeomorphism $\phi \in H$ such that

$$||\underline{f}(x) - \underline{g}(\phi(x))|| \leq \varepsilon$$

for all $x \in [a,b]$.]

3: N.B. If $||C,D|| < \varepsilon$, then there exists a $\phi \in H$ such that

$$M(\underline{f},\underline{g};\phi) < \varepsilon.$$

4: LEMMA Let C, D, C_0 be continuous curves -- then

(i) $||C,D|| \geq 0$;

(ii) $||C,D|| = ||D,C||$;

(iii) $||C,D|| \leq ||C,C_0|| + ||C_0,D||$;

(iv) $||C,D|| = 0$ iff C and D are Fréchet equivalent.

Therefore the Fréchet distance is a premetric on the set of all continuous curves with values in R^M.

<u>5:</u> THEOREM Let

$$\begin{cases} C_n \longleftrightarrow \underline{f}_n: [a_n, b_n] \to R^M \quad (n = 1,2,\ldots) \\ C \longleftrightarrow \underline{f}: [a,b] \to R^M \end{cases}$$

be continuous curves. Assume:

$$||C_n, C|| \to 0 \quad (n \to \infty).$$

Then

$$\ell(C) \leq \liminf_{n \to \infty} \ell(C_n).$$

PROOF For every n, there is a homeomorphism

$$\phi_n: [a,b] \to [a_n, b_n] \quad (\phi_n(a) = a_n, \ \phi_n(b) = b_n)$$

such that for all $x \in [a,b]$,

$$||\underline{f}(x) - \underline{f}_n(\phi_n(x))|| < ||C, C_n|| + \frac{1}{n}.$$

Let

$$D_n \longleftrightarrow \underline{f}_n \circ \phi_n: [a,b] \to R^M.$$

Then pointwise

$$\underline{f}_n \circ \phi_n \to \underline{f}$$

=>

$$\ell(C) \leq \liminf_{n \to \infty} \ell(D_n).$$

But $\ell(D_n) = \ell(C_n)$, hence

$$\ell(C) \leq \liminf_{n \to \infty} \ell(C_n).$$

In the set of continuous curves, introduce an equivalence relation by stipulating that C and D are equivalent provided C and D are Fréchet equivalent. The resulting set E_F of equivalence classes is then a metric space: If

$$\begin{cases} \{C\} \in E_F \\ \{D\} \in E_F, \end{cases}$$

then

$$||\{C\},\{D\}|| = ||C,D||.$$

<u>6:</u> <u>N.B.</u> If C, C' are Fréchet equivalent and if D, D' are Fréchet equivalent, then

$$\begin{aligned} ||C,D|| &\leq ||C,C'|| + ||C',D|| \\ &\leq ||C',D|| \leq ||C',D'|| + ||D',D|| \\ &\qquad\qquad = ||C',D'|| \end{aligned}$$

and in reverse

$$||C',D'|| \leq ||C,D||.$$

So

$$||C,D|| = ||C',D'||.$$

§5. *THE REPRESENTATION THEOREM*

Assume:

$$C \longleftrightarrow \underline{f}: [a,b] \to R^M$$

is a curve which is continuous and rectifiable.

1: THEOREM There exists a continuous curve

$$D \longleftrightarrow \underline{g}: [c,d] \to R^M$$

with the property that

$$\ell(D) = \ell(C) \quad (< +\infty)$$

and

$$\ell(D) = \int_c^d ||\underline{g}'||,$$

where $g_1, \ldots, g_M$ are absolutely continuous and in addition $\underline{f}$ and $\underline{g}$ are Fréchet equivalent.

Take $\ell(C) > 0$ and define $\underline{g}$ via the following procedure. In the first place, the domain $[c,d]$ of $\underline{g}$ is going to be the interval $[0,\ell(C)]$. This said, note that $s(x)$ is constant in an interval $[\alpha,\beta]$ iff $\underline{f}(x)$ is constant there as well. Next, for each point s_0 $(0 \leq s_0 \leq \ell(C))$ there is a maximal interval $\alpha \leq x \leq \beta$ $(a \leq \alpha \leq \beta \leq b)$ with $s(x) = s_0$. Definition: $\underline{g}(s_0) = \underline{f}(x)$ $(\alpha \leq x \leq \beta)$.

2: LEMMA

$$\begin{cases} \underline{g}(s_0-) = \underline{g}(s_0) & (0 < s_0 \leq \ell(C)) \\ \underline{g}(s_0+) = \underline{g}(s_0) & (0 \leq s_0 < \ell(C)). \end{cases}$$

Therefore

$$\underline{g}: [c,d] \to R^M$$

is a continuous curve.

<u>3:</u> SUBLEMMA Suppose that $\phi_n: [A,B] \to [C,D]$ $(n = 1,2,\ldots)$ converges uniformly to $\phi: [A,B] \to [C,D]$. Let $\Phi: [C,D] \to R^M$ be a continuous function -- then $\Phi \circ \phi_n$ converges uniformly to $\Phi \circ \phi$.

PROOF Since Φ is uniformly continuous, given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|u - v| < \delta \Rightarrow ||\Phi(u) - \Phi(v)|| < \varepsilon \quad (u,v \in [C.D]).$$

Choose N:

$$n \geq N \Rightarrow |\phi_n(x) - \phi(x)| < \delta \quad (x \in [A,B]).$$

Then

$$||\Phi(\phi_n(x)) - \Phi(\phi(x))|| < \varepsilon.$$

<u>4:</u> LEMMA $\underline{f}$ and $\underline{g}$ are Fréchet equivalent.

PROOF Approximate s by quasilinear, strictly increasing functions $s_n(x)$ $(a \leq x \leq b)$ with $s_n(a) = 0$, $s_n(b) = \ell(C)$ and

$$|s_n(x) - s(x)| < \frac{1}{n} \quad (n = 1,2,\ldots).$$

Then

$$s_n: [a,b] \to [0,\ell(C)]$$

converges uniformly to

$$s: [a,b] \to [0,\ell(C)]$$

and

$$\underline{g}: [0,\ell(C)] \to R^M$$

is continuous, so

$$\underline{g} \circ s_n \to \underline{g} \circ s$$

uniformly in [a,b], thus $\forall\ \varepsilon > 0,\ \exists\ N : n \geq N$

$$\Rightarrow\ ||\underline{g}(s_n(x)) - \underline{g}(s(x))|| < \varepsilon \quad (a \leq x \leq b)$$

or still,

$$||\underline{f}(x) - \underline{g}(s_n(x))|| < \varepsilon \quad (a \leq x \leq b).$$

Since the s_n are homeomorphisms, it follows that $\underline{f}$ and $\underline{g}$ are Fréchet equivalent.

<u>5:</u> LEMMA

$$0 \leq u < v \leq \ell(C)$$

$\Rightarrow$

$$||\underline{g}(v) - \underline{g}(u)|| = v - u$$

$\Rightarrow$

$$|\ g_m(v) - g_m(u)\ | \leq v - u \quad (1 \leq m \leq M).$$

Consequently $g_1, \ldots, g_M$ are absolutely continuous (in fact, Lipschitz).

<u>6:</u> LEMMA

$$\ell(C) = \ell(D) = \int_0^{\ell(D)} ||\underline{g}'||,$$

where $||\underline{g}'|| \leq 1$.

So

$$0 = \ell(D) - \int_0^{\ell(D)} ||g'||$$

$$= \int_0^{\ell(D)} 1 - \int_0^{\ell(D)} ||g'||$$

$$= \int_0^{\ell(D)} (1 - ||g'||)$$

implying thereby that $||g'|| = 1$ almost everywhere.

§6. *INDUCED MEASURES*

1: NOTATION BO[a,b] is the set of Borel subsets of [a,b].

Let

$$C \longleftrightarrow \underline{f}: [a,b] \to R^M$$

be a curve, continuous and rectifiable.

2: LEMMA The interval function defined by the rule

$$[c,d] \to s(d) - s(c) \quad ([c,d] \subset [a,b])$$

can be extended to a measure μ_C on BO[a,b].

3: LEMMA For m = 1,...,M, the interval function defined by the rule

$$[c,d] \to T_{f_m}[c,d] \quad ([c,d] \subset [a,b])$$

can be extended to a measure μ_m on BO[a,b].

4: FACT Given S ∈ BO[a,b],

$$\mu_m(S) \leq \mu_C(S) \leq \mu_1(S) + \cdots + \mu_M(S).$$

5: LEMMA For m = 1,...,M, the interval functions defined by the rule

$$\left[\begin{array}{l} [c,d] \to T_{f_m}^{+}[c,d] \\ \\ [c,d] \to T_{f_m}^{-}[c,d] \end{array}\right. \qquad ([c,d] \subset [a,b])$$

can be extended to measures

$$\left[\begin{array}{l} \mu_m^{+} \\ \mu_m^{-} \end{array}\right.$$

on BO[a,b].

6: NOTATION Put

$$\nu_m = \mu_m^+ - \mu_m^- \quad (m = 1,\ldots,M).$$

[Thus ν_m is a countably additive, totally finite set function on BO[a,b].]

7: RECOVERY PRINCIPLE For any $S \in$ BO[a,b],

$$\mu_C(S) = \sup_{\{P\}} \sum_{E \in P} \{\sum_{m=1}^{M} \nu_m(E)^2\}^{1/2},$$

where the supremum is taken over all partitions P of S into disjoint Borel measurable sets E.

8: FACT The set functions μ_m, μ_m^+, μ_m^-, ν_m are absolutely continuous w.r.t. μ_C.

9: NOTATION The corresponding Radon-Nikodym derivatives are denoted by

$$\beta_m = \frac{d\mu_m}{d\mu_C}, \quad \left[\begin{array}{l} \beta_m^+ = \dfrac{d\mu_m^+}{d\mu_C} \\ \\ \beta_m^- = \dfrac{d\mu_m^-}{d\mu_C} \end{array} \right., \quad \Theta_m = \frac{d\nu_m}{d\mu_C}.$$

10: CONVENTION The term almost everywhere (or measure 0) will refer to the measure space

$$([a,b],\ \mathrm{BO}[a,b],\ \mu_C).$$

11: FACT

$$\beta_m = \beta_m^+ + \beta_m^-$$

and $\qquad (m = 1,\ldots,M)$

$$\Theta_m = \beta_m^+ - \beta_m^-$$

almost everywhere.

12: NOTATION Let

$$\underline{\Theta} = (\Theta_1,\ldots,\Theta_M).$$

[Note: By definition,

$$||\underline{\Theta}(x)|| = (\Theta_1(x)^2 + \cdots + \Theta_M(x)^2)^{1/2}.]$$

13: NOTATION Given a linear orthogonal transformation $\lambda: R^M \to R^M$, let $\overline{C} = \lambda C$.

14: N.B.

$$\mu_{\overline{C}} = \mu_C.$$

15: LEMMA

$$(\overline{\nu}_1,\ldots,\overline{\nu}_M) = \lambda(\nu_1,\ldots,\nu_M).$$

16: APPLICATION

$$(\overline{\Theta}_1,\ldots,\overline{\Theta}_M) = \lambda(\Theta_1,\ldots,\Theta_M)$$

almost everywhere.

[Differentiate the preceding relation w.r.t. $\mu_{\overline{C}} = \mu_C$.]

17: LEMMA

$$|\Theta_m| \leq 1 \quad (m = 1,\ldots,M)$$

almost everywhere, so

$$||\underline{\Theta}|| \leq M^{1/2}$$

almost everywhere.

18: THEOREM

$$||\underline{\Theta}|| = 1$$

almost everywhere.

PROOF Let $0 < \delta < 1$ and let

$$S = \{x: ||\underline{\Theta}(x)|| < 1 - \delta\}.$$

Then

$$\mu_C(S) = \sup_{\{P\}} \sum_{E \in P} \{\sum_{m=1}^{M} \nu_m(E)^2\}^{1/2}.$$

But

$$\nu_m(E) = \int_E \frac{d\nu_m}{d\mu_C} d\mu_C$$

$$= \int_E \Theta_m \, d\mu_C.$$

Therefore

$$\{\sum_{m=1}^{M} \nu_m(E)^2\}^{1/2}$$

$$= \{\sum_{m=1}^{M} (\int_E \Theta_m \, d\mu_C)^2\}^{1/2}$$

$$\leq \int_E \{\sum_{m=1}^{M} \Theta_m^2\}^{1/2} d\mu_C$$

$$= \int_E ||\underline{\Theta}(x)||\, d\mu_C$$

$$\leq (1 - \delta) \int_E d\mu_C$$

$$= (1 - \delta)\mu_C(E).$$

Since

$$S = \coprod E,$$

it follows that

$$\sum_{E \in P} \{ \sum_{m=1}^{M} \nu_m(E)^2 \}^{1/2} \leq (1 - \delta)\mu_C(S).$$

Taking the supremum over the P then implies that

$$\mu_C(S) \leq (1 - \delta)\mu_C(S),$$

thus $\mu_C(S) = 0$ and $||\underline{\Theta}(x)|| \geq 1$ almost everywhere (let $\delta = \frac{1}{2}, \frac{1}{3}, \ldots$). To derive a contradiction, take $M \geq 2$ and suppose that $||\underline{\Theta}(x)|| \geq 1 + \delta > 1$ on some set T such that $\mu_C(T) > 0$ -- then for some vector

$$\underline{\xi} = (\xi_1, \ldots, \xi_M) \in R^M \quad (||\underline{\xi}|| = 1),$$

the set

$$T(\underline{\xi}) = \{x \in T: || \frac{\underline{\Theta}(x)}{||\underline{\Theta}(x)||} - \underline{\xi} || < \frac{\delta}{M^2}$$

has measure $\mu_C(T(\underline{\xi})) > 0$ (see below). Let

$$\underline{\lambda}_j = (\lambda_{j1}, \ldots, \lambda_{jM}) \quad (j = 2, \ldots, M)$$

be unit vectors such that

$$\lambda = \begin{bmatrix} \xi_1, \ldots, \xi_M \\ \lambda_{21}, \ldots, \lambda_{2M} \\ \vdots \\ \lambda_{M1}, \ldots, \lambda_{MM} \end{bmatrix}$$

is an orthogonal matrix. Viewing λ as a linear orthogonal transformation, form as above $\overline{C} = \lambda C$, hence

$$(\overline{\Theta}_1, \ldots, \overline{\Theta}_M) = \lambda(\Theta_1, \ldots, \Theta_M).$$

On $T(\underline{\xi})$,

$$|\overline{\Theta}_j| = |\lambda_{j1}\Theta_1 + \cdots + \lambda_{2M}\Theta_M|$$

$$\leq ||\underline{\Theta}|| \frac{\delta}{M^2}$$

$$\leq M^{1/2} \frac{\delta}{M^2} \leq M \frac{\delta}{M^2} = \frac{\delta}{M},$$

while

$$||\overline{\underline{\Theta}}|| \leq |\overline{\Theta}_1| + \cdots + |\overline{\Theta}_M|$$

$\Rightarrow$

$$|\overline{\Theta}_1| \geq ||\overline{\underline{\Theta}}|| - |\overline{\Theta}_2| - \cdots - |\overline{\Theta}_M|$$

$$\geq (1 + \delta) - (M - 1) \frac{\delta}{M} = 1 + \frac{\delta}{M}.$$

However

$$|\overline{\Theta}_1| \leq 1,$$

so we have a contradiction.

19: N.B. Let $\{\underline{\xi}_n : n \in N\}$ be a dense subset of the unit sphere $U(M)$ in R^M (thus $\forall$ n, $||\underline{\xi}_n|| = 1$). Given a point $x \in T$, pass to

$$\frac{\underline{\Theta}(x)}{||\underline{\Theta}(x)||} \in U(M).$$

Then there exists a $\underline{\xi}_{n_x}$:

$$\left|\left| \frac{\underline{\Theta}(x)}{||\underline{\Theta}(x)||} - \underline{\xi}_{n_x} \right|\right| < \frac{\delta}{M^2},$$

a point in the $\frac{\delta}{M^2}$ - neighborhood of

$$\frac{\underline{\Theta}(x)}{||\underline{\Theta}(x)||}$$

in $U(M)$. Therefore

$$T = \bigcup_{n=1}^{\infty} T(\underline{\xi}_n)$$

=>

$$0 < \mu_C(T) \leq \sum_{n=1}^{\infty} \mu_C(T(\underline{\xi}_n))$$

=> $\exists$ n:

$$\mu_C(T(\underline{\xi}_n)) > 0.$$

§7. *TWO THEOREMS*

Let

$$C \longleftrightarrow \underline{f}:[a,b] \rightarrow R^M$$

be a curve, continuous and rectifiable.

Let $P \in \mathcal{P}[a,b]$, say

$$P: a = x_0 < x_1 < \cdots < x_n = b.$$

<u>1:</u> DEFINITION Let $i = 1,\ldots,n$ and for $m = 1,\ldots,M$ let

$$\eta_m(x;P) = \frac{f_m(x_i) - f_m(x_{i-1})}{\mu_C([x_{i-1},x_i])},$$

where $x_{i-1} < x < x_i$ if $\mu_C([x_{i-1},x_i]) \neq 0$ and let

$$\eta_m(x;P) = 0,$$

where $x_{i-1} < x < x_i$ if $\mu_C([x_{i-1},x_i]) = 0$.

<u>2:</u> NOTATION

$$\underline{\eta}(x;P) = (\eta_1(x;P),\ldots,\eta_M(x;P)).$$

<u>3:</u> THEOREM

$$\int_a^b ||\underline{\Theta}(x) - \underline{\eta}(x;P)||^2 \, d\mu_C$$

$$\leq 2[\ell(C) - \sum_{i=1}^{n} ||\underline{f}(x_i) - \underline{f}(x_{i-1})||].$$

PROOF Given $P \in \mathcal{P}[a,b]$, let Σ' denote a sum over intervals $[x_{i-1},x_i]$, where $||\underline{\eta}(x;P)||^2 \neq 0$ and let Σ'' denote a sum over what remains. Now compute:

$$\int_a^b ||\underline{\Theta}(x) - \underline{\eta}(x;P)||^2 \, d\mu_C$$

$$= \Sigma' \int_{x_{i-1}}^{x_i} ||\underline{\Theta}(x) - \underline{\eta}(x;P)||^2 \, d\mu_C$$

$$+ \Sigma'' \int_{x_{i-1}}^{x_i} ||\underline{\Theta}(x)||^2 \, d\mu_C$$

$$= \Sigma' \int_{x_{i-1}}^{x_i} [||\underline{\Theta}(x)||^2 + ||\underline{\eta}(x;P)||^2 - 2\underline{\Theta}(x) \cdot \underline{\eta}(x;P)] \, d\mu_C$$

$$+ \Sigma'' \int_{x_{i-1}}^{x_i} ||\underline{\Theta}(x)^2|| d\mu_C$$

$$= \Sigma' \int_{x_{i-1}}^{x_i} [1 + ||\underline{\eta}(x;P)||^2 - 2\underline{\Theta}(x) \cdot \underline{\eta}(x;P)] d\mu_C$$

$$+ \Sigma'' \int_{x_{i-1}}^{x_i} 1 \, d\mu_C$$

$$= \Sigma' \, [\mu_C([x_{i-1},x_i])$$

$$+ \left[\frac{||\underline{f}(x_i) - \underline{f}(x_{i-1})||}{\mu_C([x_{i-1},x_i])} \right]^2 \mu_C([x_{i-1},x_i])$$

$$- 2 \frac{||\underline{f}(x_i) - \underline{f}(x_{i-1})||^2}{\mu_C([x_{i-1},x_i])} + \Sigma'' \, \mu_C([x_{i-1},x_i])$$

$$\leq \ell(C) - \Sigma' \frac{||\underline{f}(x_i) - \underline{f}(x_{i-1})||^2}{\mu_C([x_{i-1},x_i])}$$

$$\leq \ell(C) - \Sigma' \, ||\underline{f}(x_i) - \underline{f}(x_{i-1})||$$

$$+ \Sigma' \, ||\underline{f}(x_i) - \underline{f}(x_{i-1})|| \, (1 - \frac{||\underline{f}(x_i) - \underline{f}(x_{i-1})||}{\mu_C([x_{i-1},x_i])})$$

$$\leq \ell(C) - \Sigma' \, ||\underline{f}(x_i) - \underline{f}(x_{i-1})||$$

$$+ \Sigma' \mu_C([x_{i-1},x_i]) \, (1 - \frac{||\underline{f}(x_i) - \underline{f}(x_{i-1})||}{\mu_C([x_{i-1},x_i])})$$

$$\leq \ell(C) - \Sigma' \, ||\underline{f}(x_i) - \underline{f}(x_{i-1})||$$

$$+ \Sigma' (\mu_C([x_{i-1},x_i]) - ||\underline{f}(x_i) - \underline{f}(x_{i-1})||)$$

$$\leq \ell(C) - \Sigma' \, ||\underline{f}(x_i) - \underline{f}(x_{i-1})||$$

$$+ \Sigma' \mu_C([x_{i-1},x_i]) - \Sigma' \, ||\underline{f}(x_i) - \underline{f}(x_{i-1})||$$

$$= \ell(C) + \Sigma' \mu_C([x_{i-1},x_i]) - 2 \, \Sigma' \, ||\underline{f}(x_i) - \underline{f}(x_{i-1})||$$

$$= \ell(C) + \ell(C) - 2 \, \Sigma' \, ||\underline{f}(x_i) - \underline{f}(x_{i-1})||$$

$$= 2[\ell(C) - \Sigma' \, ||\underline{f}(x_i) - \underline{f}(x_{i-1})||]$$

$$= 2[\ell(C) - \sum_{i=1}^{n} ||\underline{f}(x_i) - \underline{f}(x_{i-1})||].$$

4: N.B. By definition, $\mu_C([x_{i-1},x_i])$ is the length of the restriction of C to $[x_{i-1},x_i]$, i.e.,

$$\mu_C([x_{i-1},x_i]) = s(x_i) - s(x_{i-1}).$$

Moreover

$$||\underline{f}(x_i) - \underline{f}(x_{i-1}|| \leq s(x_i) - s(x_{i-1}).$$

So, if $\mu_C([x_{i-1},x_i]) = 0$, then

$$||\underline{f}(x_i) - \underline{f}(x_{i-1})|| = 0 \Rightarrow \underline{f}(x_i) = \underline{f}(x_{i-1})$$

=>

$$\Sigma' \ ||\underline{f}(x_i) - \underline{f}(x_{i-1})||$$

$$= \Sigma' \ ||\underline{f}(x_i) - \underline{f}(x_{i-1})|| + \Sigma'' \ ||\underline{f}(x_i) - \underline{f}(x_{i-1})||$$

$$= \sum_{i=1}^{n} ||\underline{f}(x_i) - \underline{f}(x_{i-1})||.$$

Abbreviate

$$L^2([a,b], BO([a,b]), \mu_C)$$

to

$$L^2(\mu_C).$$

5: APPLICATION In $L^2(\mu_C)$,

$$\lim_{||P|| \to 0} \underline{\eta}(\text{—};P) = \underline{\Theta}.$$

6: SETUP

- $C_0 \longleftrightarrow \underline{f_0}: [a,b] \to R^M$

is a curve, continuous and rectifiable.

- $C_k \longleftrightarrow \underline{f_k}: [a,b] \to R^M \quad (k = 1,2,\ldots)$

is a sequence of curves, continuous and rectifiable.

Assumption: $\underline{f_k}$ converges uniformly to $\underline{f_0}$ in $[a,b]$ and

$$\lim_{k \to \infty} \ell(C_k) = \ell(C_0).$$

7: THEOREM

$$\lim_{||Q|| \to 0} \overset{b}{\underset{a}{V}} (\underline{f_k};Q) = \ell(C_k) \quad (Q \in P[a,b])$$

uniformly in k, i.e., $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ such that

$$||Q|| < \delta \Rightarrow \left|\overset{b}{\underset{a}{V}} (\underline{f_k};Q) - \ell(C_k)\right| < \varepsilon$$

for all k = 1,2,..., or still,

$$||Q|| < \delta \Rightarrow \ell(C_k) - \overset{b}{\underset{a}{V}} (\underline{f_k};Q) < \varepsilon$$

for all k = 1,2,... .

The proof will emerge in the lines to follow. Start the process by choosing $\delta_0 > 0$ such that

$$\ell(C_0) - \overset{b}{\underset{a}{V}} (\underline{f_0};P_0) < \frac{\varepsilon}{4}$$

provided $||P_0|| < \delta_0$. Consider a $P \in P[a,b]$:

$$a = x_0 < x_1 < \cdots < x_n = b$$

with $||P|| < \delta_0$. Choose $\rho > 0$ such that

$$||\underline{f_k}(c) - \underline{f_k}(d)|| < \frac{\varepsilon}{4n} \quad ([c,d] \subset [a,b])$$

for all k = 0,1,2,..., so long as $|c - d| < \rho$ (equicontinuity). Take a partition $Q \in P[a,b]$:

$$a = y_0 < y_1 < \cdots < y_m = b$$

subject to

$$||Q|| < \gamma \equiv \min_{i=1,\ldots,n} \{\rho, \frac{x_i - x_{i-1}}{2}\} \quad (\Rightarrow ||Q|| < \delta_0).$$

Put

$$\sigma_k = \sup_{a\leq x\leq b} ||\underline{f}_k(x) - \underline{f}_0(x)||$$

and let k_0 be such that

$$k > k_0 \Rightarrow \sigma_k < \frac{\varepsilon}{4n} \text{ and } |\ell(C_k) - \ell(C_0)| < \frac{\varepsilon}{4}.$$

The preparations complete, to minimize technicalities we shall suppose that each $I_j = [y_{j-1}, y_j]$ is contained in just one $I_i = [x_{i-1}, x_i]$ and write $\Sigma^{(i)}$ for a sum over all such I_j -- then

$$\begin{aligned}
\overset{b}{\underset{a}{V}} (\underline{f}_k;Q) &= \sum_{j=1}^{m} v(\underline{f}_k;I_j) \\
&= \sum_{j=1}^{m} ||\underline{f}_k(y_j) - \underline{f}_k(y_{j-1})|| \\
&= \sum_{i=1}^{n} \Sigma^{(i)} ||\underline{f}_k(y_j) - \underline{f}_k(y_{j-1})|| \\
&\geq \sum_{i=1}^{n} ||\underline{f}_k(x_i) - \underline{f}_k(x_{i-1})||.
\end{aligned}$$

<u>8:</u> SUBLEMMA Let $\underline{A}, \underline{B}, \underline{C}, \underline{D} \in R^M$ -- then

$$||\underline{C} - \underline{D}|| > ||\underline{A} - \underline{B}|| - ||\underline{A} - \underline{C}|| - ||\underline{B} - \underline{D}||.$$

[In fact,

$$\begin{aligned}
||\underline{A} - \underline{B}|| &= ||\underline{A} - \underline{C} + \underline{C} - \underline{D} + \underline{D} - \underline{B}|| \\
&\leq ||\underline{A} - \underline{C}|| + ||\underline{C} - \underline{D}|| + ||\underline{B} - \underline{D}||.]
\end{aligned}$$

Take

$$\begin{cases} \underline{C} = \underline{f}_k(x_i) \\ \underline{D} = \underline{f}_k(x_{i-1}), \end{cases} \quad \begin{cases} \underline{A} = \underline{f}_0(x_i) \\ \underline{B} = \underline{f}_0(x_{i-1}). \end{cases}$$

Then

$$||\underline{f}_k(x_i) - \underline{f}_k(x_{i-1}||$$

$$\geq ||\underline{f}_0(x_i) - \underline{f}_0(x_{i-1})||$$

$$- ||\underline{f}_0(x_i) - \underline{f}_k(x_i)|| - ||\underline{f}_0(x_{i-1}) - \underline{f}_k(x_{i-1})||,$$

thus

$$\sum_{i=1}^{n} ||\underline{f}_k(x_i) - \underline{f}_k(x_{i-1})||$$

$$\geq \ell(C_0) - \frac{\varepsilon}{4} - n\sigma_k - n\sigma_k$$

$$\geq \ell(C_0) - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} - \frac{\varepsilon}{4}$$

$$= \ell(C_0) - \frac{3\varepsilon}{4} .$$

But

$$k > k_0 \Rightarrow |\ell(C_k) - \ell(C_0)| < \frac{\varepsilon}{4}$$

$$\Rightarrow \ell(C_k) - \frac{\varepsilon}{4} < \ell(C_0).$$

Therefore

$$\ell(C_0) - \frac{3\varepsilon}{4} > \ell(C_k) - \frac{\varepsilon}{4} - \frac{3\varepsilon}{4}$$

$$= \ell(C_k) - \varepsilon.$$

Thus: $\forall\ k > k_0$,

$$\ell(C_k) - \overset{b}{\underset{a}{V}} (\underline{f}_k;Q) < \varepsilon \quad (||Q|| < \gamma).$$

Finally, for $k \leq k_0$, let γ_k be chosen so as to ensure that

$$\ell(C_k) - \overset{b}{\underset{a}{V}} (\underline{f_k};Q) < \varepsilon$$

for all partitions Q with $||Q|| < \gamma_k$. Put now

$$\delta = \min_{1,\ldots,k_0} \{\gamma_1,\ldots,\gamma_{k_0},\gamma\}.$$

Then

$$||Q|| < \delta \Rightarrow \ell(C_k) - \overset{b}{\underset{a}{V}} (\underline{f_k};Q) < \varepsilon$$

for all $k = 1,2,\ldots$.

Changing the notation (replace Q by P), $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that

$$||P|| < \delta \Rightarrow \ell(C_k) - \overset{b}{\underset{a}{V}} (\underline{f_k};P) < \varepsilon$$

for all $k = 1,2,\ldots$. Consequently

$$\int_a^b ||\underline{\Theta_k}(x) - \underline{\eta_k}(x;P)||^2 \, d\mu_{C_k}$$

$$\leq 2[\ell(C_k) - \sum_{i=1}^{n} ||\underline{f_k}(x_i) - \underline{f_k}(x_{i-1})||]$$

$$= 2[\ell(C_k) - \overset{b}{\underset{a}{V}} (\underline{f_k};P)]$$

$$< 2\varepsilon.$$

§8. *LINE INTEGRALS*

Let

$$C \longleftrightarrow \underline{f}: [a,b] \to R^M$$

be a curve, continuous and rectifiable.

Suppose that

$$F: [C] \times R^M \to R,$$

say

$$F(\underline{x},\underline{t}) \quad (\underline{x} \in [C],\ \underline{t} \in R^M).$$

1: DEFINITION F is a parametric integrand if F is continuous in $(\underline{x},\underline{t})$ and $\forall\ K \geq 0$,

$$F(\underline{x},K\underline{t}) = KF(\underline{x},\underline{t}).$$

2: EXAMPLE Let

$$F(\underline{x},\underline{t}) = (t_1^2 + \cdots + t_M^2)^{1/2}.$$

3: EXAMPLE (M = 2) Let

$$F(x_1,x_2,t_1,t_2) = x_1t_2 - x_2t_1.$$

4: N.B. If F is a parametric integrand, then $\forall\ \underline{x}$,

$$F(\underline{x},\underline{0}) = 0.$$

5: RAPPEL

$$||\underline{\Theta}|| = 1$$

almost everywhere.

<u>6:</u> LEMMA Suppose that F is a parametric integrand -- then the integral

$$I(C) \equiv \int_a^b F(\underline{f}(x), \underline{\theta}(x))\, d\mu_C$$

exists.

PROOF $[C] \times U(M)$ is a compact set on which F is bounded. Since

$$(\underline{f}(x), \underline{\theta}(x)) \in [C] \times U(M)$$

almost everywhere, the function

$$F(\underline{f}(x), \underline{\theta}(x))$$

is Borel measurable and essentially bounded w.r.t. the measure μ_C. Therefore

$$I(C) \equiv \int_a^b F(\underline{f}(x), \underline{\theta}(x))\, d\mu_C$$

exists.

[Note: The requirement "homogeneous of degree 1" in t plays no role in the course of establishing the existence of I(C). It will, however, be decisive in the considerations to follow.]

Let $P \in \mathcal{P}[a,b]$ and let ξ_i be a point in $[x_{i-1}, x_i]$ $(i = 1,\ldots,n)$.

<u>7:</u> THEOREM If F is a parametric integrand, then

$$\lim_{||P|| \to 0} \sum_{i=1}^{n} F(\underline{f}(\xi_i), \underline{f}(x_i) - \underline{f}(x_{i-1}))$$

exists and equals I(C), denote it by the symbol

$$\int_C F,$$

and call it the <u>line integral</u> of F along C.

PROOF Fix $\varepsilon > 0$ and let $B(M)$ be the unit ball in R^M. Put

$$M_F = \sup_{[C] \times B(M)} |F|.$$

Choose $\gamma > 0$:

$$\begin{cases} \|\underline{x}_1 - \underline{x}_2\| < \gamma & (\underline{x}_1,\underline{x}_2 \in [C]) \\ \|\underline{t}_1 - \underline{t}_2\| < \gamma & (\underline{t}_1,\underline{t}_2 \in B(M)) \end{cases}$$

=>

$$|F(\underline{x}_1,\underline{t}_1) - F(\underline{x}_2,\underline{t}_2)| < \frac{\varepsilon}{3\ell(C)} .$$

Introduce $\underline{\eta}(x;P)$ and set

$$g(x;P) = F(\underline{f}(\xi_i),\underline{\eta}(x;P))$$

if $x_{i-1} < x < x_i$ -- then

$$\int_a^b g(x;P)d\mu_C = \sum_{i=1}^{n} F(\underline{f}(\xi_i), \frac{\underline{f}(x_i) - \underline{f}(x_{i-1})}{\mu_C([x_{i-1},x_i])}) \mu_C([x_{i-1},x_i])$$

$$= \sum_{i=1}^{n} F(\underline{f}(\xi_i), \underline{f}(x_i) - \underline{f}(x_{i-1}))$$

modulo the usual convention if $\mu_C([x_{i-1},x_i]) = 0$. Recall now that in $L^2(\mu_C)$,

$$\lim_{\|P\| \to 0} \underline{\eta}(—;P) = \underline{\Theta},$$

hence $\underline{\eta}(—;P)$ converges in measure to $\underline{\Theta}$, so there is a $\rho > 0$ such that for all P with $\|P\| < \rho$,

$$\|\underline{\Theta}(x) - \underline{\eta}(x;P)\| < \gamma$$

except on a set S_P of measure

$$\mu_C(S_P) < \frac{\varepsilon}{3M_F} .$$

Define σ:

$$|t_1 - t_2| < \sigma \Rightarrow \|f(t_1) - f(t_2)\| < \gamma.$$

Let $\delta = \min(\sigma,\rho)$ and let P be any partition with $||P|| < \delta$ -- then

$$I(C) - \sum_{i=1}^{n} F(\underline{f}(\xi_i),\underline{f}(x_i) - \underline{f}(x_{i-1}))$$

$$= \int_a^b F(\underline{f}(x),\underline{\Theta}(x))d\mu_C - \int_a^b g(x;P)d\mu_C$$

$$= \int_a^b [F(\underline{f}(x),\underline{\Theta}(x)) - g(x;P)]d\mu_C.$$

By definition, $\delta \leq \rho$, hence

$$||\underline{\Theta}(x) - \underline{\eta}(x;P)|| < \gamma$$

except in S_P, and

$$||\underline{f}(x) - \underline{f}(\xi_i)|| < \gamma$$

since

$$|x - \xi_i| < \gamma \quad (x_{i-1} \leq x \leq x_i).$$

To complete the argument, take absolute values:

$$|I(C) - \sum_{i=1}^{n} F(\underline{f}(\xi_i),\underline{f}(x_i) - \underline{f}(x_{i-1}))|$$

$$\leq \int_a^b |F(\underline{f}(x),\underline{\Theta}(x)) - g(x;P)|d\mu_C$$

$$= \int_{[a,b]-S_P} |\ldots|d\mu_C + \int_{S_P} |\ldots|d\mu_C.$$

- On $[a,b] - S_P$ at an index i,

$$|F(\underline{f}(x),\underline{\Theta}(x)) - g(x;P)|$$

$$= |F(\underline{f}(x),\underline{\Theta}(x)) - F(\underline{f}(\xi_i),\underline{\eta}(x;P))|$$

$$\leq \frac{\varepsilon}{3\mathcal{L}(C)}.$$

Here, of course, up to a set of measure 0,

$$\underline{\Theta}(x) \in B(M) \text{ and } \underline{\eta}(x;P) \in B(M).$$

Therefore

$$\int_{[a,b]-S_P} |\cdots| \, d\mu_C \leq \frac{\varepsilon}{3\ell(C)} \ell(C) = \frac{\varepsilon}{3}.$$

- On S_P,

$$\begin{bmatrix} |F(\underline{f}(x), \underline{\Theta}(x))| \leq M_F \\ |F(\underline{f}(\xi_i), \underline{\eta}(x;P))| \leq M_F. \end{bmatrix}$$

Therefore

$$\int_{SP} |\cdots| \, d\mu_C \leq 2M_F \int_{S_P} 1 d\mu_C$$

$$= 2M_F \mu_C(S_P)$$

$$< 2M_F \frac{\varepsilon}{3M_F} = \frac{2\varepsilon}{3}.$$

So in conclusion,

$$\int_{[a,b]-S_P} |\cdots| \, d\mu_C + \int_{S_P} |\cdots| \, d\mu_C$$

$$< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon \quad (||P|| < \delta)$$

and

$$I(C) = \int_C F.$$

<u>8:</u> <u>N.B.</u> The end result is independent of the choice of the ξ_i.

<u>9:</u> THEOREM If $f_1, \ldots, f_M \in AC[a,b]$, then for any parametric integrand F,

$$\int_C F = \int_a^b F(f_1(x),\ldots,f_M(x),\ f_1'(x),\ldots,f_M'(x))\,dx,$$

the integral on the right being in the sense of Lebesgue.

PROOF The absolute continuity of the f_m implies that

$$\mu_C([c,d]) = \int_c^d ||f'||\,dx$$

for every subinterval $[c,d] \subset [a,b]$, hence μ_C is absolutely continuous w.r.t. Lebesgue measure. It is also true that ν_m is absolutely continuous w.r.t. Lebesgue measure. This said, write

$$f_m' = \frac{df_m}{dx} = \frac{d\nu_m}{dx} = \frac{d\nu_m}{d\mu_C}\,\frac{d\mu_C}{dx} = \Theta_m \frac{d\mu_C}{dx}.$$

Then

$$\begin{aligned} I(C) &= \int_a^b F(\underline{f}(x),\underline{\Theta}(x))\,d\mu_C \\ &= \int_a^b F(\underline{f}(x),\underline{\Theta}(x))\,\frac{d\mu_C}{dx}\,dx \\ &= \int_a^b F(\underline{f}(x),\underline{\Theta}(x)\,\frac{d\mu_C}{dx})\,dx, \end{aligned}$$

where

$$\frac{d\mu_C}{dx} = ||\underline{f}'|| \geq 0.$$

Continuing

$$\begin{aligned} I(C) &= \int_a^b F(f_1(x),\ldots,f_M(x),\Theta_1(x)\,\frac{d\mu_C}{dx},\ldots,\ \Theta_M(x)\,\frac{d\mu_C}{dx})\,dx \\ &= \int_a^b F(f_1(x),\ldots,f_M(x),f_1'(x)\ ,\ldots,\ f_M'(x))\,dx, \end{aligned}$$

the integrals being in the sense of Lebesgue.

Let

$$\begin{cases} C \longleftrightarrow \underline{f}: [a,b] \to R \\ D \longleftrightarrow \underline{g}: [c,d] \to R \end{cases}$$

be curves, continuous and rectifiable.

<u>10:</u> RAPPEL If C and D are Fréchet equivalent, then

$$[C] = [D] \text{ and } \ell(C) = \ell(D).$$

<u>11:</u> THEOREM If C and D are Fréchet equivalent and if F is a parametric integrand, then

$$\int_C F = \int_D F.$$

PROOF Fix $\varepsilon > 0$ and choose $\delta > 0$:

- $P \in \mathcal{P}[a,b]$ & $||P|| < \delta \Rightarrow$

$$|I(C) - \sum_{i=1}^{n} F(\underline{f}(\xi_i), \underline{f}(x_i) - \underline{f}(x_{i-1}))| < \frac{\varepsilon}{3}.$$

- $Q \in \mathcal{P}[c,d]$ & $||Q|| < \delta \Rightarrow$

$$|I(D) - \sum_{j=1}^{m} F(\underline{f}(\xi_j), \underline{f}(y_j) - \underline{f}(y_{j-1}))| < \frac{\varepsilon}{3}.$$

Fix P and Q satisfying these conditions and let k be the number of intervals in P and let ℓ be the number of intervals in Q. Fix $\gamma > 0$ such that

$$|F(\underline{x}_1,\underline{t}_1) - F(\underline{x}_2,\underline{t}_2)| < \frac{\varepsilon}{3(k+\ell)}$$

when

$$||\underline{x}_1 - \underline{x}_2|| < \gamma \quad (\underline{x}_1,\underline{x}_2 \in [C] = [D])$$

and

$$||\underline{t}_1 - \underline{t}_2|| < 2\gamma \quad (||\underline{t}_1|| \le \ell(C),\ ||\underline{t}_2|| \le \ell(D)).$$

Let $\phi:[a,b] \to [c,d]$ be a homeomorphism ($\phi(a) = c$, $\phi(b) = d$) such that

$$||\underline{f}(x) - \underline{g}(\phi(x))|| < \gamma \quad (x \in [a,b]).$$

Let

$$P^*: a = x_0^* < x_1^* < \ldots < x_r^* = b$$

be the partition obtained from P by adjoining the images under ϕ^{-1} of the partition points of Q. Let

$$Q^*: c = y_0^* < y_1^* < \ldots < y_s^* = d$$

be the partition obtained from Q by adjoining the images under ϕ of the partition points of P. So, by construction, $r = s$, either one is $\leq k + \ell$, and $y_p^* = \phi(x_p^*)$ $(p = 0,1,\ldots,q)$. Choose a point $\xi_p \in [x_{p-1}^*, x_p^*]$ and work with

$$\underline{f}(\xi_p) \text{ and } \underline{g}(\phi(\xi_p)).$$

Then

$$|I(C) - I(D)|$$

$$\leq \left|I(C) - \sum_{p=1}^{q} F(\underline{f}(\xi_p), \underline{f}(x_p^*) - \underline{f}(x_{p-1}^*))\right|$$

$$+ \sum_{p=1}^{q} \left|F(\underline{f}(\xi_p), \underline{f}(x_p^*) - \underline{f}(x_{p-1}^*)) - F(\underline{g}(\phi(\xi_p)), \underline{g}(y_p^*) - \underline{g}(y_{p-1}^*))\right|$$

$$+ \left|\sum_{p=1}^{q} F(\underline{g}(\phi(\xi_p)), \underline{g}(y_p^*) - \underline{g}(y_{p-1}^*)) - I(D)\right|.$$

Since

$$\begin{cases} ||P^*|| \leq ||P|| < \delta \\ ||Q^*|| \leq ||Q|| < \delta, \end{cases}$$

the first and third terms are each $< \frac{\varepsilon}{3}$. As for the middle term,

$$||\underline{f}(\xi_p) - \underline{g}(\phi(\xi_p))|| < \gamma$$

and

$$||\underline{f}(x_p^*) - \underline{f}(x_{p-1}^*) - \underline{g}(y_p^*) + \underline{g}(y_{p-1}^*)||$$

$$\leq ||\underline{f}(x_p^*) - \underline{g}(y_p^*)|| + ||\underline{f}(x_{p-1}^*) - \underline{g}(y_{p-1}^*)||$$

$$= ||\underline{f}(x_p^*) - \underline{g}(\phi(x_p^*))|| + ||\underline{f}(x_{p-1}^*) - \underline{g}(\phi(x_{p-1}^*))||$$

$$< \gamma + \gamma = 2\gamma.$$

Therefore the middle term is

$$< q \frac{\varepsilon}{3(k + \ell)} = \frac{q}{k + \ell} \frac{\varepsilon}{3} < \frac{\varepsilon}{3}.$$

And finally

$$|I(C) - I(D)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$\Rightarrow$

$$I(C) = I(D) \quad (\varepsilon \downarrow 0)$$

$\Rightarrow$

$$\int_C F = \int_D F.$$

<u>12:</u> SETUP

- $C_0 \longleftrightarrow \underline{f}_0 : [a,b] \to R^M$

is a curve, continuous and rectifiable.

- $C_k \longleftrightarrow \underline{f}_k : [a,b] \to R^M \quad (k = 1,2,\ldots)$

is a sequence of curves, continuous and rectifiable.

Assumption: $\underline{f}_k$ converges uniformly to $\underline{f}_0$ in [a,b] and

$$\lim_{k \to \infty} \ell(C_k) = \ell(C_0).$$

<u>13:</u> THEOREM

$$\lim_{k \to \infty} I(C_k) = I(C_0)$$

or still,

$$\lim_{k \to \infty} \int_{C_k} F = \int_{C_0} F.$$

§9. *QUASI ADDITIVITY*

1: DATA A is a nonempty set, $\mathcal{I} = \{I\}$ is a nonempty collection of subsets of A, $\mathcal{D} = \{D\}$ is a nonempty collection of nonempty finite collections $D = [I]$ of sets $I \in \mathcal{I}$, and δ is a real valued function defined on $\mathcal{D}$.

2: DEFINITIONS The sets $I \in \mathcal{I}$ are called intervals, the collections $D \in \mathcal{D}$ are called systems, and the function δ is called a mesh.

3: ASSUMPTIONS A is a nonempty topological space, each interval I has a nonempty interior, the intervals of each system D are nonoverlapping: $I_1, I_2 \in D$, $I_1 \neq I_2$

=>

$$\begin{cases} \text{int } I_1 \cap c\ell\, I_2 = \emptyset \\ c\ell\, I_1 \cap \text{int } I_2 = \emptyset. \end{cases}$$

4: ASSUMPTION For each system D, $0 < \delta(D) < +\infty$, and each $\varepsilon > 0$, there are systems with $\delta(D) < \varepsilon$.

5: REMARK In the presence of δ, one is able to convert $\mathcal{D}$ into a directed set with direction "> >" by defining $D_2 >> D_1$ iff $\delta(D_2) < \delta(D_1)$.

6: EXAMPLE Take $A = [a,b]$ and let $\mathcal{I} = \{I\}$ be the collection of all closed subintervals of [a,b]. Take for $\mathcal{D}$ the class of all partitions D of [a,b], i.e., $\mathcal{D} = \mathcal{P}[a,b]$, and let $\delta(D)$ be the norm of D.

[Note: Strictly speaking, an element of $\mathcal{P}[a,b]$ is a finite set $P = \{x_0,\ldots,x_n\}$, where

$$a = x_0 < x_1 < \cdots < x_n = b,$$

the associated element D in $\mathcal{D}$ being the set

$$[x_{i-1},x_i] \quad (i = 1,\ldots,n).]$$

7: DEFINITION An interval function is a function $\phi:\mathcal{I} \to R^M$.

[Note: Associated with ϕ are the interval functions $||\phi||$, as well as

$$\phi_m, |\phi_m|, \begin{bmatrix} \phi_m^+ \\ \\ \phi_m^- \end{bmatrix} \quad (m = 1,\ldots,M).]$$

8: NOTATION Given an interval function ϕ, a subset $S \subset A$, and a system $D = [I]$, put

$$\Sigma[\phi,S,D] = \sum_I s(I,S)\phi(I),$$

where $\sum_I$ ranges over all $I \in D$ and $s(I,S) = 1$ or 0 depending on whether $I \subset S$ or $I \not\subset S$.

[Note: Take for S the empty set $\emptyset$ -- then $I \subset \emptyset$ is inadmissible (I has a nonempty interior) and $I \not\subset \emptyset$ gives rise to zero. Therefore

$$\Sigma[\phi,\emptyset,D] = 0.]$$

9: N.B. The absolute situation is when $S = A$, thus in this case,

$$\Sigma[\phi,A,D] \equiv \Sigma[\phi,D] = \sum_I \phi(I).$$

10: DEFINITION Given an interval function ϕ and a subset $S \subset A$, the BC-integral of ϕ over S is

$$\lim_{\delta(D) \to 0} \Sigma[\phi,S,D]$$

provided the limit exists in R^M.

[Note: B = Burkill and C = Cesari.]

11: NOTATION The BC-integral of ϕ over S is denoted by

$$BC \int_S \phi.$$

12: EXAMPLE

$$BC \int_{\emptyset} \phi = \underline{0} \ (\in R^M).$$

13: DEFINITION An interval function ϕ is quasi additive on S if for each $\varepsilon > 0$ there exists $\eta(\varepsilon,S) > 0$ such that if $D_0 = [I_0]$ is any system subject to $\delta(D_0) < \eta(\varepsilon,S)$ there also exists $\lambda(\varepsilon,S,D_0) > 0$ such that for every system $D = [I]$ with $\delta(D) < \lambda(\varepsilon,S,D_0)$, the relations

$$(qa_1\text{-}S) \sum_{I_0} s(I_0,S) \ \left|\left|\sum_{I} s(I,I_0)\phi(I) - \phi(I_0)\right|\right| < \varepsilon$$

$$(qa_2\text{-}S) \sum_{I} s(I,S) \ [1 - \sum_{I_0} s(I,I_0)s(I_0,S)] \ ||\phi(I)|| < \varepsilon$$

obtain.

14: N.B. In the absolute situation, matters read as follows: An interval function ϕ is quasi additive if for each $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that if $D_0 = [I_0]$ is any system subject to $\delta(D_0) < \eta(\varepsilon)$ there exists $\lambda(\varepsilon,D_0) > 0$ such that for every system $D = [I]$ with $\delta(D) < \lambda(\varepsilon,D_0)$, the relations

$$(qa_1\text{-}A) \ \sum_{I_0} \left|\left| \sum_{I \subset I_0} \phi(I) - \phi(I_0)\right|\right| < \varepsilon$$

$$(qa_2\text{-A}) \quad \sum_{I \not\subset I_0} ||\phi(I)|| < \varepsilon$$

obtain.

[Note: The sum

$$\sum_{I \not\subset I_0} ||\phi(I)||$$

is over all $I \in D$, $I \not\subset I_0$ for any $I_0 \in D_0$.]

So, under the preceding conditions,

$$\sum_{I} \phi(I) - \sum_{I_0} \phi(I_0)$$

$$= \sum_{I_0} [\sum_{I \subset I_0} \phi(I) - \phi(I_0)] + \sum_{I \not\subset I_0} \phi(I)$$

=>

$$||\sum_{I} \phi(I) - \sum_{I_0} \phi(I_0)|| < 2\varepsilon.$$

15: THEOREM If ϕ is quasi additive on S, then

$$\text{BC} \int_S \phi$$

exists.

PROOF To simplify the combinatorics, take S = A. Given $\varepsilon > 0$, let $\eta(\varepsilon)$, D_0, $\lambda(\varepsilon, D_0)$ be per qa_1-A, qa_2-A and suppose that $D_1, D_2 \in \mathcal{D}$, where

$$\left[\begin{array}{l} \delta(D_1) < \lambda(\varepsilon, D_0) \\ \delta(D_2) < \lambda(\varepsilon, D_0). \end{array}\right.$$

Then

$$\left[\begin{array}{l} \|\sum_{I_1} \phi(I_1) - \sum_{I_0} \phi(I_0)\| < 2\varepsilon \\ \\ \|\sum_{I_2} \phi(I_2) - \sum_{I_0} \phi(I_0)\| < 2\varepsilon \end{array}\right.$$

$$\Rightarrow \qquad \|\sum_{I_1} \phi(I_1) - \sum_{I_2} \phi(I_2)\| < 4\varepsilon.$$

Therefore BC $\int_A \phi$ exists.

16: REMARK

- If the ϕ_m $(m = 1,\ldots,M)$ are quasi additive, then ϕ is quasi additive.
- If the $|\phi_m|$ $(m = 1,\ldots,M)$ are quasi additive, then $\|\phi\|$ is quasi additive.

17: DEFINITION A real valued interval function ψ is quasi subadditive on S if for each $\varepsilon > 0$ there exists $\eta(\varepsilon,S) > 0$ such that if $D_0 = [I_0]$ is any system subject to $\delta(D_0) < \eta(\varepsilon,S)$ there also exists $\lambda(\varepsilon,S,D_0) > 0$ such that for every system $D = [I]$ with $\delta(D) < \lambda(\varepsilon,S,D_0)$ the relation

$$(\text{qsa} - S) \quad \sum_{I_0} s(I_0,S) \left[\sum_{I} s(I,I_0)\psi(I) - \psi(I_0)\right]^{-} < \varepsilon$$

obtains.

18: N.B. In the absolute situation, matters read as follows: ...

$$(\text{qsa} - A) \quad \sum_{I_0} \left[\sum_{I \subset I_0} \psi(I) - \psi(I_0)\right]^{-} < \varepsilon.$$

19: LEMMA If $\psi:\mathcal{D} \to R_{\geq 0}$ is nonnegative and quasi subadditive on S, then

$$\text{BC} \int_S \psi$$

exists ($+\infty$ is a permissible value).

20: THEOREM If $\psi:\mathcal{I} \to \mathbb{R}_{\geq 0}$ is nonnegative and quasi subadditive on S and if

$$\text{BC} \int_S \psi$$

is finite, then ψ is quasi additive on S.

PROOF To simplify the combinatorics, take $S = A$. Since

$$\text{BC} \int_A \psi$$

exists and is finite, given $\varepsilon > 0$ there is a number $\mu(\varepsilon) > 0$ such that for any $D_0 = [I_0] \in \mathcal{D}$ with $\delta(D_0) < \mu(\varepsilon)$, we have

$$\left|\text{BC} \int_A \psi - \sum_{I_0} \psi(I_0)\right| < \frac{\varepsilon}{3},$$

where $\sum_{I_0}$ is a sum ranging over all $I_0 \in D_0$. Now choose D_0 in such a way that

$$\delta(D_0) < \min\{\mu(\varepsilon), \eta(\varepsilon/6)\},$$

take

$$\lambda'(\varepsilon) = \min\{\mu(\varepsilon), \lambda(\varepsilon/6, D_0)\},$$

and consider any system $D = [I]$ with $\delta(D) < \lambda'$. Since ψ is quasi subadditive,

$$\sum_{I_0} \left[\sum_{I \subset I_0} \psi(I) - \psi(I_0)\right]^- < \frac{\varepsilon}{6}.$$

On the other hand,

$$\left|\text{BC} \int_A \psi - \sum_{I} \psi(I)\right| < \frac{\varepsilon}{3}.$$

Denote by Σ' a sum over all $I \in D$ with $I \not\subset I_0$ for any $I_0 \in D_0$ -- then

$$0 \leq \sum_{I_0} \left|\sum_{I \subset I_0} \psi(I) - \psi(I_0)\right| + \Sigma'\psi(I)$$

$$= \sum_{I_0} [\sum_{I \lhd I_0} \psi(I) - \psi(I_0)]$$

$$+ 2 \sum_{I_0} [\sum_{I \lhd I_0} \psi(I) - \psi(I_0)]^-$$

$$+ \Sigma' \psi(I)$$

$$= [\sum_{I} \psi(I) - BC \int_A \psi]$$

$$- [\sum_{I_0} \psi(I_0) - BC \int_A \psi]$$

$$+ 2 \sum_{I_0} [\sum_{I \lhd I_0} \psi(I) - \psi(I_0)]^-$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + 2 \frac{\varepsilon}{6} = \varepsilon.$$

The requirements for quasi additivity are thus met.

<u>21:</u> THEOREM Suppose that $\phi: \mathcal{I} \to R^M$ is quasi additive on S -- then $||\phi||: \mathcal{I} \to R_{\geq 0}$ is quasi subadditive on S.

PROOF Fix $\varepsilon > 0$, take $S = A$, and in the notation above, introduce $\eta(\varepsilon)$, $D_0 = [I_0]$, $\lambda(\varepsilon, D_0)$, $D = [I]$ -- then the objective is to show that

$$\sum_{I_0} [\sum_{I \lhd I_0} ||\phi(I)|| - ||\phi(I_0)||]^- < \varepsilon.$$

To this end, let

$$\Phi(I_0) = \sum_{I \lhd I_0} \phi(I) - \phi(I_0).$$

Then

$$||\phi(I_0) + \Phi(I_0)|| = \sum_{I \lhd I_0} \phi(I)$$

$$= \left[\sum_{m=1}^{M} \left(\sum_{I \subset I_0} \phi_m(I)\right)^2\right]^{1/2}$$

$$\leq \sum_{I \subset I_0} \left[\sum_{m=1}^{M} \phi_m(I)^2\right]^{1/2}$$

$$= \sum_{I \subset I_0} ||\phi(I)||.$$

Meanwhile

$$\phi(I_0) = [\phi(I_0) + \Phi(I_0)] + [- \Phi(I_0)]$$

=>

$$\sum_{I \subset I_0} ||\phi(I)|| - ||\phi(I_0)||$$

$$\geq ||\phi(I_0) + \Phi(I_0)|| - ||\phi(I_0)||$$

$$\geq - ||\Phi(I_0)||$$

=>

$$\left[\sum_{I \subset I_0} ||\phi(I)|| - ||\phi(I_0)||\right]^{-} \leq ||\Phi(I_0)||$$

=>

$$\sum_{I_0} \left[\sum_{I \subset I_0} ||\phi(I)|| - ||\phi(I_0)||\right]^{-} \leq \sum_{I_0} ||\Phi(I_0)||$$

$$= \sum_{I_0} \left|\left| \sum_{I \subset I_0} \phi(I) - \phi(I_0)\right|\right|$$

$$< \varepsilon,$$

ϕ being quasi additive.

<u>22:</u> APPLICATION If $\phi: \mathcal{I} \to R^M$ is quasi additive, then the interval functions

$$I \to |\phi_m(I)| \quad (m = 1,\ldots,M)$$

are quasi subadditive.

[In fact, the quasi additivity of ϕ implies the quasi additivity of the ϕ_m and

$$||\phi_m|| = |\phi_m|.]$$

[Note: It is also true that ϕ_m^+, ϕ_m^- are quasi subadditive.]

<u>23:</u> LEMMA If $\phi : \mathcal{I} \to R^M$ is quasi additive on S and if

$$\text{BC} \int_S ||\phi|| < +\infty,$$

then ϕ is quasi additive on every subset $S' \subset S$.

PROOF First of all, $||\phi||$ is quasi subadditive on S, hence also on S'. Therefore

$$\text{BC} \int_{S'} ||\phi||$$

exists and

$$\text{BC} \int_{S'} ||\phi|| \leq \text{BC} \int_S ||\phi|| < +\infty,$$

from which it follows that $||\phi||$ is quasi additive on S'. Given $\varepsilon > 0$, determine the parameters in the definition of quasi additive in such a way that the relevant relations are simultaneously satisfied per ϕ on S and per $||\phi||$ on S', hence

$$\sum_{I_0} s(I_0,S')\left|\left|\sum_I s(I,I_0)\phi(I) - \phi(I_0)\right|\right|$$

$$\leq \sum_{I_0} s(I_0,S)\left|\left|\sum_I s(I,I_0)\phi(I) - \phi(I_0)\right|\right| < \varepsilon$$

and

$$\sum_I s(I,S')[1 - \sum_{I_0} s(I,I_0)s(I_0,S')]\,||\phi(I)|| < \varepsilon.$$

Therefore ϕ is quasi additive on S'.

24: APPLICATION If $\phi: \mathcal{I} \to R^M$ is quasi additive and if

$$BC \int_A ||\phi|| < +\infty,$$

then ϕ is quasi additive on every subset of A.

Here is a summary of certain fundamental points of this §. Work with ϕ and $||\phi||$.

- Suppose that $||\phi||$ is quasi subadditive on S and

$$BC \int_S ||\phi|| < +\infty.$$

Then $||\phi||$ is quasi additive on S.

- Suppose that ϕ is quasi additive on S -- then $||\phi||$ is quasi subadditive on S.

So: If ϕ is quasi additive on S AND if

$$BC \int_S ||\phi|| < +\infty,$$

then $||\phi||$ is quasi additive on S.

[Note: It is not true in general that $||\phi||$ quasi additive implies ϕ quasi additive.]

25: EXAMPLE Take A = [a,b] and let $\mathcal{I}$, $\mathcal{D}$, and δ be as at the beginning. Given a continuous curve

$$C \longleftrightarrow \underline{f}:[a,b] \to R^M,$$

define a quasi additive interval function $\phi: \mathcal{I} \to R^M$ by the rule

$$\begin{aligned}\phi(I) &= (\phi_1(I),\ldots,\phi_M(I)) \\ &= (f_1(d) - f_1(c),\ldots,f_M(d) - f_M(c)),\end{aligned}$$

where $I = [c,d] \subset [a,b]$, thus

$$||\phi(I)|| = ||\underline{f}(d) - \underline{f}(c)||,$$

so if $P \in \mathcal{P}[a,b]$ corresponds to

$$D \longleftrightarrow \{[x_{i-1},x_i]:i = 1,\ldots,n\},$$

then

$$\sum_{I \in D} ||\phi(I)|| = \sum_{i=1}^{n} ||\underline{f}(x_i) - \underline{f}(x_{i-1})||$$

=>

$$BC \int_A ||\phi|| = \lim_{\delta(D) \to 0} \sum_{I \in D} ||\phi(I)||$$

$$= \lim_{||P|| \to 0} \sum_{i=1}^{n} ||\underline{f}(x_i) - \underline{f}(x_{i-1})||$$

$$= \ell(C).$$

Therefore C is rectifiable iff

$$BC \int_A ||\phi|| < +\infty.$$

And when this is the case, $||\phi||$ is quasi additive on A.

[Note: A priori,

$$\ell(C) = \sup_{P \in \mathcal{P}[a,b]} \sum_{i=1}^{n} ||\underline{f}(x_i) - \underline{f}(x_{i-1})||.$$

But here, thanks to the continuity of $\underline{f}$, the sup can be replaced by lim.]

<u>26:</u> EXAMPLE Take $A = [a,b]$ and let $\mathcal{I}$ and $\mathcal{D}$ be as above. Suppose that

$$C \longleftrightarrow \underline{f}:[a,b] \to R^M$$

is a rectifiable curve, potentially discontinuous.

- Given $a \leq x_0 < b$, put

$$s^+(x_0) = \limsup_{x \downarrow x_0} ||\underline{f}(x) - \underline{f}(x_0)||$$

and let $s^+(b) = 0$.

- Given $a < x_0 \leq b$, put

$$s^-(x_0) = \limsup_{x \uparrow x_0} ||\underline{f}(x) - \underline{f}(x_0)||$$

and let $s^-(a) = 0$. Combine the data and set

$$s(x) = s^+(x) + s^-(x) \quad (a \leq x \leq b).$$

Then $s(x)$ is zero everywhere save for at most countably many x and

$$\sigma = \sum_x s(x) \leq \ell(C).$$

Take ϕ as above and define a mesh δ by the rule

$$\delta(D) = ||P|| + \sigma - \sum_{i=0}^{n} s(x_i).$$

One can then show that ϕ is quasi additive and

$$\text{BC} \int_A ||\phi|| = \ell(C).$$

<u>27:</u> NOTATION Given a quasi additive interval function ϕ, let

$$V[\phi,S] = \sup_{D \in \mathcal{D}} \Sigma[||\phi||,S,D].$$

<u>28:</u> <u>N.B.</u> By definition,

$$\text{BC} \int_S ||\phi|| = \lim_{\delta(D) \to 0} \Sigma[||\phi||,S,D],$$

so

$$\text{BC} \int_S ||\phi|| \leq V[\phi,S]$$

and strict inequality may hold.

29: LEMMA Given a quasi additive ϕ and a subset $S \subset A$, suppose that for every $\varepsilon > 0$ and any $D_0 = [I_0]$ there exists $\lambda(\varepsilon, S, D_0) > 0$ such that for every system $D = [I]$ with $\delta(D) < \lambda(\varepsilon, S, D_0)$ the relation

$$\sum_{I_0} s(I_0, S)\Big[\sum_{I} s(I, I_0)\,||\phi(I)|| - ||\phi(I_0)||\Big]^{-} < \varepsilon$$

obtains -- then

$$\mathrm{BC} \int_S ||\phi|| = V[\phi, S].$$

§10. *LINE INTEGRALS* (bis)

Through out this §, the situation will be absolute, where $A = [a,b]$ and $\mathcal{I}$, $\mathcal{D}$, and δ have their usual connotations.

If

$$C \longleftrightarrow \underline{f}: [a,b] \to R^M$$

is a curve, continuous and rectifiable, then

$$BC \int_A ||\phi|| = \ell(C).$$

And if F is a parametric integrand, then

$$\int_C F = \lim_{||P|| \to 0} \sum_{i=1}^{n} F(\underline{f}(\xi_i), \underline{f}(x_i) - \underline{f}(x_{i-1}))$$

exists, the result being independent of the ξ_i.

1: N.B. Recall the procedure: Introduce the integral

$$I(C) = \int_a^b F(\underline{f}(x), \underline{\theta}(x)) d\mu_C$$

and prove that

$$\lim_{||P|| \to 0} \sum_{i=1}^{n} F(\underline{f}(\xi_i), \underline{f}(x_i) - \underline{f}(x_{i-1}))$$

exists and equals I(C), the result being denoted by the symbol

$$\int_C F$$

and called the line integral of F along C.

There is another approach to all this which does not use measure theory. Thus define an interval function $\Phi: \mathcal{I} \to R$ by the prescription

$$\Phi(I;\xi) = F(\underline{f}(\xi), \phi(I)),$$

where $\xi \in I$ is arbitrary.

[Note: By definition,

$$\phi(I) = (\phi_1(I),\ldots,\phi_M(I))$$

$$= (f_1(d) - f_1(c),\ldots,f_M(d) - f_M(c)),$$

I being $[c,d] \subset [a,b]$. Moreover, ϕ is quasi additive.]

<u>2:</u> THEOREM Φ is quasi additive.

Admit the contention -- then

$$\lim_{\delta(D) \to 0} \sum_{I \in D} \Phi(I;\xi)$$

$$= \lim_{||P|| \to 0} \sum_{i=1}^{n} F(\underline{f}(\xi_i),\underline{f}(x_i) - \underline{f}(x_{i-1}))$$

exists, call it

$$(\xi) \int_C F.$$

<u>3:</u> <u>N.B.</u> Needless to say, it turns out that

$$(\xi) \int_C F$$

is independent of the ξ (this follows by a standard "$\varepsilon/3$" argument) (details at the end).

[Note: This is one advantage of the approach via $I(C)$ in that independence is manifest.]

To simplify matters, it will be best to generalize matters.

Assume from the outset that $\phi:\mathcal{I} \to R^M$ is now an arbitrary interval function which is quasi additive with

$$(BC) \int_A ||\phi|| < +\infty,$$

hence that $||\phi||$ is also quasi additive as well.

Introduce another interval function $\zeta: \mathcal{I} \to R^N$ and expand the definition of parametric integrand so that

$$F: X \times R^M \to R,$$

where $X \subset R^N$ is compact and $\zeta(I) \subset X$.

4: EXAMPLE To recover the earlier setup, take $N = M$, keep $\phi: \mathcal{I} \to R^M$, let $\omega: \mathcal{I} \to [a,b]$ be a choice function, i.e., suppose that $\omega(I) \in I \subset [a,b]$, let $\zeta(I) = \underline{f}(\omega(I))$, and take $X = [C] \subset R^M$.

5: CONDITION (ζ) $\forall\ \varepsilon > 0$, $\exists\ t(\varepsilon) > 0$ such that if $D_0 = [I_0]$ is any system subject to $\delta(D_0) < t(\varepsilon)$ there also exists $T(\varepsilon, D_0)$ such that for any system $D = [I]$ with $\delta(D) < T(\varepsilon, D_0)$, the relation

$$\max_{I_0} \max_{I \subset I_0} ||\zeta(I) - \zeta(I_0)|| < \varepsilon$$

obtains.

6: N.B. Owing to the uniform continuity of $\underline{f}$, this condition is automatic in the special case supra.

7: THEOREM Let F be a parametric integrand, form the interval function $\Phi: \mathcal{I} \to R$ defined by the prescription

$$\Phi(I) = F(\zeta(I), \phi(I)),$$

and impose condition (ζ) -- then Φ is quasi additive.

The proof will emerge from the discussion below but there are some preliminaries that have to be dealt with first.

Start by writing down simultaneously (qa_1-A) and (qa_2-A) for ϕ and $||\phi||$ (both are quasi additive), $\bar{\varepsilon}$ to be determined.

$$\sum_{I_0} \left|\left| \sum_{I \subset I_0} \phi(I) - \phi(I_0) \right|\right| < \bar{\varepsilon}$$

$$\sum_{I \not\subset I_0} ||\phi(I)|| < \bar{\varepsilon}$$

$$\sum_{I_0} \left| \sum_{I \subset I_0} ||\phi(I)|| - ||\phi(I_0)|| \right| < \bar{\varepsilon}$$

$$\sum_{I \not\subset I_0} \left| \, ||\phi(I)|| \, \right| < \bar{\varepsilon}$$

for $\delta(D_0) < \eta(\bar{\varepsilon})$ and $\delta(D) < \lambda(\bar{\varepsilon}, D_0)$ and in addition

$$\left| \sum_{I \in D} ||\phi(I)|| - BC \int_A ||\phi|| \right| < \bar{\varepsilon}$$

for $\delta(D) < \sigma(\bar{\varepsilon})$.

Fix $\varepsilon > 0$. Put

$$V = BC \int_A ||\phi|| \qquad (< +\infty).$$

- (F) $X \times U(M)$ is a compact set on which F is bounded:

$$|F(\underline{x},\underline{t})| \leq C \quad (\underline{x} \in X, \ \underline{t} \in U(M))$$

and uniformly continuous: $\exists \ \gamma$ such that

$$\left.\begin{array}{l} ||\underline{x} - \underline{x}'|| \\ \\ ||\underline{t} - \underline{t}'|| \end{array}\right] < \gamma \Rightarrow |F(\underline{x},\underline{t}) - F(\underline{x}',\underline{t}')| < \frac{\varepsilon}{3(V+\varepsilon)} .$$

• (α)

$$\alpha(I_0) = \frac{\phi(I_0)}{||\phi(I_0)||} \text{ if } \phi(I_0) \neq 0$$

but 0 otherwise and

$$\alpha(I) = \frac{\phi(I)}{||\phi(I)||} \text{ if } \phi(I) \neq 0$$

but 0 otherwise.

8: NOTATION Denote by

$$\Sigma_{\gamma+}^{(I_0)}$$

the sum over the $I \subset I_0$ for which

$$||\alpha(I_0) - \alpha(I)|| \geq \gamma$$

and denote by

$$\Sigma_{\gamma-}^{(I_0)}$$

the sum over the $I \subset I_0$ for which

$$||\alpha(I_0) - \alpha(I)|| < \gamma.$$

Therefore

$$\sum_{I \subset I_0} = \Sigma_{\gamma+}^{(I_0)} + \Sigma_{\gamma-}^{(I_0)} .$$

9: LEMMA

$$\frac{\gamma^2}{2} \sum_{I_0} \Sigma_{\gamma+}^{(I_0)} ||\phi(I)||$$

$$\leq \sum_{I_0} \Big|\Big| \sum_{I \lhd I_0} \phi(I) - \phi(I_0) \Big|\Big|$$

$$+ \sum_{I_0} \Big| \sum_{I \lhd I_0} ||\phi(I)|| - ||\phi(I_0)|| \Big|.$$

PROOF The inequality

$$||\alpha(I_0) - \alpha(I)|| \geq \gamma$$

implies that

$$\gamma^2 \leq ||\alpha(I_0) - \alpha(I)||^2$$

$$= (\alpha(I_0) - \alpha(I)) \cdot (\alpha(I_0) - \alpha(I))$$

$$= ||\alpha(I_0)||^2 - 2\alpha(I_0) \cdot \alpha(I) + ||\alpha(I)||^2$$

$$= 2 - 2\alpha(I_0) \cdot \alpha(I),$$

so

$$\frac{\gamma^2}{2} \leq 1 - \alpha(I_0) \cdot \alpha(I)$$

$$\Rightarrow$$

$$\frac{\gamma^2}{2} ||\phi(I)|| \leq ||\phi(I)|| - \alpha(I_0) \cdot \phi(I).$$

But for any I,

$$0 \leq ||\phi(I)|| - \alpha(I_0) \cdot \phi(I).$$

Proof: In fact,

$$||\phi(I)|| - \frac{\phi(I_0) \cdot \phi(I)}{||\phi(I_0)||}$$

$$= \frac{1}{||\phi(I_0)||} [||\phi(I)|| \, ||\phi(I_0)|| - \phi(I_0) \cdot \phi(I)].$$

Now quote Schwarz's inequality. Thus we may write

$$\frac{\gamma 2}{2}\,\Sigma_{\gamma+}^{(I_0)}\,||\phi(I)||$$

$$\leq \Sigma_{\gamma+}^{(I_0)}\,(||\phi(I) - \alpha(I_0)\cdot\phi(I))||$$

$$\leq \sum_{I\subset I_0}(||\phi(I) - \alpha(I_0)\cdot\phi(I))||$$

$$= \sum_{I\subset I_0}||\phi(I)|| - ||\phi(I_0)|| + \alpha(I_0)\cdot(\phi(I_0) - \sum_{I\subset I_0}\phi(I))$$

$$\leq \Big|\sum_{I\subset I_0}||\phi(I)|| - ||\phi(I_0)||\Big|$$

$$+ ||\phi(I_0) - \sum_{I\subset I_0}\phi(I)|| \quad \text{(Schwarz)}.$$

To finish, sum over I_0.

- (D_0) Assume

$$\delta(D_0) < \min\{t(\gamma),\eta(\varepsilon),\eta(\varepsilon\gamma^2)\}.$$

- (D) Assume

$$\delta(D) < \min\{\sigma(\varepsilon),\lambda(\varepsilon,D_0),\lambda(\varepsilon\gamma^2,D_0),T(\gamma,D_0)\}.$$

- $(\bar{\varepsilon})$ Assume

$$\bar{\varepsilon} < \min\{\gamma, \frac{\varepsilon}{3C}, \frac{\varepsilon\gamma^2}{24C}\}.$$

Then

$$\sum_{I_0}\Big|\sum_{I\subset I_0}\Phi(I) - \Phi(I_0)\Big|$$

$$= \sum_{I_0} \Big| \sum_{I\subset I_0} F(\zeta(I),\phi(I))$$

$$- \sum_{I\subset I_0} F(\zeta(I_0),\alpha(I_0))\,||\phi(I)||$$

$$+ \sum_{I\subset I_0} F(\zeta(I_0),\alpha(I_0))\,||\phi(I)||$$

$$- F(\zeta(I_0),\alpha(I_0))\,||\phi(I_0|| \Big|$$

$$= \sum_{I_0} \Big| \sum_{I\subset I_0} (F(\zeta(I),\alpha(I)) - F(\zeta(I_0),\alpha(I_0)))\,||\phi(I)||$$

$$+ \sum_{I\subset I_0} F(\zeta(I_0),\alpha(I_0))(||\phi(I)|| - ||\phi(I_0)||) \Big|$$

$$\leq \sum_{I_0} |F(\zeta(I_0),\alpha(I_0))| \; \Big| \sum_{I\subset I_0} ||\phi(I)|| - ||\phi(I_0)|| \; \Big|$$

$$+ \sum_{I_0} \sum_{I\subset I_0} |F(\zeta(I),\alpha(I)) - F(\zeta(I_0),\alpha(I_0))| \; ||\phi(I)||$$

$$= \sum_{I_0} |F(\zeta(I_0),\alpha(I_0))| \; \Big| \sum_{I\subset I_0} ||\phi(I)|| - ||\phi(I_0)|| \; \Big|$$

$$+ \sum_{I_0} \Big(\sum_{\gamma-}^{(I_0)} + \sum_{\gamma+}^{(I_0)}\Big) |F(\zeta(I_0),\alpha(I_0)) - F(\zeta(I),\alpha(I))| \; ||\phi(I)||.$$

<u>First</u>:

$$\sum_{I_0} |F(\zeta(I_0),\alpha(I_0))| \; \Big| \sum_{I\subset I_0} ||\phi(I)|| - ||\phi(I_0)|| \; \Big|$$

$$\leq C \sum_{I_0} \Big| \sum_{I\subset I_0} ||\phi(I)|| - ||\phi(I_0)|| \; \Big|$$

$$< C\bar{\varepsilon} < C\,\frac{\varepsilon}{3C} = \frac{\varepsilon}{3}\,.$$

Second: Consider

$$\sum_{I_0} \Sigma_{\gamma-}^{(I_0)} |F(\zeta(I_0),\alpha(I_0)) - F(\zeta(I),\alpha(I))|\ ||\phi(I)||.$$

Here

$$||\alpha(I_0)|| = 1,\ ||\alpha(I)|| = 1,\ ||\alpha(I_0) - \alpha(I)|| < \gamma,$$

$$||\zeta(I_0) - \zeta(I)|| < \gamma$$

=>

$$|F(\zeta(I_0),\alpha(I_0)) - F(\zeta(I),\alpha(I))| < \frac{\varepsilon}{3(V+\varepsilon)}\,.$$

The entity in question is thus majorized by

$$\frac{\varepsilon}{3(V+\varepsilon)} \sum_{I_0} \Sigma_{\gamma-}^{(I_0)} ||\phi(I)|| \leq \frac{\varepsilon}{3(V+\varepsilon)} \sum_{I\in D} ||\phi(I)||$$

$$\leq \frac{\varepsilon}{3(V+\varepsilon)}\ (V+\varepsilon) = \frac{\varepsilon}{3}\,.$$

Third:

$$\sum_{I_0} \Sigma_{\gamma+}^{(I_0)} |F(\zeta(I_0),\alpha(I_0)) - F(\zeta(I),\alpha(I))|\ ||\phi(I)||$$

$$\leq 2C \sum_{I_0} \Sigma_{\gamma+}^{(I_0)} ||\phi(I)||$$

$$\leq \frac{4C}{\gamma^2} [\sum_{I_0} \sum_{I\subset I_0} ||\phi(I) - \phi(I_0)||$$

$$+ \sum_{I_0} |\sum_{I\subset I_0} ||\phi(I)|| - ||\phi(I_0)||\ |]$$

$$\le \frac{4C}{\gamma^2}(\bar{\varepsilon} + \bar{\varepsilon})$$

$$= \frac{8C}{\gamma^2}\bar{\varepsilon}$$

$$< \frac{8C}{\gamma^2} \cdot \frac{\varepsilon\gamma^2}{24C} = \frac{\varepsilon}{3} .$$

In total then:

$$\sum_{I_0} \left| \sum_{I \subset I_0} \Phi(I) - \Phi(I_0) \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$= \varepsilon.$$

And finally

$$\sum_{I \not\subset I_0} |\Phi(I)|$$

$$= \sum_{I \not\subset I_0} | F(\zeta(I), \phi(I)) |$$

$$= \sum_{I \not\subset I_0} |F(\zeta(I), \alpha(I))| \; ||\phi(I)||$$

$$\le C \sum_{I \not\subset I_0} ||\phi(I)||$$

$$< C\bar{\varepsilon} < C \frac{\varepsilon}{3C} = \frac{\varepsilon}{3} < \varepsilon.$$

Therefore Φ is quasi additive. And since the conditions on F carry over to $|F|$, it follows that $||\Phi||$ is also quasi additive, hence

$$\text{BC} \int_A ||\Phi||$$

exists and is finite.

To tie up one loose end, return to the beginning and consider the line integrals

$$(\xi) \int_C F, \; (\xi') \int_C F,$$

the claim being that they are equal. That this is so can be seen by writing

$$|(\xi)\int_C F - (\xi')\int_C F|$$

$$= |(\xi)\int_C F - \sum_{i=1}^{n} F(\underline{f}(\xi_i),\underline{f}(x_i) - \underline{f}(x_{i-1}))$$

$$+ \sum_{i=1}^{n} F(\underline{f}(\xi_i),\underline{f}(x_i) - \underline{f}(x_{i-1}))$$

$$- \sum_{i=1}^{n} F(\underline{f}(\xi_i'),\underline{f}(x_i) - \underline{f}(x_{i-1}))$$

$$+ \sum_{i=1}^{n} F(\underline{f}(\xi_i'),\underline{f}(x_i) - \underline{f}(x_{i-1})) - (\xi')\int_C F|$$

and proceed from here in the obvious way.

<u>10:</u> EXAMPLE Take $N = 1$, $M = 1$ and define an interval function $|..|:\mathcal{I} \to R$ by sending I to its length $|I|$. Fix a choice function $\omega:\mathcal{I} \to [a,b]$. Consider a curve

$$C \longleftrightarrow f:[a,b] \to R.$$

Assume: f is continuous and of bounded variation, thus

$$\ell(C) = T_f[a,b] < +\infty.$$

Work with the parametric integrand $F(x,t) = xt$ -- then the data

$$I \to F(\zeta(I),|I|)$$

$$= F(f(\omega(I)), |I|)$$

$$= f(\omega(I))\ |I|$$

leads to sums of the form

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}),$$

hence to

$$\int_C F = \int_a^b f,$$

the Riemann integral of f.